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CLOSED DISCRETE SUBSPACES IN FIRST COUNTABLE SPACES

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Abstract. In this note, we consider the following three topological statements:

(1) NS: In every normal first countable space, closed discrete subspaces are separated.
(2) CPS: In every countably paracompact first countable T_2 space, closed discrete subspaces are separated.
(3) CMGD: In every countably metacompact first countable T_1 space, closed discrete subspaces are G_δ-sets.

We construct general combinatorial statements for each of the topological statements above. We then use the combinatorics to prove a chain of implications between the topological statements. We finish by asking if any of the implications can be reversed.

1. Normal First Countable Spaces

On the path to solving the famous Normal Moore Space Conjecture, William Fleissner showed, in [4], the classical result that if one assumes $V = L$, then in normal first countable spaces, closed discrete subspaces are separated. Later, in [6], Peter Nyikos showed that if one assumes the Product Measure Extension Axiom (PMEA), then in normal first countable spaces, closed discrete subspaces are again separated. These results motivated a flurry of activity where many researchers showed that closed discrete subspaces are separated in normal first countable spaces assuming various set-theoretical assumptions (for example, see [12]). This leads one to wonder if there is some general principle that would prove the same result and would be implied by the various set-theoretical assumptions. This would provide a nice cohesive framework to the theory of separation of closed discrete subspaces in normal first countable spaces.

In this section, we present a general combinatorial statement, due to Dennis K. Burke, that is equivalent to the following topological statement, **NS**: In every normal first countable space, closed discrete subspaces are separated. The combinatorial statement in this section will be labeled $\mathbf{N}(\omega, 2, \lambda)$ and is defined as follows.

Definition 1.1. $\mathbf{N}(\omega, 2, \lambda)$: For any family $F = \{f_\delta : \delta \in {}^\lambda 2\} \subseteq {}^\lambda\omega$ there is a function $g \in {}^\lambda\omega$ such that for all $\alpha, \beta \in \lambda$ there is a $\delta \in {}^\lambda 2$ such that the following conditions hold:

(1) $\delta(\alpha) \neq \delta(\beta)$ and
(2) $f_\delta(\alpha) \leq g(\alpha)$ and $f_\delta(\beta) \leq g(\beta)$.

In [11], Franklin D. Tall was the first to give a set of combinatorial conditions on ${}^\kappa\omega$ which was equivalent to the statement NS. The main result of this section is that the statement NS is true if and only if $\mathrm{N}(\omega, 2, \lambda)$ is true for all cardinals λ. In other words, we have the following theorem.

Theorem 1.2. *In any model of ZFC, the following are equivalent:*

(1) $\mathrm{N}(\omega, 2, \lambda)$ *is true for all cardinals* λ.
(2) *In every normal first countable space, closed discrete subspaces are separated.*

Proof: We first prove (1) $\rightarrow$ (2). To this end, suppose that X is a normal first countable space and that λ enumerates a closed discrete subspace of X. For each $\alpha \in \lambda$, let $\{U(\alpha, n) : n \in \omega\}$ be a local neighborhood base at α such that $m < n$ implies that $U(\alpha, m) \supseteq U(\alpha, n)$.

Let $\delta \in {}^\lambda 2$ denote a partition of λ into two subsets. Since X is normal, there is a function $f_\delta \in {}^\lambda\omega$ such that whenever $\alpha, \beta \in \lambda$ with $\delta(\alpha) \neq \delta(\beta)$ then $U(\alpha, f_\delta(\alpha)) \cap U(\beta, f_\delta(\beta)) = \emptyset$. This gives us a family of functions $F = \{f_\delta : \delta \in {}^\lambda 2\} \subseteq {}^\lambda\omega$.

By hypothesis, there is a $g \in {}^\lambda\omega$ as described by $\mathrm{N}(\omega, 2, \lambda)$. We claim that g is a separating function for λ. Indirectly, suppose that there are $\alpha, \beta \in \lambda$ such that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) \neq \emptyset$. Then by $\mathrm{N}(\omega, 2, \lambda)$, there is a partition $\delta \in {}^\lambda\omega$ such that (1) $\delta(\alpha) \neq \delta(\beta)$ and (2) $f_\delta(\alpha) \leq g(\alpha)$ and $f_\delta(\beta) \leq g(\beta)$ are true. However, since

g dominates f_δ on α and β and since f_δ is a separating function for δ, it must be that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) = \emptyset$, which is a contradiction. Thus, g is a separating function for λ.

