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ON INGRAM'S CONJECTURE

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Abstract. In this paper, we consider inverse limit spaces (I, f_s) where the bonding map f_s is a tent map with $s \in (\sqrt{2}, 2]$. We make a conjecture concerning homeomorphisms of (I, f_s). We show that if our conjecture is true, then Ingram's conjecture holds.

1. Introduction

In this paper, we consider inverse limit spaces

$$(I, f_s) = \varprojlim\{I \xleftarrow{f_s} I \xleftarrow{f_s} \cdots\},$$

where the inverse system spaces are each the interval $I = [0, 1]$, and the bonding map f_s is a tent map with $s \in (\sqrt{2}, 2]$. We make the following conjecture.

Pseudo-isotopy Conjecture. *Let $f_s : I \to I$ be a tent map with slope $s \in (\sqrt{2}, 2]$. Suppose $h : (I, f_s) \to (I, f_s)$ is a homeomorphism. Then there exists an integer k such that $h|_{X_s}$ and $\sigma_s^k|_{X_s}$ are pseudo-isotopic.*

Here, X_s denotes the core of the inverse limit space (I, f_s) and σ_s denotes the shift homeomorphism. By *pseudo-isotopic*, we mean that $h|_{X_s}$ and $\sigma_s^k|_{X_s}$ permute the composants of X_s in the same way. Detailed definitions of these terms are given in section 2.

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Inverse limit spaces of maps of a one-dimensional space to itself appear as attractors in dynamical systems [1], [3], [7], [13], [20], [34]. This motivates the study of such inverse systems. In the case of solenoids, which are inverse limits of circles with bonding maps z^n, there is a well-known characterization (see J. M. Aarts and R. J. Fokkink [1], and James Keesling [25]).

It is natural to try to determine when (I, f_s) and (I, f_t) are homeomorphic. Any unimodal map without wandering intervals, restrictive intervals, or periodic attractors is conjugate to a tent map (see Welington de Melo and Sebastian van Strien [26]). As conjugate maps have homeomorphic inverse limit spaces, the family of tent maps is more inclusive than it seems at first glance. The following conjecture has become well-known and widely studied.

Ingram's Conjecture. *If $1 < s < t \leq 2$, then the inverse limit spaces (I, f_s) and (I, f_t) are not homeomorphic.*

Ingram's conjecture was posed as a question in Ingram's paper [22]. Partial results exist in [5], [10], [11], [16], [19], [23], [24], [28], [30], [31], and [33].

The main result of this paper is the following.

Theorem 3.11. *If the pseudo-isotopy conjecture is true, then Ingram's conjecture is true.*

Using topological methods, Lois Kailhofer [23], [24] proved the following result.

Theorem. [24, Theorem 63] *Suppose that $s, t \in [\sqrt{2}, 2]$. Let f_s and f_t be tent maps with periodic turning points. If X_s and X_t are homeomorphic, then $s = t$.*

The theorem implies that if (I, f_s) and (I, f_t) are homeomorphic, then $s = t$ under the given assumptions. Louis Block, Slagjana Jakimovik, Kailhofer, and Keesling [10] gave a simplified proof of Kailhofer's theorem using some of Kailhofer's results [23] together with the following result.

Theorem. [10, Corollary 4.7] *Let $s \in (\sqrt{2}, 2]$ and suppose that the turning point of f_s is periodic. Let h be any homeomorphism of X_s such that h maps each endpoint to itself. Then there exists an integer N such that h is pseudo-isotopic to σ_s^N.*

Block, Jakimovik, Kailhofer, and Keesling [10] prove a stronger result about homomorphisms of the core of a tent map with periodic turning point.

Isotopy Theorem. [10] *Let $s \in (\sqrt{2}, 2]$ and suppose that the turning point of f_s is periodic. Let h be any homeomorphism of X_s. Then there exist a positive integer n and an integer k such that h^n is isotopic to σ_s^k, where σ_s is the shift homeomorphism on X_s.*

In this paper, we consider tent maps with no assumption on the orbit of the turning point. In section 3, we prove that each composant of the core mapped to itself by σ_s^k, for some integer k, has a unique point mapped to itself by σ_s^k. We also prove that if f_s and f_t are tent maps with slopes $1 < s < t \leq 2$, then for each positive integer n, the number of fixed points of f_s^n is less than or equal to the number of fixed points of f_t^n; moreover, there exists a positive integer N such that the number of fixed points of f_s^N is strictly less than the number of fixed points of f_t^N. To prove some of these results, we use some well-known, yet nontrivial results about the entropy of maps in the tent family and the entropy of maps in the quadratic family. We include only a general description of the notion of entropy and state the results we use with the appropriate references.

In section 4, we show that if the pseudo-isotopy conjecture is true, then Ingram's conjecture holds and give a sufficient condition for the pseudo-isotopy conjecture to hold.

Theorem 4.2. *Let $f_s : I \to I$, $f_s(x) = \min\{s \cdot x, s \cdot (1-x)\}$ be a tent map with slope $s \in (\sqrt{2}, 2]$. Let $h : (I, f_s) \to (I, f_s)$ be a homeomorphism. Suppose there exist a number $M > 0$ and an integer k such that $\bar{d}(h(\bar{x}), \sigma_s^k(\bar{x})) \leq M$, for all $\bar{x} \in T$. Then $h|_{X_s}$ and $\sigma_s^k|_{X_s}$ are pseudo-isotopic.*

In Theorem 4.2, T denotes the open ray entwined with the core, called the "tail." The distance referred to is the $\bar{d}$ metric defined on the tail in the same fashion as it was defined on each composant of the core of a tent map with periodic turning point [10]. Namely, for any two points $\bar{x}$ and $\bar{y}$ in the tail, there is an arc A with endpoints $\bar{x}$ and $\bar{y}$, and there exists a positive integer k such that $\pi_k|_A$ is a homeomorphism. Then we define $\bar{d}(\bar{x}, \bar{y}) = s^k \cdot |\pi_k(\bar{x}) - \pi_k(\bar{y})|$.

Observe that the $\bar{d}$ metric on the tail is consistent with the relative topology of the tail as a subspace of the inverse limit space.

2. Preliminaries

Here, we establish the terminology used in the paper. The only new term is pseudo-isotopy in Definition 2.1. All the other terms are standard.

We denote $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$, $\mathbb{R} = (-\infty, \infty)$, and $\mathbb{R}_+ = [0, \infty)$.

