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ENDS OF 4-MANIFOLDS

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ABSTRACT. We define and study a measurement of complexity for the ends of 4-manifolds. Bounds on this complexity are computed for an important family of exotic $\mathbb{R}^4$'s, and we show how a particular exotic $\mathbb{R}^4$ of high complexity can be used to corrupt the ends of 4-manifolds, resulting in a proof that many open 4-manifolds have infinitely many smooth structures.

1. INTRODUCTION

Exotic $\mathbb{R}^4$'s, i.e., manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^4$, fall into two categories, small and large, depending on whether or not they embed into S^4. The large exotic $\mathbb{R}^4$'s discussed in sections 2 and 3 of this paper arise as consequences of two theorems on the nonexistence of certain smooth 4-manifolds: For M^4 smooth, closed, and oriented, Donaldson's theorem [4] states if the intersection form q_M is negative definite, then $q_M \cong \oplus n(-1)$. Furuta's theorem [8] states that if $q_M = 2kE_8 \oplus lH$, and $k \neq 0$, then $l > 2|k|$. Here, E_8 represents the usual symmetric bilinear form on $\mathbb{Z}^8$ (see e.g., [15]), and H refers to the hyperbolic form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Furuta's theorem is often called the $\frac{10}{8}$-theorem since it has an equivalent formulation as $\frac{b_2}{|\sigma|} > \frac{10}{8}$, where σ is the signature of M and b_2 is its second Betti number. The $\frac{11}{8}$-conjecture, $\frac{b_2}{|\sigma|} \geq \frac{11}{8}$, remains open.

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Small exotic $\mathbb{R}^4$'s such as those discussed in section 4 arise as a consequence of the fact that the h-cobordism theorem of high dimensional topology holds in dimension 4 topologically, but fails smoothly. Explicit counterexamples in the smooth case lead to relatively simple descriptions of some exotic $\mathbb{R}^4$'s.

In this paper, we examine these exotic structures and quantify their complexity. Bounds for large examples are computed in section 3 and for small examples in section 4. We also show that by end summing (see [11]) with an exotic $\mathbb{R}^4$ of high complexity, infinitely many distinct smooth structures can be constructed on any 4-manifold that has at least one end homeomorphic to $\Sigma^3 \times \mathbb{R}$, where Σ^3 is any closed 3-manifold.

2. Exotic constructions

Two well-known exotic $\mathbb{R}^4$'s appear throughout this paper, and we review their constructions here. For more complete details, see [10], [11], [13], or [14]. Let $M^4 = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, and consider the intersection form $q_M = (1) \oplus 8(-1) \oplus (-1)$ with basis $\{e_0, e_1, \dots, e_9\}$, e_0 representing the (1). The element $\alpha = 3e_0 - e_1 - \cdots - e_8$ is characteristic and $\alpha^2 = 1$. One checks that in the subspace $(1) \oplus 8(-1)$, $(\alpha)^\perp \cong -E_8$. Thus in q_M, $(\alpha)^\perp \cong -E_8 \oplus (-1)$. But by Donaldson's theorem, $-E_8 \oplus (-1)$ cannot represent any closed smooth 4-manifold, so $\alpha \in H_2(\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}; \mathbb{Z})$ cannot be represented by a smoothly embedded 2-sphere. (Otherwise, a tubular neighborhood ν of that 2-sphere would be a 2-disc bundle over S^2 with Euler class $\alpha^2 = 1$, and so diffeomorphic to $\mathbb{C}P^2$ with the 4-handle removed. The boundary of $M - \nu$ would be S^3, and so capping off with a 4-ball would violate the theorem.) We can, however, represent α by a Casson handle, CH, attached to B^4 along an unknot with framing 1 [3]. Let $U = \operatorname{int}\big(B^4 \cup CH\big)$. By [6], U is homeomorphic to $\mathbb{C}P^2$ with the 4-handle removed. The (topological) core 2-sphere S of U in M carries the homology class α, and since any Casson handle can be embedded into the standard 2-handle, we have $S \subset U \subset \mathbb{C}P^2$. (See Figure 1.)

