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NON-CLASSICALITY AND QUANDLE
DIFFERENCE INVARIANTS

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ABSTRACT. Non-classical virtual knots may have upper and lower quandles which are not isomorphic. We exploit this property to define quandle difference invariants, which can detect non-classicality by comparing the numbers of homomorphisms into a finite quandle from a virtual knot's upper and lower quandles. The invariants for small-order finite quandles detect non-classicality in several interesting virtual knots. We compute the difference invariant with the six smallest connected quandles for all non-evenly intersticed Gauss codes with four crossings. For non-evenly intersticed Gauss codes with four crossings, the difference invariants detect non-classicality in 86% of codes which have non-trivial upper or lower counting invariant values.

1. INTRODUCTION

Virtual knots are equivalence classes of Gauss codes under the equivalence relation determined by Reidemeister moves. A Gauss code is *realizable* if it determines a planar knot diagram; virtual knots which include realizable Gauss codes are *classical* and those which do not are *non-classical*. See [12] and [2] for more.

Detecting the non-classicality of a virtual knot is not always simple since each classical knot type includes some Gauss codes which cannot be realized without virtual crossings. Various methods have been proposed for detecting non-classicality of Gauss codes; see [13] and [5].

In this paper, we define *quandle difference invariants* which can detect non-classicality of certain virtual knots. We show that quandle difference invariants detect non-classicality in some interesting examples of virtual knots, including one virtual knot whose non-classicality is not detected by the (upper) knot quandle or by the 1- or 2-strand bracket polynomials, though its non-classicality is detected with the 3-strand bracket polynomial in [5]. We describe an algorithm and provide an implementation in `Maple` for computing these invariants from a Gauss code and a finite quandle matrix.

In order to test the effectiveness of quandle difference invariants at detecting non-classicality, we generated a list of all Gauss codes with three and four crossings, eliminating codes which are obviously classical or trivial. From this list, we then computed the quandle difference invariant for each such code; the results are collected in Table 4.

The paper is organized as follows. Section 2 gives a brief review of virtual knots, section 3 introduces the quandle difference invariants and 2-component invariants, and section 4 contains computational results. The `Maple` code used to obtain these results is available at `www.esotericka.org/quandles`; bugfixes and improvements will be made as necessary.

2. Virtual knots and non-classicality

We begin with a definition from [12].

Definition 1. Let K be an oriented knot diagram with crossings labeled with names and signs. A *Gauss code* is a sequence of crossing labels with over/under and sign information recorded in the order encountered while following the orientation of the knot; it is defined up to cyclic permutation. A *Gauss diagram* is obtained from a Gauss code by writing the code counterclockwise around a circle and joining the two instances of each crossing with an arrow oriented "in the direction of gravity," that is, toward the under-crossing label, and noting the sign for each such arrow.

A Gauss code or diagram determines a knot diagram in a neighborhood of each crossing and specifies which strands are to be connected; thus, a knot diagram can be reconstructed from its Gauss code. We can then consider doing knot theory combinatorially by defining a *knot* to be an equivalence class of Gauss codes under the
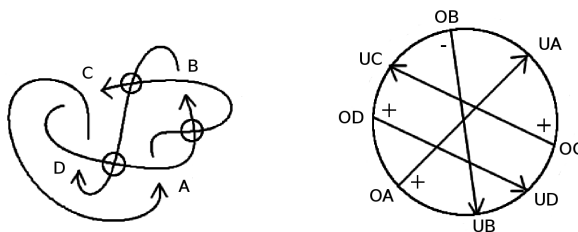
equivalence relation determined by the Reidemeister moves. However, there is no guarantee that a given Gauss code determines a planar knot diagram; we may be forced to introduce additional crossings not mentioned in the code in order to draw the diagram in the plane. These *virtual crossings* are drawn as circled self-intersections, and since they're not really there, we're permitted to arbitrarily redraw any section of an arc with only virtual crossings on its interior provided we put only virtual crossings in the interior.

Definition 2. A *virtual knot* is an equivalence class of Gauss codes under the equivalence relation generated by the Reidemeister moves.

Example 1. The Gauss code

$$UA + OB - UC + OD + OA + UB - UD + OC+$$

corresponds to the pictured virtual knot diagram and Gauss diagram.