Let X be a topological space. We say X is an *arc* if there exists a homeomorphism from X onto $[0, 1]$. A maximal connected subspace of X is called a *component* of X. A compact connected metric space is called a *continuum.*

Let X be a continuum. The *composant* of $x \in X$ is the union of all proper sub-continua of X that contain x. Composants are dense; that is, every point of a continuum X is a limit point of any composant of X [21, Theorem 3-44]. Every composant of a continuum X is the union of a countable number of proper subcontinua of X [21, Theorem 3-45].

Let X be a continuum. A sub-continuum T of X is an *end continuum* in X if for any two sub-continua A and B of X, such that $T \subset A$ and $T \subset B$, it is true that either $A \subset B$ or $B \subset A$. A point $x \in X$ is an *endpoint* of X if $\{x\}$ is an end-continuum in X. Note that any homeomorphism from one continuum to another must send endpoints to endpoints. Thus, the cardinality of the set of endpoints of a continuum is a topological invariant.

A continuum X is *indecomposable* if it is not the union of two proper sub-continua. An indecomposable continuum X has uncountably many composants [21, Theorem 3-46], and the composants of an indecomposable continuum X are disjoint [21, Theorem 3-47].

Definition 2.1. Suppose X is an indecomposable continuum and $h, g : X \to X$ are homeomorphisms. We say that h and g are *pseudo-isotopic* if for any $x \in X$, there is a proper sub-continuum $A \subset X$ such that $h(x) \in A$ and $g(x) \in A$. In other words, two homeomorphisms of an indecomposable continuum into itself are pseudo-isotopic if they permute the composants in the same way.

Definition 2.2. Let $\{X_i, d_i\}_{i=0}^{\infty}$ be a collection of compact metric spaces each with a metric d_i bounded by 1, and such that for each i, $f_i : X_{i+1} \to X_i$ is a continuous map. The *inverse limit space* of the inverse limit system $\{X_i, f_i\}_{i=0}^{\infty}$ is the set

$$\varprojlim\{X_i, f_i\}_{i=0}^{\infty} = \{\bar{x} = (x_0, x_1, \cdots) | \bar{x} \in \Pi_{i=0}^{\infty} X_i, f_i(x_{i+1}) = x_i, i \in \mathbb{N}\},$$

with a metric d given by

$$d(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}.$$

For each $i \in \mathbb{N}$, π_i denotes the projection map from $\Pi_{i=0}^{\infty} X_i$ into X_i.

Usually, the subscript/superscript notation f_i^j used in the literature on inverse limits denotes the mapping of the j-th factor space to the i-th factor space. But in this paper, as already adopted in papers considering inverse limit spaces using single bonding maps from parameterized families, we use f_s^k to denote the k-fold composition of the map f_s with itself, where s is the parameter.

An inverse limit space $\varprojlim\{X_i, f_i\}_{i=0}^{\infty}$ is a continuum if X_i is a continuum for every $i \in \mathbb{N}$ [29, Theorem 2.4].

If $X_i = X$ and $f_i = f$ for all i, the inverse limit space is denoted (X, f).

The map $\sigma : (X, f) \to (X, f)$, defined by

$$\sigma(x_0, x_1, \cdots) = (f(x_0), x_0, x_1, \cdots),$$

is called the *shift homeomorphism*, or the *induced homeomorphism*.

A continuous map $f : [a, b] \to [a, b]$ is called *unimodal* if there exists a unique point c such that $f|_{[a,c]}$ is strictly increasing and $f|_{[c,b]}$ is strictly decreasing. The point c is called a *turning point*, or a *critical point*.

Let $f : I \to I$ be a map of the interval into itself. A point x is said to be a *periodic point* of f with *period* $k > 0$ if $f^k(x) = x$ and $f^n(x) \neq x$ for $0 < n < k$. Let $x \in I$. The set of all points $y \in I$, such that $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$, is called the *ω-limit set* of x and is denoted by $\omega(x)$. In other words, $\omega(x) = \cap_{k \geq 0} \mathrm{cl}(\cup_{n \geq k} f^n(x))$.

Let x be a periodic point of f with period k. The set of points $x, f(x), f^2(x), \cdots, f^{k-1}(x)$ is called the *orbit* of x under f. The *basin* of x is the set of points whose ω-limit set contains x. We say that x is an *attracting periodic point* and that its orbit is an *attracting periodic orbit* if its basin contains an open set.

Definition 2.3. Let $f : [a, b] \to [a, b]$ be a unimodal map with turning point c. For each $x \in [a, b]$, the *forward itinerary* of x, denoted $\underline{I}(x) = b_0 b_1 b_2 \cdots$, is a sequence of R's and L's and possibly a C such that

(1) $b_i = R$ if $f^i(x) > c$,
(2) $b_i = L$ if $f^i(x) < c$, and
(3) $b_i = C$ if $f^i(x) = c$,

with the convention that the itinerary stops after the first C.

The itinerary of $f(c)$ is called the *kneading sequence* of f and it is denoted $K(f)$.

A sequence A of symbols L, R, and C is said to be *admissible* if A is either an infinite sequence of R's and L's or a finite (or empty) sequence of R's and L's followed by a C.

Every itinerary is admissible. For two admissible sequences A and B, AB denotes the concatenation of A and B, $A^n = AA \cdots A$ with A repeated n times, and $A^\infty = AA \cdots$ with A repeated indefinitely.

A finite sequence A of R's and L's is said to have *even parity* if the number of R's in A is even. Otherwise, A is said to have *odd parity.*

The *parity-lexicographical ordering* on the set of itineraries is defined as follows. Set $L < C < R$. Let $\underline{I} = w_0 w_1 \cdots$ and $\underline{I}' = v_0 v_1 \cdots$ be two distinct itineraries and let k be the first index where the itineraries differ.

(1) If $k = 0$, then $\underline{I} < \underline{I}'$ if and only if $w_0 < v_0$.
(2) If $k \geq 1$ and $w_0 w_1 \cdots w_{k-1} = v_0 v_1 \cdots v_{k-1}$ has even parity, then $\underline{I} < \underline{I}'$ if and only if $w_k < v_k$.
(3) If $k \geq 1$ and $w_0 w_1 \cdots w_{k-1} = v_0 v_1 \cdots v_{k-1}$ has odd parity, then $\underline{I} < \underline{I}'$ if and only if $v_k < w_k$.

