Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 83-93

A COUNTABLY-BASED DOMAIN REPRESENTATION OF A NON-REGULAR HAUSDORFF SPACE

HAROLD BENNETT AND DAVID LUTZER

Abstract. In this paper, we give an example of a countably-based algebraic domain D such that $\max(D)$ is Hausdorff but not regular in the relative Scott topology, and such that $\max(D)$ contains the usual space of rational numbers as a closed subspace. Our example shows that certain known results about $\max(D)$, where $\max(D)$ is regular and D is countably based, are the sharpest possible.

1. Introduction

From the viewpoint of traditional topology, a domain D with the Scott topology is not a good space. As noted in [11], it is a T_0 space that is essentially never T_1. However, its subspace $\max(D)$ of maximal elements will always be at least T_1 and has surprising properties that follow from domain-theoretic arguments using elements of $D - \max(D)$. For example, the ability to find suprema of directed subsets of $D - \max(D)$ guarantees that the subspace $\max(D)$ is always a Baire space, and more [9].

Many important results in the theory of domain representability of topological spaces begin by assuming that $\max(D)$ is a T_3-space in its relative Scott topology and that D is countably based, as the following theorem illustrates.

2000 *Mathematics Subject Classification.* Primary 54H99; Secondary 06B35, 06B30, 54D80.

Key words and phrases. domain, domain-representable space, countably-based domain, Hausdorff but not regular.

Theorem 1.1 ([8]). *Suppose that the T_3-space X is homeomorphic to the subspace* $\max(D)$ *of some countably-based domain D with the Scott topology. Then:*

(a) *X is separable and completely metrizable;*
(b) *X is a G_δ-subset of D;*
(c) $\max(D)$ *is the kernel of a measurement on D; i.e., there is a Scott continuous mapping $\mu : D \to [0,\infty)^*$ (where $[0,\infty)^*$ is the ordered set $[0,\infty)$ with its order reversed) that induces the Scott topology such that* $\max(D) = \{x \in D : \mu(x) = 0\}$;
(d) *if $M^1(X)$ is the space of mass one Borel measures on X endowed with the weak topology, and if $P^1(D)$ is the probabilistic power domain*[1] *of D, then $M^1(X)$ is homeomorphic to a subspace of* $\max(P^1(D))$ [3].

Is Theorem 1.1 the sharpest possible result? First, could one eliminate the hypothesis that D has a countable domain base? Second, could the hypothesis that X is regular be relaxed, e.g., to the hypothesis that X is Hausdorff?

The first question is easily answered. The Michael line $\mathbb{M}$, a regular space that is domain-representable [1] but neither separable nor metrizable, shows that the countable base assumption is needed if part (a) of the theorem is to hold. A second example, where (a) holds but (b) does not, can be obtained by letting $D = [0, \omega_1]$ with the usual order. We obtain a Scott domain[2] that contains the complete separable metric space $\max(D) = \{\omega_1\}$ as a dense subspace that is not a G_δ-subset of D. In this example, $\max(D)$ cannot be the kernel of a measurement so that part (c) fails without a countable base for the domain.

The second question–whether X Hausdorff is enough to prove part or all of Theorem 1.1–is harder. We answer it in the negative using the example described in the next section. The Hausdorff, non-regular space X used in the example is certainly not new, and earlier results established that this space X is domain-representable [2]. What is new is that there is a *countably-based* domain having X

[1]By $P^1(D)$, we mean the poset of all continuous valuations v on D with $v(D) = 1$, endowed with the partial order that has $v_1 \sqsubseteq v_2$ if and only if $v_1(d) \le v_2(d)$ for each $d \in D$

[2]A *Scott domain* is a continuous dcpo $(D, \sqsubseteq)$ with the property that $\sup(a, b) \in D$ whenever $a, b \in D$ and some $c \in D$ has $a, b \sqsubseteq c$.

as its space of maximal elements. This example answers a question in [8].