We now show that (2) $\rightarrow$ (1). Let $F = \{f_\delta : \delta \in {}^\lambda 2\} \subseteq {}^\lambda\omega$ be a family of functions. We define a subset $ISO \subset [\lambda \times \omega]^2$ to be the set of all elements $\{(\alpha, n), (\beta, m)\} \in [\lambda \times \omega]^2$ such that for all $\delta \in {}^\lambda 2$ the following requirement is satisfied:

$$(f_\delta(\alpha) \leq n \text{ and } f_\delta(\beta) \leq m) \rightarrow (\delta(\alpha) = \delta(\beta)).$$

The underlying set of points in our space will be $X = \lambda \cup ISO$. We topologize X by isolating the points in ISO. For each $\alpha \in \lambda$ and $k \in \omega$ the sets

$$U(\alpha, k) = \{\alpha\} \cup \{\{(\alpha, n), (\beta, m)\} \in ISO : \beta \in \lambda \text{ and } n \geq k\}$$

form a neighborhood base at $\alpha \in \lambda$. From this, it is easy to see that X is a first countable space. The following claims will establish other properties of X.

Claim 1. We claim that X is T_2. If $\alpha, \beta \in \lambda$, pick $\delta \in {}^\lambda 2$ with $\delta(\alpha) \neq \delta(\beta)$. If $k \in \omega$ such that $f_\delta(\alpha) < k$ and $f_\delta(\beta) < k$, then $U(\alpha, k) \cap U(\beta, k) = \emptyset$.

Claim 2. We claim that X is normal. It suffices to show that disjoint subsets of the closed discrete subspace λ can be separated by disjoint open sets in X. To this end, suppose that $\delta \in {}^\lambda 2$. Let

$$V_\delta = \{U(\alpha, f_\delta(\alpha)) : \alpha \in \lambda \text{ and } \delta(\alpha) = 0\}$$

and

$$W_\delta = \{U(\beta, f_\delta(\beta)) : \beta \in \lambda \text{ and } \delta(\beta) = 1\}.$$

We claim that $V_\delta \cap W_\delta = \emptyset$. To see this, pick any $\alpha, \beta \in \lambda$ with $\delta(\alpha) = 0$ and $\delta(\beta) = 1$. Consider the corresponding open neighboorhoods $U(\alpha, f_\delta(\alpha))$ and $U(\beta, f_\delta(\beta))$. If there were an isolated point $\{(\alpha, n), (\beta, m)\} \in U(\alpha, f_\delta(\alpha)) \cap U(\beta, f_\delta(\beta))$, then $f_\delta(\alpha) \leq n$ and $f_\delta(\beta) \leq m$. By the restriction we placed on the isolated points, this implies that $\delta(\alpha) = \delta(\beta)$ which is a contradiction.

By assumption, the closed discrete subspace $\lambda \subset X$ is separated. Thus, there is a $g \in {}^\lambda\omega$ such that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) = \emptyset$ for all $\alpha, \beta \in \lambda$.

Claim 3. We claim that g satisfies the conclusion of the statement $\mathrm{N}(\omega, 2, \lambda)$. To see this, suppose that $\alpha, \beta \in \lambda$. Consider the point $p = \{(\alpha, g(\alpha)), (\beta, g(\beta))\}$. If $p \in ISO$, then by the definition of our basic open sets, it must be that $p \in U(\alpha, g(\alpha)) \cap U(\beta, g(\beta))$. However, g is a separating function for λ which means that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) = \emptyset$. It follows that p is not a valid isolated point (i.e., $p \notin ISO$). Thus, there must be a $\delta \in {}^{\lambda}\omega$ such that (1) $\delta(\alpha) \neq \delta(\beta)$ and (2) $f_\delta(\alpha) \leq g(\alpha)$ and $f_\delta(\beta) \leq g(\beta)$, as desired.

This completes the proof. □

It is worth noticing that the space X, constructed in the above proof of (2) → (1), is actually a Moore space where the canonical development consists of open covers of point-order ≤ 2. This is the basis for Remark 1.3, which we state without further proof.

Remark 1.3. *In any model of ZFC, the following are equivalent:*

(1) $\mathrm{N}(\omega, 2, \lambda)$ *is true for all cardinals* λ*.*
(2) *In every normal first countable space, closed discrete subspaces are separated.*
(3) *In every (2-boundedly metacompact) normal Moore space, closed discrete subspaces are separated.*

2. Countably Paracompact First Countable Spaces

In [13], W. Stephen Watson showed that if one assumes $V = L$, then in countably paracompact first countable T_2 spaces, every closed discrete subspace is separated. This result provided a nice parallel to Fleissner's earlier result on normal first countable spaces under $V = L$. In [2], Burke showed that if one assumes the PMEA, then in countably paracompact first countable T_1 spaces, every closed discrete subspace is strongly separated. This PMEA result parallels Nyikos's earlier result on normal first countable spaces mentioned in the previous section. Finally, mirroring Alan D. Taylor's result on normal spaces, Kerry D. Smith and Paul J. Szeptycki showed in [8] that if one assumes $\diamond^*$, then in countably paracompact first countable T_2 spaces, closed discrete subspaces of size $\aleph_1$ are separated. From these results it appears that there is an intimate connection between the separation properties of normal spaces and countably paracompact spaces.