$R = \mathbb{C}P^2 - S$ is contractible and simply connected at infinity, so by [6] is homeomorphic to $\mathbb{R}^4$. But R is an open subset of $\mathbb{C}P^2$ and so inherits a smooth structure that we show to be exotic. Suppose R is diffeomorphic to $\mathbb{R}^4$. Then if C is any smooth,

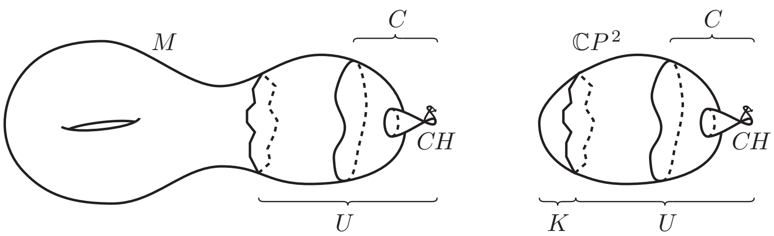

FIGURE 1. An exotic construction

compact submanifold of U satisfying $S \subset \text{int}C$, there is a smoothly embedded 3-sphere in $\mathbb{C}P^2 - \text{int}C$ that separates the compact set $K = \mathbb{C}P^2 - U$ from C. This 3-sphere also embeds smoothly in M with α (represented by S) on one side and $-E_8 \oplus (-1)$ on the other. We now cut M^4 along this 3-sphere and glue in a 4-ball. The result is a smooth, simply connected, closed 4-manifold with intersection form $-E_8 \oplus (-1)$, contradicting Donaldson's theorem.

Another (presumably different) exotic $\mathbb{R}^4$ can be constructed similarly as a subset of $3(S^2 \times S^2)$. Let M^4 be the $K3$-surface (see e.g., [13]), taken with reversed orientation so that $q_M = 2E_8 \oplus 3H$. In this case, the six elements of $H_2(M;\mathbb{Z})$ that span the $3H$ can be represented by six Casson handles, CH_i, $1 \leq i \leq 6$. Set $U = \text{int}(B^4 \cup_{i=1}^{6} CH_i)$ and we find U is homeomorphic to $\#3(S^2 \times S^2)$ with the 4-handle removed. So if S is taken to be the union of cores of the Casson handles, then $R_1 = \#3(S^2 \times S^2) - S$ is readily seen to be homeomorphic to $\mathbb{R}^4$. Suppose as before that R_1 is standard. Then we could find a smoothly embedded 3-sphere in $\#3(S^2 \times S^2) - \text{int}C$ that separates the compact set $K = \#3(S^2 \times S^2) - U$ from C. (C is, again, any smooth, compact submanifold of U satisfying $S \subset \text{int}C$.) Find this 3-sphere in M and cut it out. Glue in a 4-ball to produce a smooth (spin), simply connected 4-manifold with intersection form $2E_8$, contradicting Furuta's theorem.

Lemma 2.1 ([13])**.** *R_1 contains a compact subset K_1 that cannot be embedded into $\left(S^2 \times S^2\right) \# \left(S^2 \times S^2\right)$.*

Proof (cf. [13, 9.4.3]): Let $K = \#3(S^2 \times S^2) - U$, as described above. Fix a homeomorphism $h : \mathbb{R}^4 \to R_1$ and let $B = h\left(B^4_r\right)$ be the image of a ball of sufficiently large radius so that $K \subset \text{int}B$.

$R_1 - B$ is then homeomorphic to $S^3 \times \mathbb{R}$. Perturb the projection map $S^3 \times \mathbb{R} \to \mathbb{R}$ to obtain a smooth, proper map f, and define $K_1 = B \cup f^{-1}(-\infty, a]$ for any regular value a. Let A be a neighborhood of the end of intK_1, and suppose K_1 were embedded into $\left(S^2 \times S^2\right) \# \left(S^2 \times S^2\right)$. From the image of this embedding remove the complement of A in K_1 and denote the resulting open manifold by X_0. The end of X_0 has a neighborhood diffeomorphic to A. But $U \subset K3$ also contains an embedding of A. So we delete from $K3$ the portion of $U - A$ that contains the Casson handles, and attach X_0 by identifying the (diffeomorphic) ends. The result is a manifold whose intersection form is $2E_8 \oplus 2H$, which is impossible by Furuta's theorem. $\square$

Remark 2.1. The only information about $\left(S^2 \times S^2\right) \# \left(S^2 \times S^2\right)$ used in the proof was its intersection form. Thus K_1 cannot be embedded into any smooth 4-manifold whose intersection form is $2H$.

