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TOPOLOGICAL FUNDAMENTAL GROUPS CAN DISTINGUISH SPACES WITH ISOMORPHIC HOMOTOPY GROUPS

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Abstract. There is a natural functor π_1^{top} which imparts a topology on the based fundamental group of a space X. We exhibit examples in which π_1^{top} succeeds in distinguishing the homotopy of the spaces X and Y when such tools as the Whitehead theorem and shape theory are unsuccessful.

1. Introduction

For CW complexes the familiar Whitehead theorem [6] ensures that a map $f : X \to Y$ is a homotopy equivalence, provided f induces an isomorphism on homotopy groups. However, the conclusion of the Whitehead theorem can fail for more general spaces.

To help overcome such failure, we investigate a functor π_1^{top} [1] with the potential to distinguish the homotopy types of X and Y in the presence of a weak homotopy equivalence $f : X \to Y$. The functor π_1^{top} endows the familiar fundamental group $\pi_1(X,p)$ with the quotient topology inherited from the space of based loops over X.

We should mention that π_1^{top} is one of at least three natural topologies on the fundamental group. For example, for planar sets X, one can pull back a topology onto $\pi_1(X,p)$ via the canonical monomorphism $\phi : \pi_1(X,p) \to \phi : \check{\pi}_1(X,p)$ into the first shape group [5], the target understood as the inverse limit of discrete

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groups. The Hawaiian earring shows the shape group topology can be courser than that of π_1^{top} [4].

There is also a finer topology suggested by [2], in which, roughly speaking, one considers two loops classes $[f]$ and $[g]$ to be close if there exists a small loop α so that $f * \alpha$ is path homotopic to g. However, the Hawaiian earring shows this topology can depend on the choice of basepoint and in particular, this fine topology is not an invariant of homotopy type.

We begin by calling into question whether or not $\pi_1^{top}(X,p)$ is, in fact, a topological group. We then provide an argument (also indicated in [1]) that $\pi_1^{top}(X,p)$ is, indeed, an invariant of the homotopy type of the based underlying space X.

We then exhibit examples in which π_1^{top} succeeds in distinguishing the homotopy type of X and Y when the hypothesis of the Whitehead theorem is satisfied.

Moreover, in our second example, π_1^{top} succeeds in distinguishing X and Y, while shape theory fails in the following sense. We exhibit path connected metric compacta X and Y and a weak homotopy equivalence $f : X \to Y$ showing that the homotopy groups of X and Y are equivalently embedded in the respective homotopy shape groups. However, $\pi_1^{top}(X)$ and $\pi_1^{top}(Y)$ are not homeomorphic and thus, X and Y have distinct homotopy type.

2. Preliminaries

If X is any space and $p \in X$, let $C_p(X) = \{f : [0,1] \to X$ such that f is continuous and $f(0) = f(1) = p\}$. Endow $C_p(X)$ with the compact open topology.

The *topological fundamental group* $\pi_1(X,p)$ is the set of path components of $C_p(X)$ endowed with the quotient topology under the canonical surjection $q : C_p(X) \to \pi_1(X,p)$ satisfying $q(f) = q(g)$ if and only if f and g belong to the same path component of $C_p(X)$. Thus, a set $U \subset \pi_1(X)$ is open if and only if $q^{-1}(U)$ is open in $C_p(Y)$.

Remark 2.1. A query of Tyler Lawson (via personal correspondence March 2006) suggests a gap in the proof of Proposition 3.1 of [1] which asserts that $\pi_1^{top}(X,p)$ is a topological group under the standard group operations. (The gap is created by the general failure of the product of quotient maps to be a quotient map [7, Example 8, p. 141].) Thus, the issue at stake is whether multiplication is jointly continuous over $\pi_1(X) \times \pi_1(X)$.