Virtual crossings may be avoided by drawing our non-planar knot diagrams on a surface with genus; we may then regard the Gauss code as describing a knot in a thickened surface $\Sigma \times [0, 1]$. The equivalence relation defining virtual knots then translates into isotopy of the knot in $\Sigma \times [0, 1]$ modulo stabilization and destabilization moves [2], [14].

Classical knot theory forms a subset of virtual knot theory, since every classical knot may be considered a virtual knot without virtual crossings, and, crucially, virtually isotopic classical knots are classically isotopic [8], [14], [17]. Many invariants of classical knots extend to virtual knots by simply ignoring virtual crossings; these include the knot group, the knot quandle and biquandle, the various skein polynomials, and finite type invariants.

Given a Gauss code, one asks whether the Gauss code corresponds to a classical knot. If reconstruction yields a planar knot

diagram, then the answer is clearly yes; however, every classical knot has representative Gauss codes which determine non-planar, i.e., virtual, diagrams [17]. Thus, if a Gauss code is not obviously classical, the code may well represent either a classical or a non-classical virtual knot.

3. Quandle difference invariants

In this section, we discuss a method for detecting non-classicality of virtual knots using quandle difference invariants.

Definition 3. A *quandle* is a set Q with an operation $\triangleright : Q \times Q \to Q$ satisfying

    (i) for all $a \in Q$, $a \triangleright a = a$,
    (ii) for every $a, b \in Q$ there is a unique $c \in Q$ such that $a = c \triangleright b$, and
    (iii) for all $a, b, c \in Q$, $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

The term *quandle* is due to David Joyce; in [11], a quandle is associated to each knot diagram by a Wirtinger-type presentation. By construction, this knot quandle is an invariant of Reidemeister moves and hence virtual isotopy. While it is difficult to directly compare quandles given by presentations, we can exploit the fact that knot quandles are finitely presented to obtain a computable invariant of quandle isomorphism type by counting homomorphisms onto a finite quandle. This quandle counting invariant is studied in various papers and has been jazzed up with quandle 2-cocycles from various quandle homology theories [1], [6], [4].

In [8], it is observed that virtual knots have two knot groups, the *upper group* obtained by the usual Wirtinger presentation and the *lower group* obtained by first flipping the virtual knot over, i.e., looking at the knot diagram from the other side of its supporting surface and then using the usual Wirtinger presentation. For classical knots, flipping over is an ambient isotopy, so for classical knots, the upper and lower groups are isomorphic. However, flipping over virtual knots involves doing forbidden moves which potentially change the virtual knot type (see [16]), so in general a virtual knot can have non-isomorphic upper and lower groups. Exactly the same observation applies to quandles [12].

Theorem 1. *Let K be a virtual knot diagram and let $\mathrm{Flip}(K)$ be the virtual knot diagram obtained from K by viewing K from a base point on the other side of the supporting surface. Then $\mathrm{Flip}(K)$ is an invariant of virtual isotopy.*

Proof: Reidemeister and virtual Reidemeister moves on the flipped knot are simply Reidemeister and virtual Reidemeister moves on the original knot. □

Corollary 2. *Let Φ be any virtual knot invariant. Then*

$$\Phi_2(K) = (\Phi(K), \Phi(\mathrm{Flip}(K)))$$

is an invariant of virtual isotopy.

The two-component version of a virtual knot invariant does not reveal any additional information about K when K is classical, but it can be very helpful in distinguishing non-classical virtual knots. Obvious examples of 2-component invariants include the 2-component counting invariants from the knot group, quandle and biquandle; 2-component 2-cocycle invariants [1]; the 2-component Jones polynomial, etc.

Corollary 2 suggests the following.

Definition 4. Let T be a finite quandle and K a virtual knot. Let U denote the upper quandle of K and L the lower quandle of K. The *two-component counting invariant* is the ordered pair

$$Q2_T(K) = (|\mathrm{Hom}(U, T)|, |\mathrm{Hom}(L, T)|) .$$

The *quandle difference invariant* $QD_T(K)$ of K associated to T is the difference between the number of homomorphisms from U to T and the number of homomorphisms from L to T. That is,

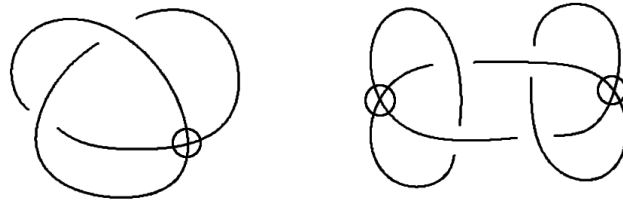$$QD_T(K) = |\mathrm{Hom}(U, T)| - |\mathrm{Hom}(L, T)|.$$

The fact that classical knots have isomorphic upper and lower quandles implies the following.