By end summing copies of R_1 together, we can create a sequence of exotic $\mathbb{R}^4$'s. Let $R_n = \natural n R_1$, i.e., the end sum of n copies of R_1. Each copy of R_1 inside R_n contains a copy of K_1, and by connecting these K_1's with neighborhoods of arcs we can form a boundary connected sum $K_n = \natural_\partial K_1$. Define $R_\infty = \natural \infty R_1$. Žarko Bižaca and John Etnyre prove that for $1 \leq n < m \leq \infty$, R_n and R_m are not diffeomorphic [1].

We now remove a technical hypothesis from a theorem in [1].

Theorem 2.2. *Let X be an open topological 4-manifold with at least one topologically collarable end, i.e., an end homeomorphic to $\Sigma^3 \times \mathbb{R}$ for some closed 3-manifold Σ^3. Then X has infinitely many distinct smooth structures.*

Proof: See Figure 2(a). Every open 4-manifold is smoothable [7], so we may assume X has a smooth structure. Let us first suppose X has exactly one end. In this case, we follow [1]. End summing with $\mathbb{R}^4$ (exotic or not) does not change the homeomorphism type of an open 4-manifold, so $X \natural R_\infty$ is homeomorphic to X. Let U be a neighborhood of the end of $X \natural R_\infty$ and let $h : \Sigma^3 \times \mathbb{R} \to U$ be a homeomorphism that embeds the nth copy of K_1 into $h(\Sigma^3 \times (n, n+1))$.

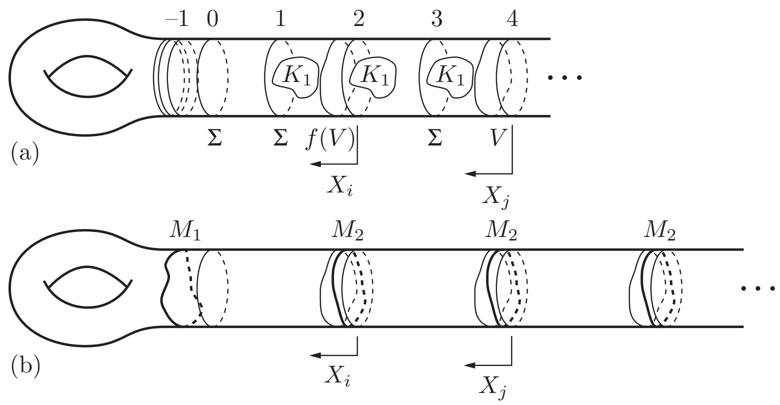

FIGURE 2. Corrupted end and periodic end

Define $X_t = (X \natural R_\infty) - h(\Sigma^3 \times [t, \infty))$, and observe that for any $t \in \mathbb{R}$, X_t is homeomorphic to X. We show that for any positive integers i, j, $1 \leq i < j$, X_i is not diffeomorphic to X_j. Thus, the collection $\{X_i\}, i \in \mathbb{Z}^+$ provides infinitely many distinct smooth structures for X. Suppose to the contrary that $f : X_j \to X_i$ is a diffeomorphism for some integers $1 \leq i < j$. Let V be a neighborhood of the end of X_j that does not intersect the copy of K_1 contained in $\Sigma^3 \times (i, i+1)$. Then $f(V) \subset X_i$ is diffeomorphic to V. Let $W = f(V) \cup (\Sigma^3 \times [i, j))$ and construct a periodic end by gluing copies of W to the end of X_j, using f to perform the gluing. See Figure 2(b).

Choose smoothly embedded, separating 3-manifolds $M_1 \hookrightarrow \Sigma^3 \times (-\infty, 0)$ and $M_2 \hookrightarrow f(V) \subset X_i$, and let Z_0 be the manifold bounded by M_1 and M_2 in X_i. Since all 3-manifolds spin bound, we may cap off Z_0 with smooth, simply connected, spin 4-manifolds to form a closed, smooth spin 4-manifold Z. Rohlin's theorem guarantees that the signature of Z is 0 mod 16 so by taking a connected sum with $K3$ surfaces of the correct orientation, we may assume Z has signature 0. Since Z is spin, its intersection form is even and hence (since $\sigma(Z) = 0$), a sum of hyperbolics (see [14]).

Say $q_Z = kH$. Then K_n cannot embed smoothly into Z when $n \geq \frac{k}{2}$ (or into any 4-manifold with the same intersection form as Z, cf. Remark 2.1). So choose $n \geq \frac{k}{2}$, and observe that K_n does embed smoothly into the periodic end constructed above. Find a copy of $M_2 \subset V$ past the embedded K_n, and let Z_1 be the manifold bounded by M_1 and this M_2. Cap off Z_1 with the same smooth, simply connected, spin 4-manifolds as above to produce Z', a manifold homotopy equivalent to Z (and hence with the same intersection form), but into which K_n embeds smoothly. This contradiction establishes that X_i is not diffeomorphic to X_j.