In this section, we give a combinatorial statement that is equivalent to the following topological statement, **CPS**: In every countably paracompact first countable T_2 space, closed discrete subspaces are separated. But first we present a definition on families of functions that we will need in our work.

Definition 2.1. Let us call a family of functions $F = \{f_\delta : \delta \in {}^\lambda\omega\} \subseteq {}^\lambda\omega$ a T_2 *family* if for all $\alpha, \beta \in \lambda$ there is a $\delta \in {}^\lambda\omega$ such that either $f_\delta(\alpha) \leq \delta(\beta)$ or $f_\delta(\beta) \leq \delta(\alpha)$.

The combinatorial statement in this section is $\mathbf{CP}(\omega, \omega, \lambda)$ and is defined below.

Definition 2.2. $\mathbf{CP}(\omega, \omega, \lambda)$: For any T_2 family $F = \{f_\delta : \delta \in {}^\lambda\omega\} \subseteq {}^\lambda\omega$, there is a function $g \in {}^\lambda\omega$ such that for all $\alpha, \beta \in \lambda$ there is a $\delta \in {}^\lambda\omega$ such that the following conditions hold:

(1) either $f_\delta(\alpha) \leq \delta(\beta)$ or $f_\delta(\beta) \leq \delta(\alpha)$ and
(2) $f_\delta(\alpha) \leq g(\alpha)$ and $f_\delta(\beta) \leq g(\beta)$.

(Notice that condition (1) is the same condition that is imposed on the definition of a T_2 family. The difference here is that both conditions (1) and (2) must hold.)

We are now ready to state and prove the main theorem of this section.

Theorem 2.3. *In any model of ZFC, the following are equivalent:*

(1) $\mathrm{CP}(\omega, \omega, \lambda)$ *is true for all cardinals* λ.
(2) *In every countably paracompact first countable* T_2 *space, closed discrete subspaces are separated.*

Proof: We first show that (1) $\rightarrow$ (2). To this end, suppose that X is a countably paracompact first countable T_2 space and that λ enumerates a closed discrete subspace of X. For each $\alpha \in \lambda$, let $\{U(\alpha, n) : n \in \omega\}$ be a local neighborhood base at α such that $m < n$ implies that $U(\alpha, m) \supseteq U(\alpha, n)$.

Let a given $\delta \in {}^\lambda\omega$ denote a specific partition of λ into countably many pieces. Since X is countably paracompact, there is a function $f_\delta \in {}^\lambda\omega$ such that whenever $\alpha, \beta \in \lambda$ and $U(\alpha, f_\delta(\alpha)) \cap U(\beta, f_\delta(\beta)) \neq \emptyset$, then $f_\delta(\alpha) > \delta(\beta)$ and $f_\delta(\beta) > \delta(\alpha)$. This gives us a family of functions $F = \{f_\delta : \delta \in {}^\lambda\omega\} \subseteq {}^\lambda\omega$. However, we wish to make F a T_2 family. Since countably paracompact first countable

T_2 spaces are regular (see [1]), for any given $\alpha \in \lambda$, apply the regularity of X to find $f \in {}^{\lambda}\omega$ such that $U(\alpha, f(\alpha)) \cap U(\beta, f(\beta)) = \emptyset$ and $f(\beta) > f(\alpha) > 1$ for all $\beta \neq \alpha$. Next define $\delta_\alpha \in {}^{\lambda}\omega$ so that $\delta_\alpha(\alpha) = 0$ and $\delta_\alpha(\beta) = f(\alpha)$ for all $\beta \neq \alpha$ and set $f_{\delta_\alpha} = f$. Note that this may *redefine* f_{δ_α} but the family F remains well-defined since for each $\alpha \in \lambda$, the chosen δ_α is picked so that $\delta_\alpha(\gamma) = 0$ if and only if $\gamma = \alpha$. Moreover, f_{δ_α} satisfies the condition at the beginning of this paragraph to be in our family F. Finally, for any $\beta \neq \alpha$, we have $f_{\delta_\alpha}(\alpha) \leq \delta_\alpha(\beta)$ and hence, F is a T_2 family.

By hypothesis, there is a $g \in {}^{\lambda}\omega$ as described by $\mathrm{CP}(\omega, \omega, \lambda)$. We claim that g is a separating function for λ. Indirectly, suppose that there are $\alpha, \beta \in \lambda$ such that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) \neq \emptyset$. Then by $\mathrm{CP}(\omega, \omega, \lambda)$, there is a partition $\delta \in {}^{\lambda}\omega$ such that (1) either $f_\delta(\alpha) \leq \delta(\beta)$ or $f_\delta(\beta) \leq \delta(\alpha)$, and (2) $f_\delta(\alpha) \leq g(\alpha)$ and $f_\delta(\beta) \leq g(\beta)$ are true. Since g dominates f_δ on α and β, it must be that both $f_\delta(\alpha) > \delta(\beta)$ and $f_\delta(\beta) > \delta(\alpha)$ hold. However this contradicts condition (1) of $\mathrm{CP}(\omega, \omega, \lambda)$. Thus, g is a separating function for λ.