The space X will also answer a family of other interrelated questions in the literature. Because $\max(D)$ is not a G_δ-subset of D, our example answers another question in [8]. Because X contains the usual space of rational numbers as a closed subspace, X also answers a question in [10]. We are indebted to Keye Martin for a central idea in our approach–his suggestion that a countably-based domain-representable space might have a closed subset that is not a Baire space. (As noted above, it was already known that the usual space of rational numbers can be a closed subspace of a domain-representable space, namely the Michael line $\mathbb{M}$, but the domain used to represent $\mathbb{M}$ is not countably based [1].)

We do not know whether Abbas Edalat's result in [3] (part (d) in the above theorem) holds for our example. The referee noted that it would be important in domain theory to decide this question.

A *poset* is a partially ordered set. Recall that a poset $(D, \sqsubseteq)$ is a *dcpo* if each nonempty directed subset of D has a supremum in D. Zorn's lemma guarantees that any dcpo contains maximal elements, and the set of all maximal elements of D is denoted by $\max(D)$. For $a, b \in D$, we write $a \ll b$ to mean that whenever E is a nonempty directed set with $b \sqsubseteq \sup(E)$, then $a \sqsubseteq e$ for some $e \in E$. The set $\{a \in Q : a \ll b\}$ is denoted by $\Downarrow(b)$, and the poset is *continuous* provided $\Downarrow(b)$ is directed and has $b = \sup(\Downarrow(b))$ for each $b \in D$. A *domain* is a continuous dcpo. In a domain, the sets $\Uparrow(a) = \{b \in D : a \ll b\}$ form a basis for a topology called the *Scott topology*. To say that a topological space X is *domain-representable* means that there is some domain $(D, \sqsubseteq)$ such that X is homeomorphic to the subspace $\max(D)$ of D, topologized using the relative Scott topology.

The authors are indebted to Keye Martin for suggesting our search for the above example and for explaining its significance in domain theory.

Throughout this paper we reserve the symbols $\mathbb{R}$, $\mathbb{P}$, and $\mathbb{Q}$ for the sets of real, irrational, and rational numbers, respectively.

2. Construction of the example

Example 2.1. *There is a domain $(D, \sqsubseteq)$ with a countable domain base B such that, with the relative Scott topology,* $\max(D)$ *is Hausdorff but not regular and contains the usual space of rational numbers as a closed subspace. Further,* $\max(D)$ *is not a G_δ-subset of D and is not the kernel of any measurement on D. Finally, each element $b \in B$ is compact in the domain-theoretic sense (i.e., has $b \ll b$) so that $(D, \sqsubseteq)$ is algebraic in the sense of* [11].

Proof: The space X that will be $\max(D)$ in our example is obtained by defining a new topology τ on $\mathbb{R}$. In τ, each rational number has its usual basic neighborhoods $(x-\epsilon, x+\epsilon)$, and any $x \in \mathbb{P}$ has τ neighborhoods of the form $(x-\epsilon, x+\epsilon) \cap \mathbb{P}$. Then (X, τ) is a Hausdorff space that is not regular, the set $\mathbb{P}$ is a dense open subset of X, and the set $\mathbb{Q}$ is a closed subset of X. Furthermore, the relative topologies $\tau|\mathbb{Q}$ and $\tau|\mathbb{P}$ are the usual metrizable topologies on $\mathbb{Q}$ and $\mathbb{P}$, respectively.

Let $\mathcal{J}$ be the collection of all closed intervals $[a, b] \subseteq \mathbb{R}$ with $a, b \in \mathbb{Q}$, and $a < b$. For any $I \in \mathcal{J}$, let $L(I)$ be the length if I. Let $\mathbb{Q} = \{r_n : n \geq 1\}$ be a fixed indexing of $\mathbb{Q}$. For any set $S \subseteq \mathbb{R}$, we will write $\text{Int}(S)$ for the interior of S in the usual topology of $\mathbb{R}$.