If X has more than one, but still finitely many ends, we use the above construction to obtain a collection $\{X_i\}, i \in \mathbb{Z}^+$ of (not necessarily distinct) smooth structures for X. In this case, X_i and X_j $(i \neq j)$ can be diffeomorphic if the diffeomorphism permutes the homeomorphic ends. But there are only finitely many such permutations, and so our infinite collection still provides infinitely many distinct smooth structures.

In the case where X has infinitely many ends, we proceed as follows: X is open and hence smoothable. Group together as one any ends not homeomorphic to $\Sigma^3 \times \mathbb{R}$. (Note that there must be at least one additional end and this "last" one is horribly noncollarable.) Now we may choose our smooth structure to respect the topological product structure on the collarable ends, so that each of our topological product ends becomes a smooth product end. To each end (of perhaps infinitely many) diffeomorphic to $\Sigma^3 \times \mathbb{R}$, end sum a copy of R_∞ and call the resulting manifold X^*. Let $\{E_s\}$ be the collection of ends of X^* diffeomorphic to $\left(\Sigma^3 \times \mathbb{R}\right) \natural R_\infty$, let U_s be a neighborhood of E_s, and define U^* (resp., E^*) to be the disjoint union of the U_s's (resp., E_s's). Fix a homeomorphism $h^* : E^* \to U^*$ so that the nth copy of K_1 in each E_s embeds into the corresponding component of $h^*(\Sigma^3 \times (n, n+1))$. Since these ends are all diffeomorphic, it is of no concern that h^* may permute them, even if there are infinitely many. Define $X_t^* = X^* - h^*\left(\coprod(\Sigma^3 \times [t, \infty))\right)$. We now claim, as in the one end case, that for integers $1 \leq i < j$, X_i^* is not diffeomorphic to X_j^*.

Suppose as before that $f^* : X_j^* \to X_i^*$ is a diffeomorphism for positive integers $i < j$. Choose diffeomorphic neighborhoods V_s of the ends of E_s that do not contain the copy of K_1 contained in $\Sigma^3 \times (i, i+1) \cap E_s$. Define $V^* = \coprod V_s$. Then $f^*(V^*)$ is diffeomorphic to V^*. (The ends may be permuted but they are all diffeomorphic, so again this is of no consequence.) We let $W^* = f^*(V^*)$ and construct a manifold homeomorphic to X but with periodic ends formed by attaching copies of W^* to the end of X_j^*, using f^* as the attaching map. Now we can find, in each of the periodic ends, smoothly embedded separating 3-manifolds $M_s \hookrightarrow \Sigma^3 \times (-\infty, 0) \subset E_s$ and $N_s \hookrightarrow f(V_t) \subset X_i^* \cap E_s$. For any s, let Z_0 be the manifold in X_i^* bounded by M_s and N_s. (All are diffeomorphic so the choice of s is immaterial.) Now the proof follows exactly as in the one end case. We cap off Z_0 with smooth, simply connected, spin 4-manifolds to obtain a closed, smooth, spin 4-manifold Z with signature 0, into which some K_n cannot embed. The construction of Z' homotopy equivalent to Z with $K_n \hookrightarrow Z'$ establishes the contradiction as before. $\square$

3. Embedded 3-manifolds

In each of the above constructions, the key property of the exotic $\mathbb{R}^4$ is a compact set K that no smoothly embedded 3-sphere separates from infinity. It is natural to ask what other smoothly embedded 3-manifolds cannot separate K from infinity. The question is relevant, for example, to the study of topological quantum field theories, in which one is concerned about the smooth splittings of 4-manifolds along 3-manifolds.