Lemma 2.4. *Let $x, x' \in [a, b]$.*

(1) *If $\underline{I}(x) < \underline{I}(x')$, then $x < x'$* [17, Lemma II.1.2].
(2) *If $x < x'$, then $\underline{I}(x) \leq \underline{I}(x')$* [17, Lemma II.1.3].

We define the *shift operation* on the set of itineraries in the following way. Let $\underline{I} = a_0 a_1 a_2 \cdots$ be an itinerary such that $\underline{I} \neq C$. Define $\sigma\underline{I} = a_1 a_2 a_3 \cdots$. If $\underline{I} = C$, $\sigma\underline{I}$ is undefined.

We use the same symbol, σ, to denote both the shift homeomorphism on an inverse limit space of a map, and the shift operation on the set of itineraries of a unimodal map. The context in which it is used should indicate the meaning of σ in each case.

Observe that $\sigma \underline{I}(x) = \underline{I}(f(x))$ if $x \neq c$.

A sequence A is called *maximal* if A is admissible and if $\sigma^k A \leq A$ for $k = 1, 2, \cdots, n$ when A is finite and has length n, and for $k = 1, 2, \cdots$ when A is infinite.

For a unimodal map $f : [a, b] \to [a, b]$ with turning point c, $f(c) > f(x)$ for all $x \in [a, b] \backslash \{c\}$; hence by Lemma 2.4, it follows that the kneading sequence is maximal.

Definition 2.5. Let $f : [a, b] \to [a, b]$ be a unimodal map. Let A be an admissible sequence. We say that A *is dominated by the kneading sequence* $K(f)$, $A << K(f)$, if for all non-negative integers k,

(1) $\sigma^k A < K(f)$ if $K(f)$ is infinite,
(2) $\sigma^k A < (DL)^\infty$ if $K(f) = DC$ and D has even parity,
(3) $\sigma^k A < (DR)^\infty$ if $K(f) = DC$ and D has odd parity.

Theorem 2.6. [17, Theorem II.3.8] *Let $f : [a, b] \to [a, b]$ be a unimodal map. Let A be an admissible sequence. If $\underline{I}(a) \leq A << K(f)$, then there exists a point $x \in [a, b]$ such that $\underline{I}(x) = A$.*

Definition 2.7. The one-parameter family of maps $f_s : I \to I$ defined by

$$f_s(x) = \min\{s \cdot x, s \cdot (1 - x)\},$$

for $x \in I$ and $s \in (\sqrt{2}, 2]$ is known as the family of tent maps.

The tent map f_s is unimodal for all $s \in (\sqrt{2}, 2]$.

Remark 2.8. For $s > 1$, if $x < x'$ are two points in I, there exists a positive integer k such that $f_s^k(x)$ is on one side of c and $f_s^k(x')$ is on the other side of c. Hence, $\underline{I}_s(x) \neq \underline{I}_s(x')$. By Lemma 2.4, it follows that $\underline{I}_s(x) < \underline{I}_s(x')$. Therefore, for $s > 1$, the map $x \to \underline{I}_s(x)$ is an injective map.

Denote $c_i = f_s^i(c)$ for $i \in \mathbb{N}$. Denote $I_L = [c_2, c], I_R = [c, c_1]$, and $J_s = [c_2, c_1]$.

The interval $J_s = [c_2, c_1]$ is invariant under f_s and if $s \in (\sqrt{2}, 2]$, then f_s is *locally eventually onto* on J_s; i.e., for every non-degenerate interval $J \subset J_s$ there exists an $n > 0$ such that $f_s^n(J) = J_s$. The interval J_s is known as the *core* of f_s. To denote the n-th coordinate in the inverse limit space, we use I_n instead of $(J_s)_n$.

For $s \in (\sqrt{2}, 2]$, the inverse limit space of (I, f_s) is equal to the union of $X_s = (J_s, f_s)$ and an open ray having X_s as its limit set. The set X_s is called *the core of the inverse limit space* (I, f_s). It is well-known and easy to prove that X_s is indecomposable for $s \in (\sqrt{2}, 2]$.

When the turning point c is periodic under f_s of period n_0, we denote $\bar{c}_i = (c_i, c_{i-1}, \cdots, c_1, c_{n_0-1}, \cdots, c_{i+1})^\infty$ for $i = 0, 1, \cdots, n_0 - 1$. In this case the endpoints of X_s are exactly $\bar{c}_i$, $i = 0, 1, \cdots, n_0 - 1$ (see, for example, Kailhofer [23]).

3. Pseudo-isotopy conjecture implies Ingram's conjecture

In this section, first we prove that the pseudo-isotopy conjecture implies Ingram's conjecture (Theorem 3.11). We start with the following result.

Theorem 3.1. *Let $f_s : I \to I$, $f_s(x) = \min\{s \cdot x, s \cdot (1-x)\}$ be a tent map with slope $s \in (\sqrt{2}, 2]$. Suppose $C \subset X_s$ is a composant with the property that there exists a positive integer k such that $\sigma_s^k(C) = C$. Then there exists a unique fixed point z of f_s^k such that the point $\bar{z} = (z, f_s^{k-1}(z), \cdots, f_s(z))^\infty \in C$.*

Proof: Let $\delta > 0$ be such that $\frac{(\frac{s^2}{2})^k}{(\frac{s^2}{2})^k - 1} \cdot \delta < \frac{c_1 - c_2}{2}$.

Let $\bar{x} = (x_0, x_1, \cdots) \in C$. Since $\sigma_s^k(C) = C$, there is a proper sub-continuum $A \subset C$ containing $\bar{x}$ and $\sigma_s^{-2k}(\bar{x})$. Consequently, for any positive integer n, the points $\sigma_s^{-2nk}(\bar{x})$ and $\sigma_s^{-2(n+1)k}(\bar{x})$ are contained in $\sigma_s^{-2nk}(A) \subset C$, a proper sub-continuum of X_s.

Let M be a large enough positive integer so that $\ell(\pi_m(A)) < \delta$ for all $m \geq M$. This is possible since A is a proper sub-continuum of X_s, and $\ell(\pi_n(A)) \to 0$ as $n \to \infty$.

As we chose δ such that $\delta < \frac{c_1 - c_2}{2}$, we have that for all $m \geq M$,

$$\ell(\pi_{m+2k}(A)) \leq \frac{1}{(\frac{s^2}{2})^k} \cdot \ell(\pi_m(A)).$$

The expression $\frac{s^2}{2}$ in the above inequality comes from the fact that, for sufficiently small subintervals K of $[c_2, c_1]$, we might have the critical point $c \in K$ or $c \in f_s(K)$, but not both. If for example $c \in K$, then length of $f_s(K)$ is at least $\frac{s}{2}$ times the length of K. In any case, the length of $f_s^2(K)$ is at least $\frac{s^2}{2}$ times the length of K.