We now prove (2) $\rightarrow$ (1). Let $F = \{f_\delta : \delta \in {}^{\lambda}\omega\} \subseteq {}^{\lambda}\omega$ be a T_2 family. We define a subset $ISO \subset [\lambda \times \omega]^2$ to be the set of all elements $\{(\alpha, n), (\beta, m)\} \in [\lambda \times \omega]^2$ such that for all $\delta \in {}^{\lambda}\omega$ the following requirement is satisfied:

$$(f_\delta(\alpha) \leq n \text{ and } f_\delta(\beta) \leq m) \rightarrow (f_\delta(\alpha) > \delta(\beta) \text{ and } f_\delta(\beta) > \delta(\alpha)).$$

For $\alpha \in \lambda$ and $k \in \omega$, set

$$U(\alpha, k) = \{\alpha\} \cup \{\{(\alpha, n), (\beta, m)\} \in ISO : \beta \in \lambda \text{ and } n \geq k\}.$$

We topologize our set $X = \lambda \cup ISO$ by declaring each point in ISO isolated and by using $\{U(\alpha, k) : k < \omega\}$ as a local neighborhood base for each $\alpha \in \lambda$. This defines a first countable T_1 topology on X.

Claim 1. We claim that X is T_2. If $\alpha, \beta \in \lambda$, pick $\delta \in {}^{\lambda}\omega$ such that either $f_\delta(\alpha) \leq \delta(\beta)$ or $f_\delta(\beta) \leq \delta(\alpha)$ (such a δ exists since F is a T_2 family). If there is an isolated point $\{(\alpha, n), (\beta, m)\} \in U(\alpha, f_\delta(\alpha)) \cap U(\beta, f_\delta(\beta))$, then by the requirement of ISO, it must be that both $f_\delta(\alpha) > \delta(\beta)$ and $f_\delta(\beta) > \delta(\alpha)$, which is a contradiction.

Claim 2. We claim that X is countably paracompact. Since the nonisolated points in X form the closed discrete subspace λ, it suffices to show that every countable partition of λ can be expanded to a locally finite collection of open sets. To see this, let $\delta \in {}^{\lambda}\omega$. For all $n \in \omega$, let

$$U_n = \bigcup\{U(\alpha, f_\delta(\alpha)) : \alpha \in \lambda \text{ and } \delta(\alpha) = n\}.$$

We claim that the collection $\{U_n : n \in \omega\}$ is locally finite in X. To see this, suppose $\alpha \in \lambda$. If $\beta \in \lambda$ such that $U(\alpha, f_\delta(\alpha)) \cap U(\beta, f_\delta(\beta)) \neq \emptyset$, then there is an $\{(\alpha, n), (\beta, m)\} \in U(\alpha, f_\delta(\alpha)) \cap U(\beta, f_\delta(\beta))$. Then, $f_\delta(\alpha) \leq n$ and $f_\delta(\beta) \leq m$, which implies that $f_\delta(\alpha) > \delta(\beta)$. Hence, $U(\alpha, f_\delta(\alpha))$ is a neighborhood of $\alpha \in \lambda$ that meets at most finitely many U_n's. Of course, each isolated point $\{(\alpha, n), (\beta, m)\}$ is in at most two U_n's. Thus, $\{U_n : n \in \omega\}$ is locally finite and X is countably paracompact.

By assumption, the closed discrete subspace λ is separated in X. So let $g \in {}^{\lambda}\omega$ such that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) = \emptyset$ for all $\alpha, \beta \in \lambda$.

Claim 3. We claim that g satisfies the conclusion of the statement CP$(\omega, \omega, \lambda)$. To see this, suppose that $\alpha, \beta \in \lambda$. Consider the point $p = \{(\alpha, g(\alpha)), (\beta, g(\beta))\}$. If $p \in ISO$, then by the definition of our basic open sets, it must be that $p \in U(\alpha, g(\alpha)) \cap U(\beta, g(\beta))$. However, g is a separating function for λ which means that $U(\alpha, g(\alpha)) \cap U(\beta, g(\beta)) = \emptyset$. It follows that p is not a valid isolated point; i.e., $p \notin ISO$. Thus, there must be a $\delta \in {}^{\lambda}\omega$ such that (1) either $f_\delta(\alpha) \leq \delta(\beta)$ or $f_\delta(\beta) \leq \delta(\alpha)$ and (2) $f_\delta(\alpha) \leq g(\alpha)$ and $f_\delta(\beta) \leq g(\beta)$, as desired.