We define three sets A, B, and C by

$$A = \{(I, 1, n) : I \in \mathcal{J},\ n \geq 1,\ L(I) < n^{-1}\},$$

$$B = \{(I, 2, n) : n \geq 1,\ I \in \mathcal{J},\ L(I) < n^{-1},\ I \cap \{r_1, r_2, \cdots, r_n\} = \emptyset\},$$

$$\text{and}\, C = \{(x, 1) : x \in \mathbb{R}\} \cup \{(y, 2) : y \in \mathbb{P}\}.$$

Let $D = A \cup B \cup C$, and for $d_1, d_2 \in D$, define $d_1 \sqsubseteq d_2$ if and only if one of the following holds:

a) $d_1 = d_2$;
b) $d_1 = (I, 1, m),\ d_2 = (J, 1, n),\ m < n$, and $J \subseteq \text{Int}(I)$;
c) $d_1 = (I, 1, m),\ d_2 = (J, 2, n),\ m < n$, and $J \subseteq \text{Int}(I)$;
d) $d_1 = (I, 2, m),\ d_2 = (J, 2, n),\ m < n$, and $J \subseteq \text{Int}(I)$;
e) $d_1 = (I, 1, m),\ d_2 = (x, 1),\ x \in \mathbb{R}$, and $x \in \text{Int}(I)$;
f) $d_1 = (I, 1, m),\ d_2 = (x, 2),\ x \in \mathbb{P}$, and $x \in \text{Int}(I)$;
g) $d_1 = (x, 1)$ and $d_2 = (x, 2)$ for $x \in \mathbb{P}$;
h) $d_1 = (I, 2, m),\ d_2 = (x, 2)$ for $x \in \mathbb{P}$ and $x \in \text{Int}(I)$;
i) $d_1 = (I, 1, m),\ d_2 = (x, 2)$ for $x \in \mathbb{P}$ and $x \in \text{Int}(I)$.

Then $\sqsubseteq$ is a partial order on D. For future reference, we record that the following prohibited relationships never occur in $(D, \sqsubseteq)$:

(*) $d_1 \sqsubseteq d_2$, where $d_1 \in B$ and $d_2 \in A$;
(**) $d_1 \sqsubseteq d_2$, where $d_1 = (I, 2, m)$ and $d_2 = (x, 1)$ with $x \in \mathbb{R}$;
(***) $d_1 \sqsubseteq d_2$, where $d_1 \in C$ and $d_2 \in A \cup B$.

The rest of the proof is a sequence of lemmas that establish various properties of $(D, \sqsubseteq)$. In what follows, if d is an ordered triple or ordered pair, we will write $\pi_i(d)$ for the i^{th} coordinate of d.

Lemma 2.2. *If $E \subseteq D$ is a nonempty directed subset of D, then* $\sup(E) \in D$.

Proof of Lemma: It is enough to consider the case where E does not contain any maximal element of itself. Then $E \cap C = \emptyset$, so $E \subseteq A \cup B$. First consider the case where $E \subseteq A$. Then every $e \in E$ has the form $e = (I, 1, n)$. Because E has no maximal element, we may choose distinct $e_i \in E$ with $e_i \sqsubseteq e_{i+1}$ for all $i \geq 1$. Write $e_j = (I_j, 1, n_j)$ with $n_1 < n_2 < \cdots$. Observe that $\mathcal{D}(E) = \{\pi_1(e) : e \in E\}$ is a directed family of nonempty compact subsets of $\mathbb{R}$, so that $\bigcap\{\pi_1(e) : e \in E\} \neq \emptyset$. Because $L(I_j) < \frac{1}{n_j}$ and $I_j \in \mathcal{D}(E)$, we know that there is a real number x such that $\bigcap\{\pi_1(e) : e \in E\} = \{x\}$. Then $d_1 = (x, 1) \in D$. Fix $\hat{e}_1 = (I_1, 1, n_1) \in E$ and find $\hat{e}_2 \in E - \{\hat{e}_1\}$ with $\hat{e}_1 \sqsubseteq \hat{e}_2$. Writing $\hat{e}_2 = (I_2, 1, n_2)$, we have $I_2 \subseteq \mathrm{Int}(I_1)$. Then $I_2 = \pi_1(\hat{e}_2) \in \mathcal{D}(E)$ gives $x \in I_2 \subseteq \mathrm{Int}(I_1)$ which shows that $\hat{e}_1 \sqsubseteq (x, 1)$. Hence, $d_1 = (x, 1)$ is an upper bound for the set E in D.