Note also that for all $m \geq M$ and all $n \geq 0$, $\pi_m(\sigma_s^{-2nk}(\bar{x})) = \pi_{m+2nk}(\bar{x})$. Hence, for all $m \geq M$ and all $n \geq 0$,

$$d(\pi_m(\sigma_s^{-2nk}(\bar{x})), \pi_m(\sigma_s^{-2(n+1)k}(\bar{x}))) < \frac{1}{(\frac{s^2}{2})^{nk}} \cdot \delta.$$

Since the series $\sum_{n=0}^{\infty} \frac{1}{(\frac{s^2}{2})^{nk}}$ converges, therefore the sequence $\{\pi_m(\sigma_s^{-2nk}(\bar{x}))\}_{n=0}^{\infty}$ is Cauchy; hence, it converges to some z_m. By continuity of f_s, $f_s(z_{m+1}) = z_m$ for all $m \geq M$. Define $z_i = f_s^{M-i}(z_M)$ for all $0 \leq i < M$. Then $\bar{z} = (z_0, z_1, z_2, \cdots)$ is a point in the inverse limit space (I, f_s). Moreover, the sequence $\{\sigma_s^{-2nk}(\bar{x})\}$ converges to $\bar{z}$, and $\sigma_s^{-2k}(\bar{z}) = \bar{z} = \sigma_s^{2k}(\bar{z})$.

Due to the fact that $\sigma_s^{-2nk}(\bar{x}) \in \sigma_s^{-2nk}(A) \cap \sigma_s^{-2(n-1)k}(A)$, we have that $\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A)$ is connected; hence, $\text{cl}(\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A))$ is connected.

We know that for all m, it is true that

$$\ell(\pi_m(\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A))) \leq \sum_{n=0}^{\infty} \ell(\pi_m(\sigma_s^{-2nk}(A))).$$

Since for each positive integer n and all m, $\pi_m(\sigma_s^{-2nk}(A)) = \pi_{m+2nk}(A)$, it follows that for each positive integer n and all m,

$$\sum_{n=0}^{\infty} \ell(\pi_m(\sigma_s^{-2nk}(A))) = \sum_{n=0}^{\infty} \ell(\pi_{m+2nk}(A)).$$

As for all $m \geq M$, $\ell(\pi_{m+2k}(A)) < \frac{1}{(\frac{s^2}{2})^k} \cdot \ell(\pi_m(A))$, it follows that for all $m \geq M$,

$$\ell(\pi_m(\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A))) \leq \sum_{n=0}^{\infty} \ell(\pi_{m+2nk}(A))$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{(\frac{s^2}{2})^{nk}} \cdot \ell(\pi_m(A)) = \frac{(\frac{s^2}{2})^k}{(\frac{s^2}{2})^k - 1} \cdot \delta < \frac{c_1 - c_2}{2}.$$

Therefore, for all $m \geq M$, $\ell(\pi_m(\text{cl}(\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A)))) < \frac{c_1-c_2}{2}$, which tells us that $\text{cl}(\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A))$ is a proper sub-continuum of X_s. Note that $\bar{z} \in \text{cl}(\cup_{n=0}^{\infty}\sigma_s^{-2nk}(A)) \subset C$.

Since $\bar{z}$ is a fixed point for the map σ_s^{2k}, there exists a point $z \in I$ such that $\bar{z} = (z, f_s^{2k-1}(z), \cdots, f_s(z))^{\infty}$. Therefore,

$$z, f_s^{2k-1}(z), \cdots, f_s(z) \in I$$

are fixed points for the map f_s^{2k}. In fact, we claim that $z \in I$ is a fixed point for the map f_s^k. Suppose not. Then z and $f_s^k(z)$

are distinct points in I, and they are some $\epsilon > 0$ distance apart. It follows that the points $\bar{z} = (z, f_s^{2k-1}(z), \cdots, f_s(z))^\infty$ and $\bar{z}' = \sigma_s^k(\bar{z}) = (f_s^k(z), f_s^{k-1}(z), \cdots, f_s(z), z, f_s^{2k-1}(z), \cdots, f_s^{k+1}(z))^\infty$ are two distinct points in the composant C. There exists a proper sub-continuum $A \subset C$ which contains both $\bar{z}$ and $\bar{z}'$. Observe that $\sigma_s^{-k}(\bar{z}) = \bar{z}'$ and $\sigma_s^{-k}(\bar{z}') = \bar{z}$. Hence, for each positive integer n, $\sigma_s^{-nk}(A)$ contains both $\bar{z}$ and $\bar{z}'$. Thus, for each positive integer n, $\pi_{nk}(A)$ contains both z and $f_s^k(z)$. But, since A is a proper sub-continuum of X_s, $\ell(\pi_n(A)) \to 0$ as $n \to \infty$, which contradicts the assumption that z and $f_s^k(z)$ are a fixed distance $\epsilon > 0$ apart.

By an argument similar to the one in the paragraph above, it is easy to prove that $\bar{z}$ is a unique fixed point of σ_s^k in C. □

Theorem 3.2. *Let $f_s : I \to I$, $f_s(x) = \min\{s \cdot x, s \cdot (1-x)\}$ be a tent map with slope $s \in (\sqrt{2}, 2]$. Suppose $C \subset X_s$ is a composant for which there exists an integer k such that $\sigma_s^k(C) = C$. Assume that either the turning point c is not periodic, or if c is periodic of period n_0, then $\bar{c}_i \notin C$ for each $i \in \{0, 1, \cdots, n_0 - 1\}$. Then there is a continuous bijection from the real line R to C.*

Proof: Let $\bar{z} = (z_0, z_{k-1}, \cdots, z_1)^\infty$, where $z_i = f_s^i(z_0)$, $i \in \{0, 1, \cdots, k-1\}$, be the unique fixed point of σ_s^k in C. Let $\varepsilon = \frac{1}{2} \cdot \min\{|z_i - c|, |z_i - c_1|, |z_i - c_2|, |z_i - c_3| : i = 0, 1, \cdots, k-1\}$. Then the symmetric interval $J_0 = [z_0 - \varepsilon, z_0 + \varepsilon] \subset [c_2, c_1]$ and $c_3 \notin J_0$.