This completes the proof. □

3. Countably Metacompact First Countable Spaces

In [3], Burke showed that if one assumes the PMEA, then in countably metacompact first countable T_1 spaces, every closed discrete subspace is a G_δ. This result showed that the countable metacompactness property provided a mild form of separation of closed discrete subspaces in the class of first countable T_1 spaces. In [7], Nyikos showed that if $V = L$ then in a countably metacompact locally countable T_1 space, every closed discrete subspace is a G_δ. He then asked if his result would still hold under $V = L$ if one replaced the locally countable condition with the first countable property. Later, in [9], Szeptycki constructed a counterexample to Nyikos's question by constructing a countably metacompact first countable regular space with a closed discrete subspace that is not a G_δ. In this section, we study closed discrete subspaces in countable metacompact first countable T_1 spaces.

Here, we present a combinatorial statement that is equivalent to the following topological statement, **CMGD**: In every countably metacompact first countable T_1 space, closed discrete subspaces are G_δ-sets. The combinatorial statement in this section will be labeled **CM**$(\omega, \omega, \lambda)$ and is defined below.

Definition 3.1. **CM**$(\omega, \omega, \lambda)$: For any family $F = \{f_\delta : \delta \in {}^\lambda\omega\} \subseteq {}^\lambda\omega$ there is a sequence of functions $\{g_n \in {}^\lambda\omega : n \in \omega\}$ such that whenever $\{\alpha_n : n \in \omega\} \in [\lambda]^\omega$ satisfying the condition for all $i, j \in \omega$ $(i \neq j \to \alpha_i \neq \alpha_j)$, then there is a $\delta \in {}^\lambda\omega$ and an $S \in [\omega]^\omega$ such that the following conditions hold:

(1) $|\{\delta(\alpha_n) : n \in S\}| = \omega$ and
(2) $f_\delta(\alpha_n) \leq g_n(\alpha_n)$ for every $n \in S$.

We are now ready to state and prove the main theorem of this section.

Theorem 3.2. *In any model of ZFC, the following are equivalent:*

(1) $\mathrm{CM}(\omega, \omega, \lambda)$ *is true for all cardinals* λ.
(2) *In every countably metacompact first countable* T_1 *space, closed discrete subspaces are* G_δ*-sets.*

Proof: We first show that (1) $\to$ (2). Suppose X is a countably metacompact first countable T_1 space and λ enumerates a closed discrete subspace of X. For each $\alpha \in \lambda$, let $\{U(\alpha, n) : n \in \omega\}$ be a monotonically decreasing local neighborhood base at α with $\{\alpha\} = \cap_{n\in\omega} U(\alpha, n)$.

For every partition $\delta \in {}^\lambda\omega$, let $f_\delta \in {}^\lambda\omega$ code a point finite open expansion of $\{A_n : n \in \omega\}$, where $A_n = \{\alpha \in \lambda : \delta(\alpha) = n\}$. This gives us a family of functions $F = \{f_\delta : \delta \in {}^\lambda\omega\} \subseteq {}^\lambda\omega$.

By assumption, there is a sequence of functions $\{g_n \in {}^\lambda\omega : n \in \omega\}$ that satisfies the conclusion of $\mathrm{CM}(\omega, \omega, \lambda)$. For each $\alpha \in \lambda$ and $n \in \omega$, let $h_n(\alpha) = \max\{g_n(\alpha), n\}$. Then, for each $n \in \omega$, set $U_n = \cup\{U(\alpha, h_n(\alpha)) : \alpha \in \lambda\}$. We claim that $\lambda = \cap_{n\in\omega} U_n$ and hence that λ is a G_δ. Clearly, we have $\lambda \subset U_n$ for every $n \in \omega$. To see that $\lambda = \cap_{n\in\omega} U_n$ suppose that there is a point $x \in X - \lambda$ with $x \in \cap_{n\in\omega} U_n$. Then there is an infinite set of points $\{\alpha_n : n \in \omega\} \in [\lambda]^\omega$ such that $x \in U(\alpha_n, h_n(\alpha_n))$ for each $n \in \omega$.

By CM(ω,ω,λ), there is a partition $\delta \in {}^{\lambda}\omega$ and an $S \in [\omega]^{\omega}$ such that (1) $|\{\delta(\alpha_n) : n \in S\}| = \omega$ and (2) $f_\delta(\alpha_n) \leq g_n(\alpha_n)$ for every $n \in S$. Now for every $n \in S$, since $f_\delta(\alpha_n) \leq g_n(\alpha_n) \leq h_n(\alpha_n)$, it follows that $U(\alpha_n, f_\delta(\alpha_n)) \supseteq U(\alpha_n, g_n(\alpha_n)) \supseteq U(\alpha_n, h_n(\alpha_n))$. Hence, $x \in U(\alpha_n, f_\delta(\alpha_n))$ for every $n \in S$. However, since f_δ codes a point finite open expansion of the partition δ, conclusion (1) of CM(ω,ω,λ) is contradicted. This completes the proof.