Problem. Suppose X is any topological space. Is $\pi_1^{top}(X,p)$ a topological group with the standard group operations? Suppose X is the Hawaiian earring (HE), the union of a null sequence of simple closed curves joined at a common point p. Is $\pi_1^{top}(HE,p)$ a topological group?

Remark 2.2. The foregoing problem cannot be settled easily by embedding $\pi_1^{top}(HE)$ in the inverse limit of free groups [4].

Remark 2.3. Corollary 3.4 of [1] asserts that π_1^{top} is a functor from the homotopy category of spaces to the category of topological groups. However, the proof that π_1^{top} is an invariant of homotopy type of X does not require knowledge that multiplication in $\pi_1^{top}(X)$ is jointly continuous and can be carried out as follows.

Summary. Because $\Pi : C_p(X) \to \pi_1(X,p)$ is a quotient map, if Z is any space and $R : C_p(X) \to Z$ is constant on the path components, then there exists a map $r : \pi_1(X,p) \to Z$ such that $r(\Pi) = R$. In particular, if $\alpha : [0,1] \to X$ is any path, and if $g : X \to Y$ is any map, then the induced isomorphism $\hat{\alpha} : \pi_1(X,\alpha(0)) \to \pi_1(X,\alpha(1))$ is a homeomorphism, and the induced homomorphism $g^* : \pi_1(X,p) \to \pi_1(Y,g(p))$ is continuous. (Recall the isomorphism $\hat{\alpha}$ is determined by conjugation with the (fixed) path α.) Consequently, if $f : X \to Y$ is a homotopy equivalence, then the standard proof that $f^* : \pi_1(X,p) \to \pi_1(Y,f(p))$ is an isomorphism [7, Theorem 11.3]) shows f^* is a homeomorphism. In particular, the topology of $\pi_1^{top}(X,p)$ is invariant under the homotopy type of the (based) underlying space X.

3. Definitions

If A_n is a sequence of planar compacta, then $A_n \to A$ in the *Hausdorff metric* if for each $\varepsilon > 0$ there exists N so that if $n \geq N$, then for each $a \in A$ there exists $a_n \in A_n$ such that $|a - a_n| < \varepsilon$ and for each $a_n \in A_n$ there exists $a \in A$ such that $|a - a_n| < \varepsilon$.

For path connected spaces, if $\pi_n(X)$ denotes the *nth* homotopy group, the map $f : X \to Y$ is a *weak homotopy equivalence* if $f_n^* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for $n \geq 1$.

A *Peano continuum* is a path connected locally path connected metric space. The space X is *discrete* if each subset of X is open.

Given a sequence of maps $f_n : X_n \to X_{n-1}$, the *inverse limit space* $\lim_{\leftarrow} X_n = \{(x_1, x_2, ..)|f_n(x_n) = x_{n-1}\}$, and inherits the product topology from $X_1 \times X_2 ....$

We discuss a minimum of shape theory in order to formulate a sense in which shape theory can fail to distinguish the spaces X and Y.

Suppose $X \subset R^2$ and X is closed and path connected and $p \in X$. Suppose $X = \cap_{n=1}^{\infty} U_n$ where U_n is open and $U_{n+1} \subset U_n$. Then inclusion $j_k : X \hookrightarrow U_k$ determines a map $j : X \to \lim_{\leftarrow} U_k$ and a canonical homomorphism from the *nth* homotopy groups $\phi_n : \pi_n(X,p) \to \lim_{\leftarrow} \pi_n(U_k,p)$. The group $\lim_{\leftarrow} \pi_n(U_k,p)$ is the *nth shape homotopy group* of X and is denoted $\breve{\pi_n}(X,p)$. Moreover, the group $\breve{\pi_n}(X,p)$ admits a natural topology as the inverse limit of discrete spaces $\pi_1(U_n,p)$.