Let J_1 be the component of $f_s^{-1}(J_0)$ in $I_1 = \pi_1(X_s)$ which contains z_{k-1}. Then f_s maps J_1 linearly onto J_0. Let J_2 be the component of $f_s^{-1}(J_1)$ in $I_2 = \pi_2(X_s)$ which contains z_{k-2}. And so on, for each positive integer n, let J_n be the component of $f_s^{-1}(J_{n-1})$ in $I_n = \pi_n(X_s)$ which contains $z_{(-n \mod k)}$. Our choice of ε such that for each positive integer n, $c_3 \notin J_n$ ensures that f_s maps J_n linearly onto J_{n-1}; that is, no piece of J_n gets cut out by "pulling back" J_n to the left of c.

Let $\bar{J} = \varprojlim\{J_n, f_s\}_{n=0}^\infty$. Then $\bar{J}$ is a proper sub-continuum of C containing $\bar{z}$. Note that for each positive integer n, $\pi_n|_{\bar{J}}$ is a homeomorphism, and for each positive integer n, $f_s|_{J_n}$ is a homeomorphism. Hence, $\bar{J}$ is an arc in C.

Consider the sequence $\bar{J}, \sigma_s^k(\bar{J}), \sigma_s^{2k}(\bar{J}), \cdots$ of arcs in C. We claim that it is an increasing sequence.

Let $\bar{x} = (x_0, x_1, x_2, \cdots) \in \bar{J}$. In order to prove that $\bar{x} \in \sigma_s^k(\bar{J})$, we need to show that there exists a point $\bar{y} \in \bar{J}$ such that $\sigma_s^k(\bar{y}) = \bar{x}$.

Let $\bar{y} = \sigma_s^{-k}(\bar{x}) = (x_k, x_{k+1}, \cdots)$. Then $\sigma_s^k(\bar{y}) = \bar{x}$. It only remains to show that $\bar{y} \in \bar{J}$. As $x_k \in J_k$ and $J_k = [z_0 - \frac{\varepsilon}{s^k}, z_0 - \frac{\varepsilon}{s^k}]$, we have that $|x_k - z_0| \leq \frac{\varepsilon}{s^k} < \varepsilon$. Hence, $x_k \in J_0$. It follows that $\bar{y} \in \bar{J}$.

We proved that $\bar{J} \subset \sigma_s^k(\bar{J})$; hence, for each positive integer n,

$$\sigma_s^{nk}(\bar{J}) \subset \sigma_s^{(n+1)k}(\bar{J}).$$

Let $A \subset C$ be a proper sub-continuum containing $\bar{z}$. Then $\ell(\pi_n(A)) \to 0$ as $n \to \infty$. Hence, there exists a positive integer m such that $\pi_{mk}(A) \subset [z_0 - \varepsilon, z_0 + \varepsilon]$. Thus, $A \subset \sigma_s^{mk}(\bar{J})$. It follows that the composant C is an increasing union of arcs. □

Theorem 3.3. *Let $1 < t \leq 2$ be such that the turning point c is periodic under the tent map f_t. Suppose $K(f_t) = DC$ where D is a finite sequence of R's and L's. Let B be an R or an L such that $DB < DC$. Let $y \in [f_t^2(c), f_t(c)]$ be a point such that c is not in the orbit of y under f_t. Then $\underline{I}_t(y) < (DB)^\infty$.*

Proof: Suppose $\underline{I}_t(y) \geq (DB)^\infty$. Since $(DB)^\infty \leq \underline{I}_t(y) < DC$, we know that $\underline{I}_t(y) = DBY$, where Y is some sequence of R's and L's. Let $Y = D_1X$, where D_1 and X are sequences of R's and L's and the length of D_1 is the same as the length of D. Since $\sigma^k \underline{I}_t(y) < DC$ for all $k \geq 0$, we have that $D_1 \leq D$. But $\underline{I}_t(y) \geq (DB)^\infty$ and DB has even parity by the way B was chosen; hence, $D_1 \geq D$. Therefore, $D_1 = D$. By induction, we get that $\underline{I}_t(y) = (DB)^\infty$. Then for each $j \geq 0$, the points $f_t^j(y)$ and $f_t^j(f_t(c))$ are in the same closed interval $[f_t^2(c), c]$ or $[c, f_t(c)]$. This contradicts the fact that f_t is a tent map with $t > 1$. □

Corollary 3.4. *Let $1 < s < t \leq 2$ be such that the turning point c is periodic under the tent map f_t. Suppose $K(f_t) = DC$ where D is a finite sequence of R's and L's. Let B be an R or an L such that $DB < DC$. Let $y \in [f_s^2(c), f_s(c)]$ be a point such that c is not in the orbit of y under f_s. Then $\underline{I}_s(y) < (DB)^\infty$.*

Proof: Since $K(f_s) < K(f_t)$, the assumption that $\underline{I}_s(y) \geq (DB)^\infty$ gives us that $(DB)^\infty \leq \underline{I}_s(y) < DC$. The same argument as in the proof of Theorem 3.3 leads to a contradiction. □

The one-parameter family of maps $f_\mu : I \to I$ defined by $f_\mu(x) = \mu \cdot x \cdot (1 - x)$, for $x \in I$ and for $0 \leq \mu \leq 4$, is known as the *quadratic family.*

The *topological entropy* of a map is a quantitative measure of the complexity of the system modeled by iterating that map. It measures how many distinct orbits of a given length there are for this map, that is, how this number grows with the length. For a precise definition of entropy, see [9, Section VIII.1] or [2, Section 4.1]. For a comprehensive study of the concept of topological entropy, we refer the reader to [9, Chapter VIII] and [2, Chapter 4].

The topological entropy of a tent map f_s with $s > 1$ is $h(f_s) = \log s$ [2, Corollary 4.3.13].

For tent maps f_s with slope $s > 1$, the topological entropy is strictly increasing as a function of the slopes. For the quadratic family, the topological entropy is an increasing function of the slopes, but no “real” proof is known; all known proofs of this fact use complexification of the maps. We will state those results for easier referral.

Theorem 3.5 (Sullivan, Milnor, Douady, and Hubbard). [26, Chapter II, Theorem 10.1] *If the turning points of the quadratic maps f_μ and $f_{\mu'}$ are eventually periodic and their kneading invariants are equal, then $\mu = \mu'$.*

Corollary 3.6. [26, Chapter II, Corollary 1] *Let $f_\mu : I \to I$ be a unimodal family consisting of C^1 maps depending continuously on the parameter μ. If the assumption that the turning points of f_μ and $f_{\mu'}$ are eventually periodic and their kneading invariants are equal implies that $\mu = \mu'$, then*

$$\mu \to K(f_\mu)$$

is monotone.