To see that d_1 is the least of all upper bounds of E, suppose that $d \in D$ and $e \sqsubseteq d$ for each $e \in E$. Because E contains elements with arbitrarily large third coordinate, we see that $d \notin A \cup B$. Hence, d has the form $d = (z, 1)$ for some $z \in \mathbb{R}$, or else $d = (z, 2)$ for some $z \in \mathbb{P}$. In either case, because $e \sqsubseteq d$ for each $e \in E$, we know that $z \in \bigcap\{\pi_1(e) : e \in E\} = \{x\}$ so that $z = x$. But then in either case, we have $d_1 \sqsubseteq d$, as required.

It remains to consider the case where $E \cap B \neq \emptyset$. Choose $e_0 \in E \cap B$. We claim that E must contain elements of the form $(J, 2, n) \in B$ with arbitrarily large third coordinate. Using the fact that E contains no maximal element, we may choose distinct $e_i \in E$ with $e_i \sqsubseteq e_{i+1}$. Then the points e_i have arbitrarily large third coordinate (but e_j might not be in B). For each $i \geq 1$, use directedness of E to choose some $\hat{e}_i \in E$ with $e_0, e_i \sqsubseteq \hat{e}_i$. Then the points $\hat{e}_i$ have arbitrarily large third coordinates. We know that $e_0 = (I_0, 2, m)$ so that in order to avoid the prohibited relation (*), we must have $\hat{e}_i = (J_i, 2, n_i)$, showing that $\hat{e}_i \in E \cap B$. Consequently, $J_i \cap \{r_1, r_2, \cdots, r_{n_i}\} = \emptyset$ for arbitrarily large n_i. Once again, we see that $\mathcal{D}(E) = \{\pi_1(e) : e \in E\}$ is a directed collection of nonempty compact subsets of $\mathbb{R}$ that contains sets of arbitrarily small length, so that $\bigcap\{\pi_1(e) : e \in E\} = \{x\}$ for some $x \in \mathbb{R}$. Because $J_i \in \{\pi_1(e) : e \in E\}$ and $J_i \cap \{r_1, r_2, ..., r_{n_i}\} = \emptyset$ for arbitrarily large n_i, we conclude that $x \in \mathbb{P}$. Let $d_2 = (x, 2) \in D$. As in the previous case, d_2 is an upper bound for E and any upper bound d for E must have $d \in C$ and $\pi_1(d) = x$. Noting that the relationship $\hat{e}_i \sqsubseteq (x, 1)$ was prohibited in (**), we conclude that $d = (x, 2) = d_2$, so that $d_2 = \sup(E)$ as required to complete the proof of Lemma 2.2.

Note that the proof of Lemma 2.2 shows that if a directed set $E \subseteq D$ contains no maximal element and contains at least one element of B, then the set $E \cap B$ is infinite and has elements with arbitrarily large third coordinate.

Lemma 2.3. *For any* $d \in A \cup B,\ d \ll d$.

Proof of Lemma: We must show that if $d \sqsubseteq \sup(E)$ for a nonempty directed subset $E \subseteq D$, then $d \sqsubseteq e$ for some $e \in E$. In case E contains a maximal element e_0 of itself, then $d \sqsubseteq \sup(E) = e_0 \in E$ as required, so suppose no element of E is maximal in E. There are several cases to be considered.

Case 1. Consider the case where $d \in A$. Then $d = (I, 1, m)$. As in the proof of Lemma 2.2, there is a point $x \in \mathbb{R}$ with $\{x\} = \bigcap\{\pi_1(e) : e \in E\}$. There are two subcases.