A homotopy equivalence $f : X \to Y$ suggests a commutative diagram such that Γ_n is both an isomorphism and a homeomorphism.

$$\begin{array}{ccc} \pi_n(X) & \overset{\phi_X}{\to} & \breve{\pi_n}(X) \\ f^* \downarrow & & \Gamma_n \downarrow \\ \pi_n(Y) & \overset{\phi_Y}{\to} & \breve{\pi_n}(Y). \end{array}$$

However, Theorem 5.3 shows that a weak homotopy equivalence $f : X \to Y$ can determine a comparable diagram and this motivates the following definition.

Definition 3.1. Suppose D^2 and D^1 are path connected planar continua and suppose for each $n \geq 0$ there exists $\Gamma_n : \breve{\pi_n}(D^2) \to \breve{\pi_n}(D^1)$ such that Γ_n is both an isomorphism and a homeomorphism. Suppose $\phi_n^i : \pi_n(D^i) \to \breve{\pi_n}(D^i)$ denotes the canonical homomorphism and suppose there exists a weak homotopy equivalence $f : D^2 \to D^1$ such that $\phi_n^2 f_n^* = \Gamma_n \phi_n^1$ for each n. Then the *shape homotopy groups do not distinguish* D^2 and D^1.

Remark 3.2. In general, shape theory has the potential to distinguish a weak homotopy equivalence $f : X \to Y$ from a genuine homotopy equivalence. (Consider, for example, inclusion of a one point set into the "topologist's sine curve" ∂U, as in Theorem 5.3.)

4. Example 1: π_1^{top} versus the Whitehead theorem

We exhibit a weak homotopy equivalence $j : X \to Y$ such that j^* is not a homeomorphism of π_1^{top}. Let $p = (0,0)$. For $n \in \{1,2,3,...\}$, let the simple closed curve $C_n \subset R^2$ denote the boundary of the triangle with vertices $\{(0,0),(\frac{1}{n},1),(\frac{1}{n}-\frac{1}{3n^3},1-\frac{1}{2n})\}$. Let $X = \cup_{n=1}^{\infty} C_n$. Note $\overline{X}$ is compact and $\{p\} \cup \overline{X} \backslash X$ is the line segment $L = [(0,0),(0,1)]$. Let $Y = \overline{X}$ and let $j : X \to Y$ denote the inclusion map.

Remark 4.1. If $n \neq 1$, then $\pi_n(X) = \pi_n(Y) = 0$ [3].

Lemma 4.2. *The homomorphism induced by the inclusion of* $j^* : \pi_1^{top}(X,p) \to \pi_1^{top}(Y,p)$ *is an isomorphism but not a homeomorphism. Thus,* j *is not a homotopy equivalence. Moreover,* $\pi_1^{top}(X)$ *and* $\pi_1^{top}(Y)$ *are not homeomorphic and thus,* X *and* Y *do not have the same homotopy type.*

Proof: Let $E^2 \subset R^2$ denote the closed unit disk. To prove j^* is one to one, suppose $f : \partial E^2 \to X$ is inessential in Y and suppose $f(0) = f(1) = p$. Let $F : E^2 \to Y$ satisfy $F_{\partial E^2} = f$. Consider the open set $U = F^{-1}(Y \backslash p) \subset E^2$. If U_i is a component of U such that $F(U_i) \subset L \backslash p$, then $F(\partial U_i) = p$. Thus, we may redefine F over each U_i to be the constant p. This new function is continuous and shows f is inessential in X and hence, j^* is one to one.

To prove j^* is onto, suppose f is any loop in Y based at p. Since $im(f)$ is a Peano continuum, $im(f) \cap X$ is a strong deformation retract of $im(f) \cup L$ via linear contraction along L. Applying the deformation retraction to $im(f)$ shows f is path homotopic to a loop in X. Thus, j^* is an isomorphism.

To see that $\pi_1^{top}(X,p)$ is discrete, note X is locally contractible. Consequently, if $f_n \to f$ in $C_p(X)$, then f_n will eventually be path homotopic to f, and in particular, $\pi_1^{top}(X,p)$ is discrete.