We now prove (2) $\to$ (1). Let $F = \{f_\delta : \delta \in {}^{\lambda}\omega\} \subseteq {}^{\lambda}\omega$ be a family of functions. We define a subset $ISO \subseteq [\lambda \times \omega]^{\omega}$ to be the set of all $\{(\alpha_i, n_i) : i \in \omega\} \in [\lambda \times \omega]^{\omega}$ such that for all $i, j \in \omega$ ($i \neq j \to \alpha_i \neq \alpha_j$) and that satisfy the following requirement:

$$\forall \delta \in {}^{\lambda}\omega\ \forall S \in [\omega]^{\omega}((f_\delta(\alpha_i) \leq n_i, \forall i \in S) \to (|\{\delta(\alpha_i) : i \in S\}| < \omega)).$$

For each $\alpha \in \lambda$ and $k \in \omega$, let $U(\alpha, k)$ be the following set

$$\{\alpha\} \cup \{\{(\alpha_i, n_i) : i \in \omega\} \in ISO : \exists i \in \omega(\alpha = \alpha_i \text{ and } n_i \geq k)\}.$$

We topologize our set $X = \lambda \cup ISO$ by declaring each point in ISO isolated and by using $\{U(\alpha, k) : k < \omega\}$ as a local neighborhood base for each $\alpha \in \lambda$. This defines a first countable T_1 topology on X.

Claim 1. We claim that X is T_1. It suffices to show that each point in ISO is closed. To see this, suppose that $p = \{(\alpha_i, n_i) : i \in \omega\} \in ISO$ and that $\alpha \in \lambda$. If $\alpha \neq \alpha_i$ for every $i \in \omega$, then $U(\alpha, 0)$ is a neighborhood of α that does not contain p. On the other hand, since $i \neq j \to \alpha_i \neq \alpha_j$, then for every $i \in \omega$, $U(\alpha_i, n_i + 1)$ is a neighborhood of α_i that does not contain p. This proves the claim.

Claim 2. We claim that X is countably metacompact. Since the nonisolated points in X form the closed discrete subspace λ, it suffices to show that every countable partition of λ can be expanded to a point finite collection of open sets. To see this, let $\delta \in {}^{\lambda}\omega$. For every $n \in \omega$, let

$$V_n = \cup\{U(\alpha, f_\delta(\alpha)) : \alpha \in \lambda \text{ and } \delta(\alpha) = n\}.$$

We claim that the collection $\{V_n : n \in \omega\}$ is point finite in X. If not, there is an infinite subset $\{\alpha_i : i \in \omega\}$ of λ with the property that $i \neq j \to \delta(\alpha_i) \neq \delta(\alpha_j)$ and a point $p = \{(\beta_j, m_j) : j \in \omega\} \in ISO$ with $p \in \cap_{i\in\omega} U(\alpha_i, f_\delta(\alpha_i))$. Let $S = \{j \in \omega : \beta_j = \alpha_i \text{ for some } i \in \omega\}$. Then S is an infinite subset of ω, that is $S \in [\omega]^{\omega}$, and there is a one-to-one correspondence $c : S \to \omega$ such that $\beta_j = \alpha_{c(j)}$ for every $j \in S$. Then for every $j \in S$, we have $f_\delta(\beta_j) = f_\delta(\alpha_{c(j)}) \leq m_j$ and hence, since $p \in ISO$, it must be that $|\{\delta(\alpha_{c(j)}) : j \in S\}| < \omega$ which is a contradiction. Thus, $\{V_n : n \in \omega\}$ is point finite and X is countably metacompact.

By assumption, λ is a G_δ. Thus, there are functions $\{g_n : n \in \omega\}$ such that if $U_n = \cup\{U(\alpha, g_n(\alpha)) : \alpha \in \lambda\}$ then $\cap_{n\in\omega} U_n = \lambda$.

Claim 3. The sequence $\{g_n : n \in \omega\}$ satisfies the conclusion of the statement CM$(\omega, \omega, \lambda)$. To see this, suppose that $\{\alpha_n : n \in \omega\} \in [\lambda]^\omega$ satisfies the condition $\forall i, j \in \omega\ (i \neq j \rightarrow \alpha_i \neq \alpha_j)$. Consider the point $p = \{(\alpha_n, g_n(\alpha_n)) : n \in \omega\}$. If $p \in ISO$, then by the definition of $U(\alpha_n, g_n(\alpha_n))$ it must be that $p \in U(\alpha_n, g_n(\alpha_n))$ for every $n \in \omega$. However, since $\{g_n : n \in \omega\}$ codes a G_δ subset of X that is equal to λ, there is an $n \in \omega$ with $p \notin U(\alpha_n, g_n(\alpha_n))$. We are thus forced to conclude that $p \notin ISO$. Hence, it must be that there is a $\delta \in {}^\lambda\omega$ and an $S \in [\omega]^\omega$ such that (1) $|\{\delta(\alpha_n) : n \in S\}| = \omega$ and yet (2) $f_\delta(\alpha_n) \leq g_n(\alpha_n)$ for every $n \in S$.