Theorem 3.7. *Let f_{μ_1} be a map in the quadratic family with kneading sequence DC. Let B be an L or an R such that $DB < DC$. Then there is an open interval (μ_2, μ_1) such that for all $\mu \in (\mu_2, \mu_1)$, $K(f_\mu) = (DB)^\infty$ and $h(f_\mu) = h(f_{\mu_1})$.*

Proof: Since $K(f_{\mu_1}) = DC$, the turning point is periodic under f_{μ_1}. Let n denote the period of c. Then $(f^n_{\mu_1})' = 0$ at each point in the orbit of c. There exists $\delta > 0$ such that for each $\mu \in (\mu_1 - \delta, \mu_1 + \delta)$ and each point x within distance δ of any point in the orbit of c, we have that $|(f^n_\mu)'(x)| < 1$. Thus, for each $\mu \in (\mu_1 - \delta, \mu_1 + \delta)$, f_μ has A? unique attracting periodic orbit within distance δ of any point in the orbit of c, and as $k \to \infty$, $f^k_\mu(c)$ approaches this orbit.

Let $\mu_2 = \mu_1 - \delta$. Consider $\mu \in (\mu_2, \mu_1)$. By Theorem 3.5 and Corollary 3.6, $K(f_\mu) \leq K(f_{\mu_1})$. Let x_μ be the point near c in the attracting periodic orbit. Then $I_\mu(f_\mu(x_\mu)) = (DA)^\infty$, where A is an L or an R depending on which side of c, x_μ is on. By construction, $K(f_\mu) = I_\mu(f_\mu(c)) = I_\mu(f_\mu(x_\mu)) = (DA)^\infty$. But, by Theorem 3.5 and Corollary 3.6, $K(f_\mu)$ is monotone as a function of μ. Thus, $(DA)^\infty = (DB)^\infty$.

The entropy depends only on the kneading sequence. Moreover, the entropy is a continuous function of the parameter in the quadratic family [2, Corollary 4.5.5]. Hence, the entropy is constant on $(\mu_2, \mu_1]$. Thus, $h(f_\mu) = h(f_{\mu_1})$ for all $\mu \in (\mu_2, \mu_1]$. □

Note that the proof of Theorem 3.7 shows that there is an open interval (μ_1, μ_3) such that for all $\mu \in (\mu_1, \mu_3)$, $K(f_\mu) = (D\bar{B})^\infty$ and $h(f_\mu) = h(f_{\mu_1})$, where $\bar{B}$ is an L or an R such that $D\bar{B} > DC$.

Theorem 3.8. *Let $\sqrt{2} < s < t \leq 2$ be such that for each of the tent maps f_s and f_t, the turning point is periodic. Suppose $K(f_s) = D_sC$ and $K(f_t) = D_tC$, where D_s and D_t are some finite sequences of R's and L's. Let B_s be an R or an L so that $D_sB_s < D_sC$ and let B_t be an R or an L so that $D_tB_t < D_tC$. Then $\sigma^k(D_sB_s)^\infty < (D_tB_t)^\infty$ for each $k \geq 0$.*

Proof: By [17, Theorem III.1.1], there exists a parameter μ_s such that the kneading sequence of the quadratic map f_{μ_s} is $K(f_{\mu_s}) = D_sC$. By Theorem 3.5 and Corollary 3.6, for μ_1 slightly smaller than μ_s, $K(f_{\mu_1}) = (D_sB_s)^\infty$. By Theorem 3.7, the entropy of the quadratic map f_{μ_1} is $h(f_{\mu_1}) = h(f_{\mu_s})$, and we know that $h(f_{\mu_s}) = \log s$; hence, $h(f_{\mu_1}) = \log s$. Similarly, for μ_2 slightly smaller than μ_t, $K(f_{\mu_2}) = (D_tB_t)^\infty$ and the entropy of the quadratic map f_{μ_2} is $h(f_{\mu_2}) = h(f_{\mu_t}) = \log t$. Since $s < t$, $h(f_{\mu_1}) = \log s < \log t = h(f_{\mu_2})$. Since in the quadratic family, the entropy depends monotonically on the kneading sequence, it follows that $(D_sB_s)^\infty < (D_tB_t)^\infty$. As $(D_sB_s)^\infty$ is the kneading sequence of the map f_{μ_1}, we know that $(D_sB_s)^\infty$ is shift maximal; hence, $\sigma^k(D_sB_s)^\infty < (D_tB_t)^\infty$ for each $k \geq 0$. □

We adopt the following notation. If k is a positive integer, we let $F(f_s^k)$ denote the number of fixed points of f_s^k in $J_s = [c_2, c_1]$.

Theorem 3.9. *Let f_s and f_t be tent maps with slopes $1 < s < t \leq 2$. Then for any positive integer n, $F(f_s^n) \leq F(f_t^n)$.*

Proof: Let n be a fixed positive integer. There are four cases to distinguish. In each of these cases we will show that there is an injective function from the set of fixed points of f_s^n to the set of fixed points of f_t^n.

Case 1. For each of the tent maps f_s and f_t, the turning point is not periodic. Then both $K(f_s)$ and $K(f_t)$ are infinite.

Let y be a fixed point of f_s^n. Then c is not in the orbit of y under f_s. The forward itinerary of y under f_s is $\underline{I}_s(y) = S^\infty$ for some sequence S of length n of L's and R's. Since $\sigma^k S^\infty < K(f_s) < K(f_t)$ for all k, by Definition 2.5, we have that $S^\infty << K(f_t)$. By Theorem 2.6, there is a point z with $\underline{I}_t(z) = S^\infty$. Since z and $f_t^n(z)$ have the same itinerary, by Remark 2.8, it follows that $f_t^n(z) = z$. By Remark 2.8, it also follows that z is the unique point with $\underline{I}_t(z) = S^\infty$.

Case 2. The turning point c is periodic under f_s, and c is not periodic under f_t. Then $K(f_t)$ is infinite and $K(f_s) = DC$ for some finite sequence D of R's and L's.