On the other hand, there exist essential loops $f_n : [0,1] \to C_n$ converging to an (inessential) loop $f : [0,1] \to L$. Since $\Pi : C_p(Y) \to \pi_1^{top}(Y,p)$ is continuous, it follows that $[f_n] \to [f]$ and hence, $\pi_1^{top}(Y,p)$ is not discrete. Thus, j is not a homotopy equivalence, and X and Y do not have the same homotopy type. □

5. Example 2: π_1^{top} versus shape theory

We exhibit spaces D^2 and D^1 and a weak homotopy equivalence $f : D^2 \to D^1$ such that the shape groups do not distinguish D^2 and D^1 in the sense of Definition 3.1 but such that π_1^{top} succeeds in distinguishing the homotopy type of D^1 and D^2.

The space D^i is the union of a contractible continuum and countably many large loops joined at a common point p. In particular, D^1 is the space Y from Lemma 4.2 and D^2, roughly speaking, is obtained as follows. Starting with $\overline{U}$, the closure of the bounded complementary domain of a standard topologist's sine curve, attach loops C_n^* "converging" to a nonseparating "$\sin(\frac{1}{x})$ subcontinuum" $A \subset \partial U$.

To be more precise, let $A = \{[-2,0] \times \{0\}\} \cup \{(x,y) \in R^2 | x = \sin(\frac{1}{y}) - 1 \text{ and } 0 \leq y \leq 1\}$. Let $p = (0,0)$. Next, construct a sequence of simple closed curves C_n^* converging to A in the Hausdorff metric and satisfying the following:

(1) $C_n^* \cap A = \{(0,0)\}$ and $C_n^* \cap C_m^* = \{(0,0)\}$ if $n \neq m$.
(2) If $0 < y < 1$, then $((\infty,\infty) \times \{y\}) \cap C_n^* = \{(x_1,y),(x_2,y)\}$ with $\sin(\frac{1}{y}) - 1 < x_1 < x_2$ and $0 < x_2 - x_1 < \frac{1}{n}$.
(3) $((-\infty,\infty) \times \{1\}) \cap C_n^* = \{(\frac{1}{n},1)\}$
(4) If $y \notin [0,1]$, then $((-\infty,\infty) \times \{y\}) \cap C_n^* = \emptyset$.
(5) Each curve C_n^* is the union of finitely many line segments, none of which are nontrivial horizontal segments.
(6) The curve $C_{n+1}^* < C_n^*$ in the sense that for $0 < y \leq 1$, $C_{n+1}^* \cap \{(-\infty,\infty) \times \{y\}\} < C_n^* \cap \{(-\infty,\infty) \times \{y\}\}$.

Define the continuum $B = A \cup C_1^* \cup C_2^* \ldots$.

To create D^2, "thicken" A as follows. Consider the union of three line segments $\beta = [(-2,0),(-10,0)] \cup [(-10,0),(-10,1))] \cup [(-10,1),(\sin(1)-1,1)]$. Let U denote the bounded component of $R^2 \backslash (\beta \cup B)$. Let $D^2 = B \cup \overline{U}$.

Define a map $f : B \to D^1$ such that $f(A) = L$ and such that f maps C_n^* homeomorphically onto C_n as follows. For $(x,y) \in A$, let $f(x,y) = (0,y)$. Let $f(\frac{1}{n},1) = (\frac{1}{n},1)$. If $0 < y < 1$, map

$C_n^* \cap (R \times y)$ onto $C_n \cap (R \times y)$ such that f preserves the order of the x coordinates.

Extend $f : D^2 \to D^1$ such that $f(x, y) = (0, y)$ for $(0, y) \in \overline{U}$. (Such extension is continuous by the pasting Lemma [7, p. 108].)