This completes the proof. □

4. Relationships

In this section, we will show that there is a chain of implications among the three topological statements that we studied in the previous sections. Moreover, we will consider the following additional topological statement and show its relationship to the previous statements.

CPSS: In every countably paracompact first countable T_2 space, closed discrete sets are *strongly* separated.

Theorem 4.1. *In any model of ZFC, the following holds:*

$$\text{CMGD} \rightarrow \text{CPSS} \rightarrow \text{CPS} \rightarrow \text{NS}.$$

Proof: All but one of the implications stated in this theorem are easily proved. For example, by definition, it is obvious that CPSS implies CPS. To see that CPS→NS, suppose in some model of ZFC that there is a normal first countable space X with a closed discrete non-separated subspace D. If we isolate the points of $X - D$, then we obtain a space Y that is countably paracompact. However, in Y, the points of D are still not separated. Thus, Y is a countably paracompact first countable T_2 space with a non-separated closed discrete subspace. Hence, by contraposition, CPS→NS.

The real challenge, and not an inherently obvious implication, is to show that CMGD→CPSS. We prove this by showing that if the combinatorial statement $\mathrm{CM}(\omega,\omega,\lambda)$ is true for all cardinals λ, then so is the topological statement CPSS. Then by Theorem 3.2, it follows that CMGD→CPSS.

To this end, suppose that X is a countably paracompact first countable T_2 space. However, in this situation, instead of using λ to enumerate a closed discrete subspace of X, we will use λ to enumerate the entire space X. Since the statement $\mathrm{CM}(\omega,\omega,\lambda)$ is true for all cardinals λ, it will follow that in any countably paracompact first countable T_2 space X of arbitrary cardinality closed discrete subspaces are strongly separated.

Let D be a closed discrete subspace of λ $(= X)$ and let $\delta \in {}^{\lambda}\omega$. Since X is countably paracompact, there is a $h_\delta \in {}^{D}\omega$ that codes a locally finite open expansion of $\{D_n : n \in \omega\}$ where $D_n = \{\alpha \in D : \delta(\alpha) = n\}$. That is, for each $n \in \omega$, if we set $U_n = \bigcup\{U(\alpha, h_\delta(\alpha)) : \delta(\alpha) = n\}$, then the family $\{U_n : n \in \omega\}$ is a locally finite family of open subsets of X with $D_n \subseteq U_n$ for each $n \in \omega$. By local finiteness, for each point $\alpha \in X$, there is a neighborhood $U(\alpha, j_\delta(\alpha))$ that meets at most finitely many of the U_n's. For each $\alpha \in D$, define $f_\delta(\alpha) = \max\{h_\delta(\alpha), j_\delta(\alpha)\}$ and for each $\alpha \in X - D$, define $f_\delta(\alpha) = j_\delta(\alpha)$. This gives us a family of functions $F = \{f_\delta : \delta \in {}^{\lambda}\omega\} \subseteq {}^{\lambda}\omega$. By $\mathrm{CM}(\omega,\omega,\lambda)$, there is a sequence of functions $\{g_n \in {}^{\lambda}\omega : n \in \omega\}$ such that whenever $\{\alpha_n : n \in \omega\} \in [\lambda]^{\omega}$ there is a $\delta \in {}^{\lambda}\omega$ and an $S \in [\omega]^{\omega}$ such that the following conditions hold:

(1) $|\{\delta(\alpha_n) : n \in S\}| = \omega$ and
(2) $f_\delta(\alpha_n) \leq g_n(\alpha_n)$ for every $n \in S$.

We claim that the sequence $\{g_n \in {}^{\lambda}\omega : n \in \omega\}$ will show that the closed discrete subspace D is a regular G_δ subset of X and that D is separated. Moreover, it is known that in a countably paracompact regular space, two closed sets can be separated if one of them is a regular G_δ subset (see [5]). From this and our claim, we can conclude that D is strongly separated in X.

We first show that D is a regular G_δ subset. By our usual means, for each $n \in \omega$, let

$$V_n = \bigcup\{U(\alpha, g_n(\alpha)) : \alpha \in D\}.$$

To show that D is a regular G_δ, we will prove that $D = \bigcap_{n\in\omega} \overline{V_n}$.