Definition 3.1. Let R be homeomorphic to $\mathbb{R}^4$. Define $e(R)$ to be the supremum, taken over all compact codimension zero submanifolds K of R, of the minimal first Betti number of any smooth 3-manifold Σ^3 that separates K from infinity, i.e.,

$$e\,(R) = \sup_K \left(\min_\Sigma \left(b_1\,(\Sigma) \right) \right) \in \{0, 1, 2, \dots, \infty\}.$$

For example, if $R = \mathbb{R}^4$ then $e\,(R) = 0$, since for any compact K there is a smoothly embedded 3-sphere that separates K from infinity. We may think of $e(R)$ as a measurement of complexity of the exotic $\mathbb{R}^4$, and there is a natural extension to the complexity of any open 4-manifold, i.e., consider the supremum of complexities of the ends. We note that e is subadditive under end sum, i.e., for R_1, R_2 homeomorphic to $\mathbb{R}^4$, $e(R_1 \natural R_2) \leq e(R_1) + e(R_2)$. Definition 3.1 is equivalent to the engulfing index defined by Bižaca and Robert E. Gompf. (See section 4 of this paper and also [2].)

We begin our analysis by showing $e\,(R)$ is nontrivial. Let R be the exotic $\mathbb{R}^4$ created in section 2 as a subset of $\mathbb{C}P^2$. We have already seen that the compact set K cannot be separated from infinity by S^3. We now show that K cannot be separated from infinity by any rational homology sphere. Let $\Sigma^3 \hookrightarrow R \subset \mathbb{C}P^2$ be a smoothly embedded 3-manifold that separates K from infinity, and suppose Σ^3 has the same rational homology as S^3. By construction, K can be taken to contain a topological 3-sphere that is also a subset of $U = \mathrm{int}(B^4 \cup S)$. (Recall that S is the topological core of the Casson handle that carries the generator for homology in $\mathbb{C}P^2$.) Find this 3-sphere in $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, as per the original construction of R. Σ^3 is then found smoothly embedded into $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, and we define $M^4 \subset \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ to be the 4-manifold bounded by Σ^3 that does not contain S.

Let N^4 be the 4-manifold contained in R that is bounded by Σ^3 and define $X^4 = M \cup_\Sigma N$ so that X^4 is closed and smooth. We examine the Mayer-Vietoris sequence of M and N (coefficients to be taken in $\mathbb{Q}$).

$$\cdots \to H_2(\Sigma) \xrightarrow{\varphi_*} H_2(M) \oplus H_2(N) \xrightarrow{\psi_*} H_2(X) \xrightarrow{\eta_*} H_1(\Sigma) \to \cdots$$

$H_2(\Sigma) \cong H_1(\Sigma) \cong 0$, so ψ_* is an isomorphism. The image of $H_2(\Sigma)$ in $H_2(M)$ is zero, so the intersection form $q_M \cong -E_8 \oplus (-1)$. Now since $\Sigma \subset R$ and $H_2(N) = \mathrm{im}(H_2(\Sigma)) = 0$, N has the homology type of the 4-ball. So X has the intersection form $-E_8 \oplus (-1)$, which is impossible by Donaldson's theorem. Thus, $e(R) > 0$.

Similarly, we may show that $e(R_1) > 0$, where R_1 is the exotic $\mathbb{R}^4$ created in section 2 as a subset of $\#3(S^2 \times S^2)$, but since, in this case, the manifolds carry spin structures, we will be able to improve our bound for $e(R)$. Suppose $\Sigma \hookrightarrow R_1 \subset \#3(S^2 \times S^2)$ is a smoothly embedded rational homology 3-sphere that separates K_1 from infinity. K_1 contains a topological 3-sphere which is also a subset of $U = \mathrm{int}(B^4 \cup S)$. (Here, S is the union of cores of the Casson handles.) Find this 3-sphere in the $K3$ surface so that Σ can be found smoothly embedded into $K3$, and define $M \subset K3$ to be the manifold bounded by Σ that does not contain S. Let N be the manifold contained in R_1 that is bounded by Σ.

Define $X^4 = M \cup_\Sigma N$, so that X^4 is closed, smooth and spin. We again examine the Mayer-Vietoris sequence of M and N.

$$\cdots \to H_2(\Sigma) \xrightarrow{\varphi_*} H_2(M) \oplus H_2(N) \xrightarrow{\psi_*} H_2(X) \xrightarrow{\eta_*} H_1(\Sigma) \to \cdots$$

$H_2(\Sigma) \cong H_1(\Sigma) \cong 0$, so ψ_* is an isomorphism. The image of $H_2(\Sigma)$ is again zero, so the intersection form $q_M \cong 2E_8$. Thus, if q_N contains fewer than three hyperbolics, X would violate Furuta's theorem. In fact, since $\Sigma \subset R_1$ and $H_2(N) = \mathrm{im}(H_2(\Sigma)) = 0$, N has the homotopy type of the 4-ball. So X has intersection form $2E_8$. The contradiction establishes that $e(R_1) > 0$.