Theorem 3. *If a virtual knot K is classical, then $QD_T(K) = 0$ for every finite quandle T.*

Corollary 4. *If $QD_T(K) \neq 0$ for any finite quandle T, then K is not classical. In particular, if $QD_T(K) \neq 0$ for any finite quandle T, then K is not the unknot.*

Moreover, since the individual components of QD_T are invariants of virtual isotopy, QD_T is itself an invariant of virtual knots; $QD_T(K) \neq QD_T(K')$ implies K is not virtually isotopic to K'. In the present work, however, we focus on using the quandle difference invariant to detect non-classicality of virtual knots. We know already that the quandle difference invariant does not always detect non-classicality.

Example 2. The knot on the left is Kauffman's virtual trefoil. It is known to be non-trivial and non-classical [12]. However, its upper and lower quandles are both isomorphic to the free quandle on one generator, that is, the knot quandle of the unknot. The knot on the right is a variant of the Kishino knot; it has rotational symmetry about the horizontal axis, so its upper and lower quandles have identical presentations.

Definition 5. We say a virtual knot is *pseudoclassical* if it is virtually isotopic to its flip. In particular, a pseudoclassical virtual knot has isomorphic upper and lower quandles.

Clearly, the quandle difference invariant cannot detect the non-classicality of pseudoclassical virtual knots for any finite quandle T. However, the quandle difference invariants are effective at detecting non-classicality in many virtual knots. Using the smallest connected quandle, for example, we detect the non-classicality (and hence non-triviality) of the non-alternating Kishino knot in Figure 1. It took only two tries to find a quandle difference invariant which detects the non-classicality of the virtual knot K_D in Figure 1; this knot is not distinguished from the unknot by either the Jones polynomial or 2-strand bracket polynomial, though the 3-strand bracket polynomial does the trick [5].

$$K = \quad T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$QD_T(K) = -6$$

$$K_D = \quad T_2 = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$
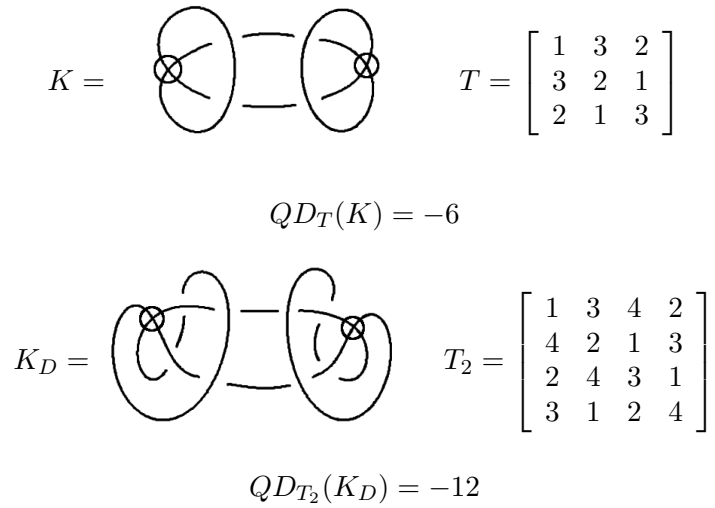
$$QD_{T_2}(K_D) = -12$$

FIGURE 1. QD_T detects non-classicality of virtual knots K and K_D

To compute the upper and lower quandle presentations from a Gauss diagram, we divide the outer circle at each arrowhead to

$$x_{i+1} \leftarrow \Big| - x_i \quad \Rightarrow x_i \triangleright y = x_{i+1}$$

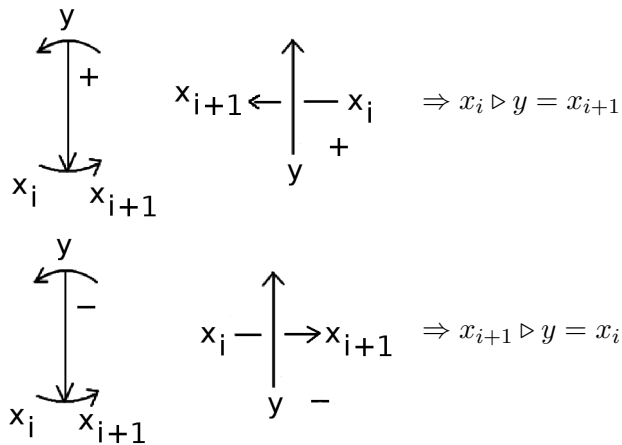$$x_i - \Big| \rightarrow x_{i+1} \quad \Rightarrow x_{i+1} \triangleright y = x_i$$