In subcase 1-a, we have $E \subseteq A$. Then the proof of Lemma 2.2 shows that $\sup(E) = (x, 1)$ so that $d \sqsubseteq \sup(E)$ gives $(I, 1, m) \sqsubseteq (x, 1)$ and therefore $x \in \mathrm{Int}(I)$. Because $\pi_1(e)$ is a compact subset of $\mathbb{R}$ for each $e \in E$, we know that some $e_1 = (J_1, 1, n_1) \in E$ has $x \in J_1 = \pi_1(e_1) \subseteq \mathrm{Int}(I)$. Because E contains no maximal element of itself, we may choose distinct $e_i \in E$ with $e_1 \sqsubseteq e_2 \sqsubseteq \cdots$, and we have $e_i = (J_i, 1, n_i)$ for $i \geq 1$ because $E \subseteq A$. Then $x \in \pi_1(e_{i+1}) \subseteq \mathrm{Int}(\pi_1(e_i)) \subseteq \cdots \subseteq J_1 \subseteq \mathrm{Int}(I)$ so that for sufficiently large k, we have $\pi_1(e_k) \subseteq \mathrm{Int}(I)$ and $n_k > m$. Therefore, $d = (I, 1, m) \sqsubseteq e_k \in E$ as required to complete subcase 1-a.

In subcase 1-b, we have $E \cap B \neq \emptyset$. Then, as in the proof of Lemma 2.2, we know that E contains points of the form $e = (J, 2, n) \in B$ with arbitrarily large values of n, and $\bigcap\{\pi_1(e) : e \in E\} = \{y\}$ for some $y \in \mathbb{P}$ so that $\sup(E) = (y, 2)$. From $d = (I, 1, m) \in A$ and $d \sqsubseteq \sup(E)$, we have $(I, 1, m) \sqsubseteq (y, 2)$ so that $y \in \text{Int}(I)$. Then, choosing $e_i \in E \cap B$ with $\pi_3(e_i) \to \infty$ and $e_i \sqsubseteq e_{i+1}$, we find some $e_k \in E \cap B$ with $d \sqsubseteq e_k$ as required in subcase 1-b.

Case 2. Consider the case where $d \in B$, say $d = (I, 2, m)$, and $d \sqsubseteq \sup(E)$. Because the relationship in (**) is prohibited, it cannot happen that $E \subseteq A$ because, from the proof of Lemma 2.2, if $E \subseteq A$, then $\sup(E)$ has the form $(x, 1)$ so that $d \sqsubseteq \sup(E)$ which would give $(I, 2, m) \sqsubseteq (x, 1)$, a prohibited relationship of type (**). Therefore, $E \cap B \neq \emptyset$ so that $\sup(E) = (y, 2)$ for some $y \in \mathbb{P}$. (See Case 2 in the proof of Lemma 2.2.) Then $y \in \text{Int}(I)$ and, as in subcase 1-b above, we may find $e \in E \cap B$ with $\pi_1(e) \subseteq \text{Int}(I)$ and $\pi_3(e) > m$. Therefore, $d \sqsubseteq e \in E$ as required.