With a little more care, we can establish a better lower bound for $e(R_1)$. Instead of assuming Σ is a homology sphere, we only suppose $b_1(\Sigma) \leq 2$. By duality, $H_2(\Sigma) \cong H_1(\Sigma)$. The idea is that $H_2(\Sigma)$ can only contribute at most a rank two subspace to each of $H_2(M)$ and $H_2(N)$. So $H_2(X)$ can inherit at largest a rank four subspace from $H_*(\Sigma)$: roughly speaking, half from $H_2(\Sigma)$ (via ψ_*) and half from $H_1(\Sigma)$. This subspace has signature zero and, since X is spin, must be a sum of hyperbolics. Then $H_2(X)$ would correspond to the intersection form $2E_8 \oplus 2H$ (at best), which would contradict Furuta's theorem and establish $e(R_1) \geq 3$. Recalling the definitions of R_n and K_n from section 2, we apply this method.

Theorem 3.2. $e(R_n) \geq 2n + 1$.

Proof (See Figure 3): Let Σ be a 3-manifold that separates K_n from infinity. Define $M \subset \#nK3$ to be the 4-manifold bounded by Σ that does not contain S, the union of cores of the Casson handles that carry the $3nH$ in homology. Define $N \subset \#3n(S^2 \times S^2)$ to be the 4-manifold bounded by Σ that is contained in R_n, and let $X = M \cup_\Sigma N$. Note that all 4-manifolds in our construction are spin, and so have even intersection form and signature 0 mod 16.

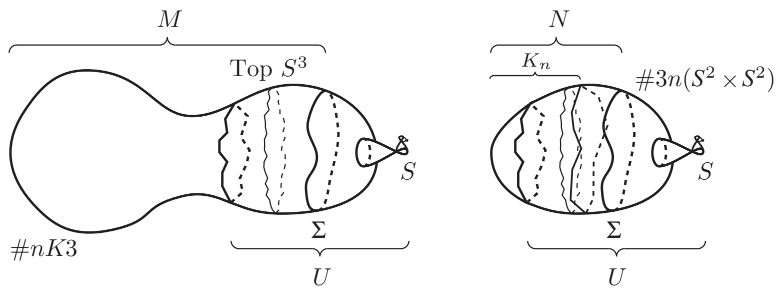

FIGURE 3. Calculation of $e(R_n)$

Now suppose $H_1(\Sigma)$ has rank $\leq 2n$. We use the above Mayer-Vietoris sequence, and the exact sequences of the pairs (M, Σ) and (N, Σ) (again, coefficients in $\mathbb{Q}$).

$$\begin{aligned} \cdots \quad &\to \quad H_2(\Sigma) \xrightarrow{i_*} H_2(M) \xrightarrow{\kappa_*} H_2(M, \Sigma) \xrightarrow{\partial_*} H_1(\Sigma) \to \cdots \\ \cdots \quad &\to \quad H_2(\Sigma) \xrightarrow{j_*} H_2(N) \xrightarrow{\lambda_*} H_2(N, \Sigma) \xrightarrow{\partial_*} H_1(\Sigma) \to \cdots \end{aligned}$$

Duality implies $H_2(\Sigma) \cong H_1(\Sigma)$. The end of $\mathrm{int}(M)$ (similarly for N) contains a topological 3-sphere that, along with Σ, bounds a 4-manifold A that is a subset (topologically) of $\mathbb{R}^4$. See Figure 4. So any H_2 in A can be pushed into Σ. Thus, the form corresponding to $H_2(M, \Sigma)$ is $2nE_8$, and $H_2(N, \Sigma) \cong 0$. So $H_2(M)$ corresponds to the form $2nE_8 \oplus i_*(H_2(\Sigma))$, and $H_2(N)$ to the form $j_*(H_2(\Sigma))$. Now, $\mathrm{rank}(i_*), \mathrm{rank}(j_*) \leq 2n$, so $\mathrm{rank}(\mathrm{im}\psi_*) \leq 2n$. By exactness, $\mathrm{im}\psi_* = \ker \eta_*$, and since $b_1(\Sigma) \leq 2n$, the form corresponding to $H_2(X)$ must be a subform of $2nE_8 \oplus \Gamma$, for some Γ of (even) rank $\leq 4n$. Γ has signature 0, and since X is spin, Γ is a sum of (at most $2n$) hyperbolics, i.e., $H_2(X)$ is a subform of $2nE_8 \oplus 2nH$. But the form for $H_2(X)$ necessarily contains $2n$ E_8 summands, which contradicts Furuta's theorem. Note that we have actually proved any exotic $\mathbb{R}^4$ that contains K_n satisfies $e(R) \geq 2n+1$, even if it is not diffeomorphic to R_n. $\square$