Let y be a fixed point of f_s^n. Then $\sigma^k \underline{I}_s(y) \leq K(f_s) < K(f_t)$ for all k. Since $K(f_t)$ is infinite, this means that $\underline{I}_s(y) << K(f_t)$. As in Case 1, there exists a unique point z with $\underline{I}_t(z) = \underline{I}_s(y)$ and z is a fixed point of f_t^n.

Case 3. The turning point c is not periodic under f_s, and c is periodic under f_t. Then $K(f_s)$ is infinite and $K(f_t) = DC$ for some finite sequence D of R's and L's.

Let y be a fixed point of f_s^n. Then c is not in the orbit of y under f_s. The forward itinerary of y under f_s is $\underline{I}_s(y) = S^\infty$ for some sequence S of length n of L's and R's, and $\sigma^k S^\infty < K(f_s) < K(f_t)$ for all k. By Corollary 3.4, $\sigma^k S^\infty < (DB)^\infty$ where B is chosen to be an R or an L so that $DB < DC$. Thus, $S^\infty << K(f_t)$. It follows that there exists a unique point z with $\underline{I}_t(z) = S^\infty$ and $f_t^n(z) = z$.

Case 4. For each of the tent maps f_s and f_t, the turning point is periodic. Then $K(f_s) = D_s C$ and $K(f_t) = D_t C$ for some finite sequences D_s and D_t of R's and L's.

Let y be a fixed point of f_s^n. If c is not in the orbit of y under f_s, then the forward itinerary of y under f_s is $\underline{I}_s(y) = S^\infty$ for some sequence S of length n of L's and R's, and $\sigma^k S^\infty < K(f_s) < K(f_t)$ for all k. By Corollary 3.4, $\sigma^k S^\infty < (D_s B_s)^\infty$ where B_s is chosen to be an R or an L so that $D_s B < D_s C$. Thus, $S^\infty << K(f_t)$.

It follows that there exists a unique point z with $\underline{I}_t(z) = S^\infty$ and $f_t^n(z) = z$.

If c is not fixed by f_s^n, we are done.

Suppose c is fixed by f_s^n. Then the length of D_sC divides n. Choose B_s to be an R or an L so that $D_sB_s < D_sC$ and choose B_t to be an R or an L so that $D_tB_t < D_tC$. By Theorem 3.8, $\sigma^k(D_sB_s)^\infty < (D_tB_t)^\infty$ for each $k \geq 0$. Thus, $(D_sB_s)^\infty << K(f_t)$. It follows that there exists a unique point z with $\underline{I}_t(z) = (D_sB_s)^\infty$ and $f_t^n(z) = z$. Note that no point y has $\underline{I}_s(y) = (D_sB_s)^\infty$.

Thus, we proved that in all of the above cases there is an injective function from the set of fixed points of f_s^n to the set of fixed points of f_t^n. □

Theorem 3.10. *Let f_s and f_t be tent maps with slopes $1 < s < t \leq 2$. Then, there exists a positive integer N such that $F(f_s^N) < F(f_t^N)$.*

Proof: Let $s < t_1 < t$ be such that c is periodic under the tent map f_{t_1}. Let N be the period of c under f_{t_1}. In the proof of Theorem 3.9, we establish an injective function from the set of fixed points of f_s^N to the set of fixed points of $f_{t_1}^N$. But the turning point is not in the orbit of fixed points of $f_{t_1}^N$ which we obtained. Thus, since c is a fixed point of $f_{t_1}^N$, we have that $F(f_s^N) < F(f_{t_1}^N)$. Finally, by Theorem 3.9, we have $F(f_{t_1}^N) \leq F(f_t^N)$. □

Theorem 3.11. *If the pseudo-isotopy conjecture is true, then Ingram's conjecture is true.*

Proof: Suppose the pseudo-isotopy conjecture is true. If the result is proved for $\sqrt{2} < s < t \leq 2$, then it can be extended to $1 < s < t \leq 2$ in the following way. For $s \in (1, \sqrt{2}]$, there are two intervals J_1 and J_2 in the core J_s of f_s with pairwise disjoint interiors such that $f_s^2|_{J_1}$ and $f_s^2|_{J_2}$ are topologically conjugate to $f_{s^2}|_{J_{s^2}}$. It follows that for $s \in (1, \sqrt{2}]$, $X_s = \varprojlim\{J_s, f_s\}_{i=0}^\infty$ is determined by $X_{s^2} = \varprojlim\{J_{s^2}, f_{s^2}\}_{i=0}^\infty$. Therefore, it is enough to consider tent maps with slopes in $(\sqrt{2}, 2]$.

Assume there is a homeomorphism $h : (I, f_s) \to (I, f_t)$. By Theorem 3.10, there exists a positive integer N such that $F(f_s^N) < F(f_t^N)$.

Consider the following diagram.

$$\begin{array}{ccc} (I, f_s) & \xrightarrow{\sigma_s^N} & (I, f_s) \\ h \downarrow & & \downarrow h \\ (I, f_t) & \xrightarrow[h\circ\sigma_s^N\circ h^{-1}]{} & (I, f_t) \end{array}$$

Since we assume that the pseudo-isotopy conjecture is true, there exists a positive integer k such that $h \circ \sigma_s \circ h^{-1}$ and σ_t^k are pseudo-isotopic. Hence, $h \circ \sigma_s^N \circ h^{-1}$ and σ_t^{kN} are pseudo-isotopic.

By Theorem 3.1, the number of composants mapped to themselves by σ_s^N is equal to $F(f_s^N)$, and the number of composants mapped to themselves by σ_t^{kN} is equal to $F(f_t^{kN})$. Now, since σ_s^N and $h \circ \sigma_s^N \circ h^{-1}$ are conjugate, and since $h \circ \sigma_s^N \circ h^{-1}$ and σ_t^{kN} are pseudo-isotopic, it follows that the number of composants of X_s mapped to themselves by σ_s^N is equal to the number of composants of X_t mapped to themselves by σ_t^{kN}. Hence, $F(f_s^N) = F(f_t^{kN})$. But this contradicts the fact

$$F(f_t^{kN}) \geq F(f_t^N) > F(f_s^N).$$

Thus, we proved that if the pseudo-isotopy conjecture is true, then for $\sqrt{2} < s < t \leq 2$, the inverse limit spaces (I, f_s) and (I, f_t) are homeomorphic if and only if $s = t$. □

4. Sufficient condition for the pseudo-isotopy conjecture

Finally, we give a sufficient condition for the pseudo-isotopy conjecture to hold. We use the concept of $\bar{d}$ metric developed in [10].