Lemma 2.4. *It never happens that $d_1 \ll d_2$ for $d_1, d_2 \in C$.*

Proof of Lemma: Suppose $d_1 \ll d_2$. Because $d_i \in C$ and $d_1 \sqsubseteq d_2$, we know that $\pi_1(d_1) = \pi_1(d_2) \in \mathbb{R}$. First consider the case where $x \in \mathbb{P}$ and $d_1 = (x, 1), d_2 = (x, 2)$. Find a sequence J_i of closed intervals with rational endpoints, with $x \in J_{i+1} \subseteq \text{Int}(J_i)$ and $L(J_i) < \frac{1}{i}$, and having $J_i \cap \{r_1, \cdots, r_i\} = \emptyset$. Let $e_i = (J_i, 2, i)$ and $E = \{e_i : i \geq 1\}$. Then $E \subseteq B$ is directed and has $\sup(E) = (x, 2)$. However, there is no $e_i \in E$ with $d_1 = (x, 1) \sqsubseteq e_i$ because that would be a prohibited relationship of type (***). Consequently, we know that $(x, 1) \ll (x, 2)$ fails for all $x \in \mathbb{P}$. But then $(x, 1) \ll (x, 1)$ must also fail for $x \in \mathbb{P}$, because $(x, 1) \ll (x, 1) \sqsubseteq (x, 2)$ would give $(x, 1) \ll (x, 2)$. Similarly, $(x, 2) \ll (x, 2)$ must fail whenever $x \in \mathbb{P}$. The remaining case to consider is where $d_1 = d_2 = (x, 1)$ for some $x \in \mathbb{Q}$. Find closed intervals J_i with rational endpoints, $L(J_i) < \frac{1}{i}$, and $x \in J_{i+1} \subseteq \text{Int}(J_i)$, and write $e_i = (J_i, 1, i)$. Then the set $E = \{e_i : i \geq 1\}$ is a directed subset of A with $\sup(E) = (x, 1)$, and no member $e_i \in E$ has $(x, 1) \sqsubseteq e_i$ because that would be a prohibited relationship of type (***). Hence, $(x, 1) \ll (x, 1)$ fails for each $x \in \mathbb{Q}$.

Lemma 2.5. *The dcpo $(D, \sqsubseteq)$ is continuous.*

Proof of Lemma: We must show that the set $\Downarrow(d)$ is directed and has d as its supremum for each $d \in D$. If $d \in A \cup B$, then Lemma 2.3 tells us that $d \in \Downarrow(d)$, so d is a common extension of any pair of elements of $\Downarrow(d)$ and $d = \sup(\Downarrow(d))$, as required.

Now consider the case where $d \in C$. Lemma 2.4 guarantees that $\Downarrow(d) \subseteq \{p \in A \cup B : p \sqsubseteq d\}$. Conversely, if $p \in A \cup B$ and $p \sqsubseteq d$, then Lemma 2.3 shows that $p \ll p \sqsubseteq d$ so that $p \in \Downarrow(d)$. Thus, $\Downarrow(d) = \{p \in A \cup B : p \sqsubseteq d\}$.

If $d = (x, 1)$ for some $x \in \mathbb{R}$, we may choose a sequence J_i of closed intervals with rational endpoints in such a way that $x \in \mathrm{Int}(J_{i+1}) \subseteq J_{i+1} \subseteq \mathrm{Int}(J_i)$ and $L(J_i) < \frac{1}{i}$. Then $e_i = (J_i, 1, i) \in \Downarrow(d)$ and we have $\sup\{e_i : i \geq 1\} = d$. To see that $\Downarrow(d)$ is directed, suppose $p_k = (I_k, 1, n_k) \in \Downarrow(d)$ for $k = 1, 2$. Then $x \in \mathrm{Int}(I_1) \cap \mathrm{Int}(I_2)$. Let $n_3 = n_1 + n_2$ and find a closed interval $[a, b]$ with $a, b \in \mathbb{Q}$, $|b-a| < \frac{1}{n_3}$ and with $x \in (a, b) \subseteq [a, b] \subseteq \mathrm{Int}(J_1) \cap \mathrm{Int}(J_2)$. Then $([a, b], 1, n_3) \in \Downarrow(d)$ is a common extension of p_1 and p_2. Finally, suppose $d = (x, 2)$ for some $x \in \mathbb{P}$. Find the points e_i, as in the first part of this paragraph, with the added restriction that $\pi_1(e_i) \cap \{r_1, \cdots, r_i\} = \emptyset$. Then $e_i \in \Downarrow(d)$ and $\sup\{e_i : i \geq 1\} = d$. To prove that $\Downarrow(d)$ is directed, we begin with $p_1, p_2 \in \Downarrow(d)$ and find n_3, a and b as above with the additional restriction that $[a, b] \cap \{r_1, \cdots r_{n_3}\} = \emptyset$. Then $([a, b], 2, n_3) \in B \cap \Downarrow(d)$ is a common extension of p_1 and p_2.