Let $X^* = \{0, 0\} \cup C_1^* \cup C_2^* \ldots$

The following technical facts are used in the proof of Theorem 5.3.

Lemma 5.1. *Let $p = \{(0,0)\}$. Suppose $\beta : [a,b] \to \overline{U}$ is a path such that $\beta(a) \in p$. Suppose the sequence of paths $\beta_n : [a,b] \to D^2$ satisfies $\beta_n \to \beta$ uniformly and $\varepsilon > 0$. Let $E = X^* \cap \{R \times [\varepsilon, 1]\}$. Then there exists N such that if $n \geq N$ then $im(\beta_n) \cap E = \emptyset$.*

Proof: First observe that if C is a subcontinuum of $\overline{U}$ and if γ denotes the component of $C \cap A$ containing p, and if γ is nonempty and nontrivial, then C is not locally path connected. Next note $im(\beta_n) \to im(\beta)$ in the Hausdorff metric. To obtain a contradiction, suppose there exists $(x_{n_k}, y_{n_k}) \in im(\beta_{n_k})$ for a subsequence $\beta_{n_1}, \beta_{n_2}, \ldots$. Then $im(\beta_{n_k})$ contains an arc γ_{n_k} connecting (x_{n_k}, y_{n_k}) to $\beta_{n_k}(0)$. Recall $C_n^* \to A$ in the Hausdorff metric. Thus, if γ is a subsequential limit of γ_{n_k}, then $\gamma \subset A$ and γ is nonempty and nontrivial and $p \in \gamma$. On the other hand, $\gamma \subset im(\beta)$ and $im(\beta)$ is a locally path connected continuum and we have a contradiction. □

Remark 5.2. Suppose $1 > \varepsilon > 0$ and $U = X^* \cap \{R \times (1 - \varepsilon, \infty)\}$. Then $X^* \backslash U$ is contractible, and if $\alpha \in C_p(X^*)$ and $V = \alpha^{-1}(U)$, then the homotopy path class of α is determined by α_V, the restriction of α to V.

Theorem 5.3. *The map $f : D^2 \to D^1$ is a weak homotopy equivalence and shows that the shape homotopy groups do not distinguish D^2 and D^1 in the sense of Definition 3.1.*

Proof: Let $E^2 \subset R^2$ denote the closed unit disk. To see that f is a weak homotopy equivalence, note D^i is path connected and thus, $\pi_n(D^i) = 0$ if $n \neq 1$ [3].

To prove $f^* : \pi_1(D^2, p) \to \pi_1(D^1, p)$ is one to one, suppose $\alpha : \partial E^2 \to D^2$ is a based loop such that $f(\alpha)$ is inessential in D^1. Recall the space X from Lemma 4.2 and note f maps X^* homeomorphically onto X. Note $im(\alpha)$ is a Peano continuum. Let U denote the union of the bounded components of $R^2 \backslash im(\alpha)$. Let

$E = X \cup im(\alpha) \cup U$. Notice $E \subset D^2$, and X^* is a strong deformation retract of E. Thus, α is path homotopic to a loop β in X^* such that $f(\beta)$ is inessential in D^1. Recalling the proof of Lemma 4.2, we obtain a map $F : E^2 \to D^1$ extending $f(\beta)$ such that $im(F) \subset X$. The map $f^{-1}(F)$ extends β and shows α is inessential in D^2.

In similar fashion, the proof of Lemma 4.2 also shows that f^* is onto as follows. Given a loop α in D^1, we homotop α in D^1 to a loop $\alpha\hat{}$ such that $im(\alpha\hat{}) \subset X$. Now let $\beta\hat{} = f_{X^*}^{-1}(\alpha\hat{})$. Thus, $f(\beta) = \alpha\hat{}$.