Let $T = (I, f_s) \backslash X_s$ denote the "tail" of the inverse limit space (I, f_s) of the tent map f_s with $\sqrt{2} < s \leq 2$.

Observe that $f_s((c_1, 1]) = [0, c_2)$. Since f_s reaches its maximum c_1 at c, no points from $I = [0, 1]$ map into $(c_1, 1]$. Thus, if $\bar{x} = (x_0, x_1, \cdots) \in (I, f_s)$ and $x_m \in [0, c_2)$ for some positive integer m, then $x_k \in [0, c_2)$ for each positive integer $k \geq m$.

Lemma 4.1. *T is an open ray with endpoint $\bar{0} = (0, 0, \cdots) \in (I, f_s)$. (I, f_s) is a compactification of T.*

Proof: First we prove that T is an open ray with endpoint $\bar{0} = (0, 0, \cdots)$. Denote $x_0 = c_2$. Let $J_0 = [0, x_0]$. There exists a point $x_1 \in J_0$ such that $f_s(x_1) = x_0$. Let $J_1 = [0, x_1]$. Then f_s maps J_1 linearly onto J_0. There exists a point $x_2 \in J_1$ such that $f_s(x_2) = x_1$.

Let $J_2 = [0, x_2]$. Then f_s maps J_2 linearly onto J_1. We form $J_3, J_4, \cdots$ in the same manner. Let $\bar{J} = \varprojlim\{J_n, f_s\}_{n=0}^{\infty}$. Then $\bar{J}$ is an arc in (I, f_s) with endpoints $\bar{0}$ and $\bar{x} = (x_0, x_1, \cdots)$. Note that $\pi_0^{-1}([0, c_2)) = \bar{J}\backslash\{\bar{x}\}$. Therefore, $\bar{J}\backslash\{\bar{x}\}$ is open in (I, f_s). Since $T = \cup_{k=0}^{\infty}\sigma_s^k(\bar{J}\backslash\{\bar{x}\}) = \cup_{k=0}^{\infty}\sigma_s^k(\bar{J})$ is a union of increasing rays with a common endpoint $\bar{0}$, it follows that T is an open ray with endpoint $\bar{0}$.

Next we prove that T is dense in (I, f_s). Let $\bar{x} = (x_0, x_1, \cdots) \in X_s$. For any positive integer m, there exist a positive integer k and $y \in [0, c_2)$ such that $f_s^k(y) = x_m$. Then the points $\bar{x}$ and $\bar{y}_m = (x_0, x_1, \cdots, x_m, f_s^{k-1}(y), \cdots, f_s(y), y, \cdots)$ agree on the first $m + 1$ coordinates and $\bar{y}_m \in T$. Thus, the sequence of points $\{\bar{y}_m\}_{m=0}^{\infty}$ in T converges to the point $\bar{x} \in X_s$. □

We can define the metric $\bar{d}$ on T as follows. Let $\bar{x}$ and $\bar{y}$ be points in T. By Lemma 4.1, T is a ray; hence, there is an arc $A \subset T$ with endpoints $\bar{x}$ and $\bar{y}$. Then there exists a nonnegative integer k such that $\pi_k|_A$ is a homeomorphism. Define

$$\bar{d}(\bar{x}, \bar{y}) = s^k|\pi_k(\bar{x}) - \pi_k(\bar{y})|.$$

This is analogous to how $\bar{d}$ was defined when the turning point of the tent map was periodic [10]. There is a difference: $\bar{d}$ in the present case is consistent with the subspace topology of T. In [10], the $\bar{d}$ metric on a composant of the core X_s was not consistent with the subspace topology of that composant.

Theorem 4.2. *Let $f_s : I \to I$, $f_s(x) = \min\{s \cdot x, s \cdot (1 - x)\}$ be a tent map with slope $s \in (\sqrt{2}, 2]$. Let $h : (I, f_s) \to (I, f_s)$ be a homeomorphism. Suppose there exist a number $M > 0$ and an integer k such that $\bar{d}(h(\bar{x}), \sigma_s^k(\bar{x})) \leq M$, for all $\bar{x} \in T$. Then $h|_{X_s}$ and $\sigma_s^k|_{X_s}$ are pseudo-isotopic.*

Proof: Let $\bar{x} \in X_s$. By Lemma 4.1, the tail T is dense in the inverse limit space (I, f_s); hence, there exists a sequence $\{\bar{x}_n\}_{n=1}^{\infty}$ in T converging to the point $\bar{x}$. Then the sequence $h(\bar{x}_n)$ converges to the point $h(\bar{x})$ and $\sigma_s^k(\bar{x}_n)$ converges to the point $\sigma_s^k(\bar{x})$. Consider the unique arcs $A_n \subset T$ with endpoints $h(\bar{x}_n)$ and $\sigma_s^k(\bar{x}_n)$. By assumption, $\bar{\ell}(A_n) = \bar{d}(h(\bar{x}_n), \sigma_s^k(\bar{x}_n)) \leq M$ for each n. Let $m > 0$ be an integer such that $M < s^m \cdot (f_s(c) - f_s^2(c))$. Then for each n, $\ell(\pi_m(A_n)) \leq \frac{M}{s^m} < f_s(c) - f_s^2(c)$.

Let $\mathcal{C}((I, f_s))$ denote the space of nonempty sub-continua of the inverse limit space (I, f_s) with the Hausdorff metric. The projection $\pi_m : (I, f_s) \to I$ induces a continuous map $\pi_m : \mathcal{C}((I, f_s)) \to \mathcal{C}(I)$. Since $\mathcal{C}((I, f_s))$ is a compact metric space, the sequence $\{A_n\}$ of arcs in T has a convergent subsequence $\{A_{n_j}\}$ converging to a sub-continuum $A \in \mathcal{C}((I, f_s))$. In fact, $A \subset X_s$ by Lemma 4.1. Note that $h(\bar{x}), \sigma_s^k(\bar{x}) \in A$. Since $\pi_m : \mathcal{C}((I, f_s)) \to \mathcal{C}(I)$ is continuous, $\pi_m(A)$ has length at most $\frac{M}{s^m} < f_s(c) - f_s^2(c)$. Thus, A must be a proper sub-continuum of X_s. Therefore, $h(\bar{x})$ and $\sigma_s^k(\bar{x})$ are in the same composant of X_s. By Definition 2.1, $h|_{X_s}$ and $\sigma_s^k|_{X_s}$ are pseudo-isotopic. □

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