Lemma 2.6. *The set of maximal elements of D is*

$$\max(D) = \{(x, 1) : x \in \mathbb{Q}\} \cup \{(y, 2) : y \in \mathbb{P}\}$$

and the function h, defined by $h(x) = (x, 1)$ if $x \in \mathbb{Q}$ and $h(y) = (y, 2)$ if $y \in \mathbb{P}$, is a homeomorphism from X onto $\max(D)$ *where the latter space carries the relative Scott topology.*

Proof of Lemma: It is clear that $\max(D)$ is the set described above, and that the function h is 1-1 and onto.

To verify that h is continuous at a rational number x, suppose $h(x) = (x, 1) \in \Uparrow(d)$. Then d must have the form $d = (J, 1, n)$ because the relation $(J, 2, n) \sqsubseteq (x, 1)$ is prohibited by (**). Writing $J = [a, b]$, we see that (a, b) is a neighborhood of x in the space X. To see that $h[(a, b)] \subseteq \Uparrow(d)$, suppose $z \in (a, b)$. Then $d = (J, 1, n) \sqsubseteq h(z)$ so that, from Lemma 2.3, $d \ll d \sqsubseteq h(z)$. Therefore, $h(z) \in \Uparrow(d)$

as required. To verify that h is continuous at the irrational number y, suppose $(y, 2) \in \Uparrow(d)$ where $d = ([c, d], i, n)$ with $i \in \{1, 2\}$. Then $(c, d) \cap \mathbb{P}$ is a neighborhood of y in the space X, and for each $z \in (c, d) \cap \mathbb{P}$, we have $h(z) = (z, 2) \in \Uparrow(d)$. Therefore, h is continuous.

To show that h is an open mapping, suppose U is a neighborhood of the point x in the space (X, τ). We must find $d \ll h(x)$ with $\max(D) \cap \Uparrow(d) \subset h[U]$. If $x \in \mathbb{Q}$, there is an interval $J = [a, b]$ with rational endpoints and length < 1 having $x \in (a, b) \subseteq [a, b] \subseteq U$. Then $d = (J, 1, 1) \in D$ has $d \ll (x, 1) = h(x)$. To see that $\max(D) \cap \Uparrow(d) \subseteq h[U]$, suppose $(z, i) \in \Uparrow(d)$. Then $z \in \mathrm{Int}(J) = (a, b) \subseteq U$ so that $(z, i) = h(z) \in h[U]$, as required. The case in which $x \in \mathbb{P}$ is analogous, with the additional proviso that $[a, b]$, is not allowed to contain the rational number r_1.

At this point, we know that D is a domain and that X is homeomorphic to $\max(D)$ in the relative Scott topology. The other properties of D mentioned in the statement of Example 2.1 are verified as follows. The proof of Lemma 2.5 shows that the countable set $A \cup B$ is a domain basis for $(D, \sqsubseteq)$, so that, in the terminology of [11], X is an ω-continuous dcpo. Lemma 2.3 shows that every element d of the basis $A \cup B$ is compact in the sense of domain theory (i.e., has $d \ll d$), so that D is algebraic in the sense of [11]. To see that $\max(D)$ is not a G_δ subset of D, recall a result of Martin ([7, Theorem 2.32]) that in a G_δ subset of a locally compact sober [3] space, each closed subset is a Baire space. With the Scott topology, D is locally compact and sober [4], so that if $\max(D)$ were a G_δ-subset of D, then $\max(D)$ could not have the usual space of rational numbers as a closed subset. And this concludes the proof. □

Remark 2.7. *What Example 2.1 is not.*

---

[3]A T_0-space is *sober* if every irreducible closed set is the closure of a single point, where a closed set is said to be irreducible if it cannot be written as the union of two closed, proper subsets.