Next we "thicken" $\cup_{i=1}^{n} C_i^*$ and $\cup_{i=1}^{n} C_i$ with planar open sets U_n and V_n, respectively, so that $D^2 = \cap_{n=1}^{\infty} U_n$ and $D^1 = \cap_{n=1}^{\infty} V_n$ and so that Definition 3.1 is satisfied as follows. Let F_n denote the free group on letters $\{w_1, ..w_n\}$ and let $\gamma_n : F_n \to F_{n-1}$ denote the canonical epimorphism deleting all occurrences of the letter w_n. If, to the letter w_i, we associate counterclockwise loops respectively in C_i^* and C_i, then we can choose U_n and V_n such that the homomorphisms induced by inclusion $i_n^* : \pi_1(U_n) \to \pi_1(U_{n-1})$ and $j_n^* : \pi_1(V_n) \to \pi_1(V_{n-1})$ are canonically equivalent to γ_n. □

Theorem 5.4. *The spaces $\pi_1^{top}(D^2, p)$ and $\pi_1^{top}(D^1, p)$ are not homeomorphic. Hence, $f : D^2 \to D^1$ is not a homotopy equivalence, and D^2 and D^1 are not homotopy equivalent.*

Proof: To see that $\pi_1^{top}(D^2, p)$ and $\pi_1^{top}(D^1, p)$ are not homeomorphic, recall in D^1 there exist essential loops $\beta_n : [0,1] \to C_n$ such that $\beta_n \to \beta$ with β inessential. Thus, since $\Pi : C_p(D^2) \to \pi_1^{top}(D^2, p)$ is continuous, the sequence $[\beta_n]$ converges to $[\beta]$ in $\pi_1^{top}(D^1, p)$ and in particular, $\pi_1^{top}(D^1, p)$ is not a discrete space.

Now we show $\pi_1^{top}(D^2, p)$ is discrete by proving that $C_p(D^2)$ has open path components.

Since f is a weak homotopy equivalence, it follows from Lemma 4.2 that inclusion $\kappa : X^* \to D^2$ is also a weak homotopy equivalence. In particular, if $\alpha \in C_p(D^2)$, then the path class of α is determined by α_C where $C = \alpha^{-1}(X^*)$.

Now suppose $\alpha \in C_p(D^2)$ and $\alpha_n \to \alpha$ in $C_p(D^2)$. First note there exist finitely many components $[a_i, b_i]$ of $\alpha^{-1}(\overline{U})$ such that $\alpha([a_i, b_i]) \cap \{R \times (\frac{1}{2}, \infty) \neq \emptyset$. Moreover, since κ is a weak homotopy equivalence for sufficiently large n, Lemma 5.1 shows that $\alpha_{n[a_i,b_i]}$ makes no essential contribution to the path class of α_n.

Now let $V = \alpha^{-1}(X^* \cap (\frac{1}{2} \times \infty))$. Let $V_n = \alpha_n^{-1}(X^* \cap (R \times (\frac{2}{3} \times \infty))$. For sufficiently large n, we have $V_n \subset S(\alpha)$. It follows from Remark 5.2 that the path class of α_n is determined by the restriction to $S(\alpha)$. Moreover, since $\alpha_n \to \alpha$ uniformly over $\overline{S(\alpha)}$, α_n and α eventually determine the same word in the free group on infinitely many generators $F(w_1, w_2, ...)$. Hence, α and α_n are eventually path homotopic. □

Remark 5.5. It follows from Theorem 5.3 and Lemma 4.2 that inclusion $\kappa : X^* \hookrightarrow D^2$ is a weak homotopy equivalence inducing a homeomorphism $\kappa^* : \pi_1^{top}(X^*, p) \to \pi_1^{top}(D^2, p)$. However, shape theory shows κ is **not** a homotopy equivalence. Since X^* has the homotopy type of an open planar set, $\breve{\pi}_1(X^*, p)$ is the (countable) direct limit of free groups. On the other hand, $\breve{\pi}_1(D^2, p)$ is the (uncountable) inverse limit of free groups.

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