By way of contradiction, suppose that there is an $\alpha \in X - D$ with $\alpha \in \bigcap_{n\in\omega} \overline{V_n}$. Since X is regular, we may assume that $\bigcap_{n\in\omega} \overline{U(\beta, g_n(\beta))} = \{\beta\}$ for every $\beta \in \lambda$. (We may need to refine the g_n's, but this may be done without violating the statement $\mathrm{CM}(\omega, \omega, \lambda)$.) Then there is an infinite subset $\{\alpha_n : n \in \omega\} \subset D$ such that

$$(*)\ U(\alpha, g_n(\alpha)) \cap U(\alpha_n, g_n(\alpha_n)) \neq \emptyset \text{ for every } n \in \omega.$$

By $\mathrm{CM}(\omega, \omega, \lambda)$, there is a partition $\delta \in {}^\lambda\omega$ and an $S \in [\omega]^\omega$ such that (1) $|\{\delta(\alpha_n) : n \in S\}| = \omega$ and (2) $f_\delta(\alpha_n) \leq g_n(\alpha_n)$ for every $n \in S$. Since f_δ codes a locally finite open expansion of $\{D_n : n \in \omega\}$ (as defined above), there is a positive integer $n_0 \in \omega$ such that $U(\alpha, n_0)$ meets at most finitely many of the U_n's. Since g_n dominates f_δ at α_n for every $n \in S$ (by property (2) of $\mathrm{CM}(\omega, \omega, \lambda)$) and since $|\{\delta(\alpha_n) : n \in S\}| = \omega$ (by property (1) of $\mathrm{CM}(\omega, \omega, \lambda)$), it follows that $U(\alpha, n_0)$ meets at most finitely many members of the set $\{U(\alpha_n, g_n(\alpha_n)) : n \in S\}$. Let $m_0 \in \omega$ be the maximum $n \in S$ such that $U(\alpha, n_0) \cap U(\alpha_n, g_n(\alpha_n)) \neq \emptyset$. However, since $U(\alpha, g_n(\alpha))$ is a decreasing neighborhood base at α and since S is an infinite subset of ω, there is an $n \in S$ with $n > \max\{m_0, n_0\}$ such that $U(\alpha, g_n(\alpha)) \cap U(\alpha_n, g_n(\alpha_n)) = \emptyset$. But this contradicts $(*)$ above. Thus, D is a regular G_δ subset of X.

To show that D is separated, we note that the above proof shows that *every* closed discrete subspace of X is a regular G_δ-set. In particular, if $A, B \subset D$ with $A \cap B = \emptyset$, then A and B are disjoint regular G_δ subsets of X. By countable paracompactness, A and B can be separated in X. Thus, the closed discrete set D is normalized. Moreover, if $\delta \in {}^\lambda\omega$ is a partition of D into countably many pieces D_n, then there is a disjoint open expansion $\{W_n : n \in \omega\}$ of $\{D_n : n \in \omega\}$.

We now construct a separation of D. We note that the above proof also does not make use of the fact that $\alpha \in X - D$. In fact, for any $\alpha \in D$, if we set $D' = D - \{\alpha\}$ and we redefine V_n using only the points in D', then $\alpha \notin \bigcap_{n\in\omega} \overline{V_n}$. That is, for each $\alpha \in D$, there is some n such that $\alpha \notin \overline{\bigcup\{U(\gamma, g_n(\gamma)) : \gamma \in D, \gamma \neq \alpha\}}$. So, let $n_\alpha \in \omega$ be the first n such that

$$U(\alpha, g_n(\alpha)) \cap (\bigcup\{U(\gamma, g_n(\gamma)) : \gamma \in D, \gamma \neq \alpha\}) = \emptyset.$$

For each $n \in \omega$, let $D_n = \{\alpha \in D : n_\alpha = n\}$. Then the points of D_n are separated by the function g_n. By the previous paragraph, the subsets D_n are separated. Hence, the entire set of points of D are separated.

This completes the proof. □

Theorem 4.1 provides an interesting connection between many well-known results. In particular, in [3], Burke showed that assuming the PMEA, that both CPSS and CMGD were true. By our results, in any model of set theory where CMGD holds, it follows that each of the statements CPSS, CPS, and NS holds as well.

An interesting and much studied connection is whether any one of the reverse implications of Theorem 4.1 holds. More specifically, in many models of set theory where researchers have proven the statement NS is true, later results have shown that CPS is also true in these models. So, the following natural question arises.

Question 4.2. In any model of set theory, does NS → CPS?

It is not known whether countably paracompact first countable T_2 collectionwise Hausdorff spaces are strongly collectionwise Hausdorff in ZFC (a question of Szyptecki). In fact, Nyikos has asked whether this statement is true assuming $V = L$. We ask if this statement is true in any model of set theory.

Question 4.3. In any model of set theory, does CPS → CPSS?

Finally, we note that it is *not* true that CPS → CMGD for in [9], Szeptycki constructed a countably metacompact first countable T_1 space with a closed discrete subspace that is not a G_δ-set assuming $V = L$. Thus, we know that the topological statements from Theorem 4.1 are not all equivalent. This does not leave out the following possibility.

Question 4.4. In any model of set theory, does CPSS → CMGD?

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