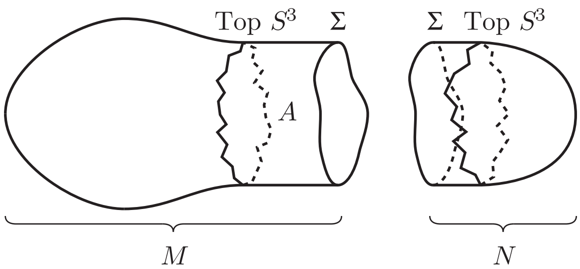

FIGURE 4. Any H_2 in A can be pushed into Σ

Corollary 3.1. $e(R_\infty) = \infty$.

Proof: Suppose $e(R_\infty) = s < \infty$. Then for any compact set $K \subset R_\infty$, K can be separated from infinity by a 3-manifold Σ with $b_1(\Sigma) \leq s$. In particular, $K_s \subset K_\infty$ is compact, but as seen above, K_s cannot be separated from infinity by a 3-manifold with $b_1 < 2s + 1$, implying $e(R_\infty) > 2s$. The contradiction establishes $e(R_\infty) = \infty$. □

4. Small exotic constructions

Many small exotic $\mathbb{R}^4$'s arise as a consequence of the topological success and smooth failure of the h-cobordism theorem. A topological h-cobordism between $K3\#\overline{\mathbb{C}P^2}$ and a surgered version of the same manifold results in the exotic $\mathbb{R}^4$ in Figure 5(b), which we denote R. (For details, see [9] or [13].)

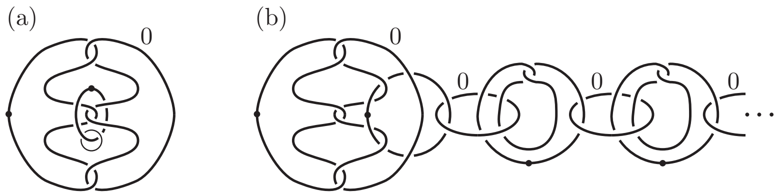

Figure 5. A small exotic R^4

To see directly that R is homeomorphic to $\mathbb{R}^4$, replace the Casson handle in Figure 5(b) with a standard 2-handle. We then have Figure 5(a), where the unmarked 2-handle has framing 0. That 2-handle now cancels the 1-handle it goes over, and the remaining 1-2 handle pair cancels to leave B^4. Freedman's work then implies that R is homeomorphic to $\mathbb{R}^4$.

We now compute a bound on the complexity of R. In this setting, Bižaca and Gompf refer to the complexity as the *engulfing index* of R in [2], where additional examples may be found. Define Y_n to be the compact submanifold of R obtained by cutting off the Casson handle after the nth stage, so that Y_2 is as pictured in Figure 6(a).

Begin with the handle slide indicated by the arrow, and isotope the picture to obtain 6(b). Another handle slide and isotopy produce 6(c). Sliding 1-handles twice then yields 6(d), which is the