[4]We know that the space D is sober in the light of Theorem 2.20 in [11], which asserts that a continuous poset is sober if and only if it is a dcpo.

Our domain D is not a Scott domain; i.e., it is not true that $\sup(d_1, d_2) \in D$ for any pair $d_1, d_2 \in D$ that have a common extension in D. More important, there is no way to represent our space (X, τ) as $\max(S)$ where S is a Scott domain because the space of maximal elements of a Scott domain is completely regular.

Our example is not an ideal domain in the sense of [10] because if $x \in \mathbb{P}$, then $p = (x, 1)$ is neither maximal nor compact. In fact, for uncountably many $x \in X$ (indeed, for each $x \in \mathbb{P}$), we have $\Uparrow((x, 1)) = \emptyset$ even though $(x, 1)$ is not maximal in D.

Something like that must be true in any domain E that has a countable domain base and has (X, τ) homeomorphic to $\max(E)$. We claim that the set $E_0 = \{e \in E - \max(E) : \Uparrow(e) = \emptyset\}$ must be uncountable. For contradiction, suppose E_0 is countable. Because our space X is separable, there is a countable set $C_1 \subseteq \max(E)$ that intersects every non-empty $\Uparrow(e)$ for $e \in E - \max(E)$. For each $e \in E_0$, we choose $x(e) \in \max(E)$ with $e \sqsubseteq x(e)$. Letting $C = C_1 \cup \{x(e) : e \in E_0\}$, we would have a countable subset of X with the property that for each $e \in E - \max(E)$ some $x \in C$ has $e \sqsubseteq x$. According to Lemma 5.2 of [10], it follows that $\max(E)$ is a G_δ-subset of E. Now the argument concerning G_δ-subsets of locally compact sober T_0-spaces that we used above in D may be applied to obtain a contradiction. Consequently, our space X has no ideal domain model in the sense of [10].

Question 2.8. *Suppose that X is a Hausdorff second countable space that is domain-representable. Can X be represented as $\max(D)$ where D is a countably-based domain?*

We note that for T_3-spaces, Question 2.8 has an affirmative answer, as can be seen from [5], [4], [8].

References

[1] Harold Bennett and David Lutzer, *Domain-representable spaces*, Fund. Math. **189** (2006), no. 3, 255–268.

[2] ———, *Representability of certain complete spaces.* To appear.

[3] Abbas Edalat, *When Scott is weak on the top,* Math. Structures Comput. Sci. **7** (1997), no. 5, 401–417.

[4] Abbas Edalat and Reinhold Heckmann, *A computational model for metric spaces,* Theoret. Comput. Sci. **193** (1998), no. 1-2, 53–73.

[5] Jimmie Lawson, *Spaces of maximal points,* Math. Structures Comput. Sci. **7** (1997), no. 5, 543–555.

[6] Keye Martin, *Domain theoretic models of topological spaces,* Electron. Notes Theor. Comput. Sci. **13** (1998), no. 1, 173–181.

[7] ———, *Nonclassical techniques for models of computation,* Topology Proc. **24** (1999), summer, 375-405. (2001)

[8] ———, *The regular spaces with countably based models,* Theoret. Comput. Sci. **305** (2003), no. 1-3, 299–310.

[9] ———, *Topological games in domain theory,* Topology Appl. **129** (2003), no. 2, 177–186.

[10] ———, *Ideal models of spaces.* To appear in Theoretical Computer Science.

[11] Keye Martin, M. W. Mislove, and G. M. Reed, *Topology and domain theory,* in Recent Progress in General Topology, II. Ed. M. Husek and J. van Mill. Amsterdam: North-Holland, 2002. 371–394.

(Bennett) Department of Mathematics & Statistics; Texas Tech University; Lubbock, TX 79409-1042

E-mail address: harold.bennett@ttu.edu

(Lutzer) Mathematics Department; College of William & Mary; P.O. Box 8795; Williamsburg, VA 23187-8795

E-mail address: lutzer@math.wm.edu