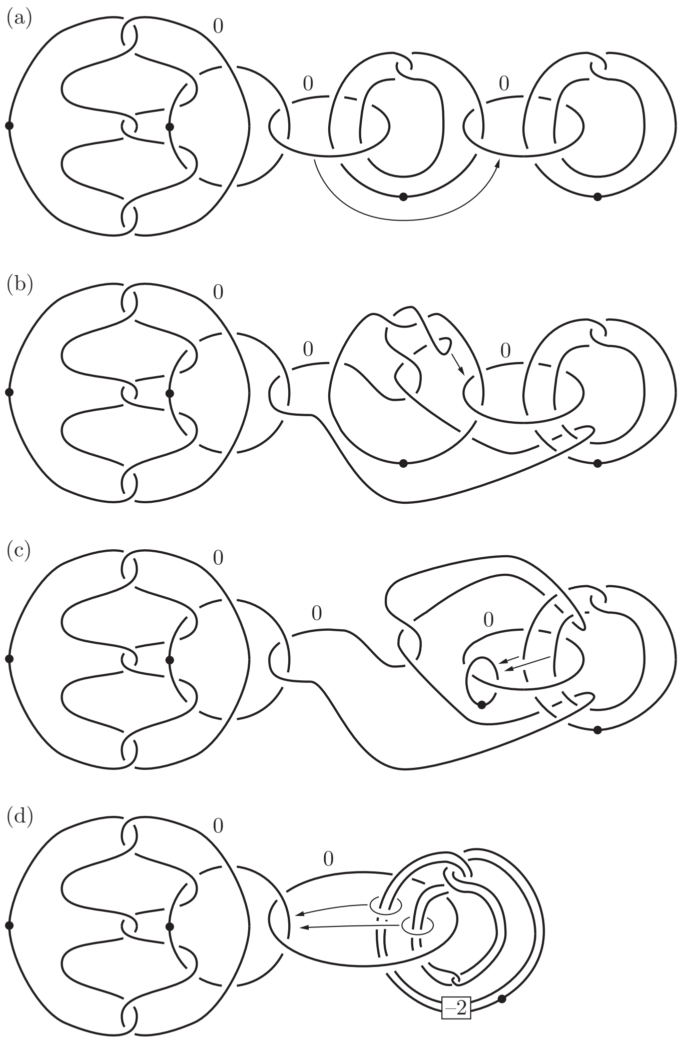

FIGURE 6. Y_2

same as 6(a) with one fewer 1-2 handle pair and the rightmost 1-handle replaced by its untwisted Whitehead double. Iterating this sequence of moves, but doubling the number of 1-handle slides in 6(c) for each iteration allows us to cancel 1-2 handle pairs of Y_n, obtaining the n -fold untwisted Whitehead double of the unknot for the rightmost dotted circle. Sliding that dotted circle 2^n times over the middle dotted circle as indicated in 6(d) and canceling the remaining handle pair give us Figure 7(a), which is isotopic to 7(b), where the n-fold double is drawn for convenience as a thickened strand.

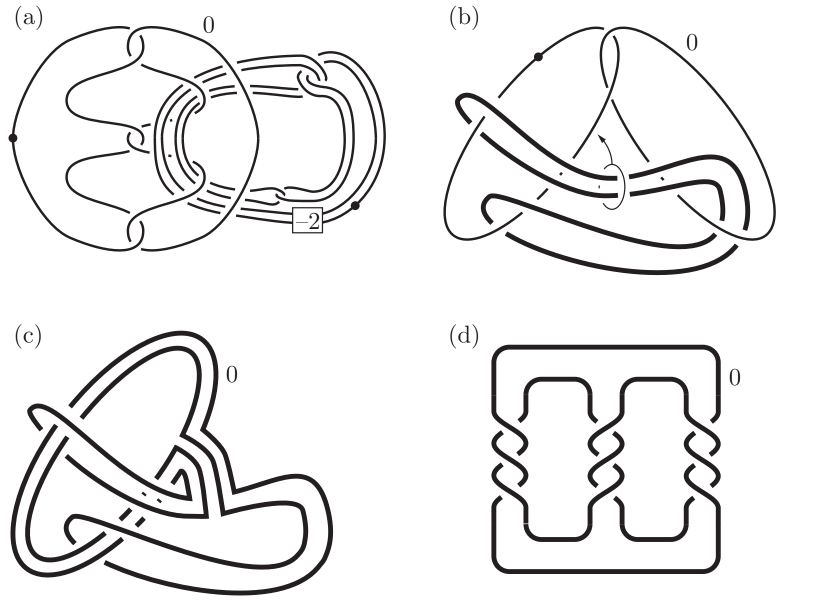

FIGURE 7. Calculation of the boundary of Y_n

We now wish to determine the boundary of Y_n. Surger the 1-handles by replacing the dotted circles with 0-framed circles, and slide each strand of the n-fold double over the left circle twice as indicated in 7(b). The right circle is now a 0-framed meridian of the left, and so the two can be canceled to leave 7(c), which is isotopic to the n-fold untwisted Whitehead double of the $(-3, 3, -3)$ pretzel knot pictured in 7(d).

The upshot of this calculation is that ∂Y_n is the 3-manifold obtained by 0-surgery on a knot in S^3, so $H_1(\partial Y_n; \mathbb{Z}) = \mathbb{Z}$. Since every compact subset of R is contained in some Y_n, we have $e(R) \leq 1$.

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