FIGURE 2. Quandle relations from a Gauss diagram

obtain a list of arcs in the diagram, and the quandle relations at each crossing are determined according to the crossing sign. As Figure 2 shows, the rule is that if the arrowhead dividing x_i from x_{i+1} has its tail on arc y, the relation determined is $x_i \triangleright y = x_{i+1}$ if the crossing sign is $+$ and $x_{i+1} \triangleright y = x_i$ if the crossing sign is $-$.
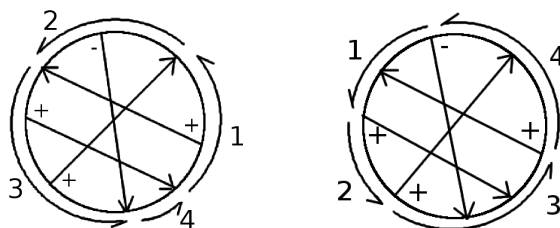
Dividing the same Gauss diagram at the arrowtails and applying the same procedure yields a presentation of the lower quandle L. Alternatively, we can obtain a presentation of the lower quandle by reversing the direction of the arrows, then getting the upper quandle presentation of the resulting diagram.

Example 3. Let K be the virtual knot from Example 1. Then K has upper quandle presentation

$$U = \langle 1, 2, 3, 4 \mid 1 \triangleright 3 = 2, 2 \triangleright 1 = 3, 4 \triangleright 2 = 3, 4 \triangleright 3 = 1 \rangle$$

and lower quandle presentation

$$L = \langle 1, 2, 3, 4 \mid 1 \triangleright 3 = 2, 2 \triangleright 4 = 3, 3 \triangleright 1 = 4, 1 \triangleright 3 = 4 \rangle.$$

From these presentations, we can compute the quandle difference invariant $QD_T(K)$ for any choice of finite target quandle T by systematically considering all assignments of elements of T to each generator in the quandle presentation and checking whether the assignment satisfies the required relations in the target quandle.

4. Computational results

Here, we describe our computational procedure for computing the quandle difference invariant for any choice of Gauss code and finite target quandle and give some computational results.

`Maple` code implementing the algorithms below is available in the file `quandledifference.txt` at the second listed author's website `www.esotericka.org/quandles`.

We compute the quandle difference invariant $QD_T(K)$ using the quandle matrix notation from [9]. Specifically, let $T = \{x_1 \ldots, x_n\}$ be a finite quandle. Then the matrix of T, M_T, is the matrix abstracted from the operation table of T by dropping the xs, keeping only the subscripts. A list of all finite quandles with up to five elements appears in [9]; lists of all finite quandles with six, seven, and eight elements, along with software for computing such lists, are available at `www.esotericka.org/quandles`.[1] The file `quandles-maple.txt` contains `Maple` code for counting homomorphisms from a knot quandle to finite quandle specified by a matrix.

We represent a finitely presented quandle with a *quandle presentation matrix* as described in [10]. That is, we put

$$M_{ij} = \left\{ \begin{array}{ll} k, & x_i \triangleright x_j = x_k, \\ 0, & \text{else.} \end{array} \right.$$

If a Gauss code has a negative undercrossing label x_i such that the next undercrossing label x_{i+1} has sign $+$ and both overcrossings lie on the same arc x_j, then the quandle relations determined are $x_i \triangleright x_j = x_{i-1}$ and $x_i \triangleright x_j = x_{i+1}$, which determine conflicting entries for the i, j position in the presentation matrix. In this situation, we can do a type I Reidemeister move to obtain a presentation with no conflicting entries; the program `gfix` finds and corrects this problem.

Example 4. The virtual knot diagram in Example 1 has upper and lower quandle presentation matrices

$$U = \left[\begin{array}{cccc} 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array} \right] \quad \text{and} \quad L = \left[\begin{array}{ccccc} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right].$$

We represent Gauss codes in `Maple` as vectors with (appropriately?) Gaussian integer entries, using the convention

$$OX+ \leftrightarrow X, \quad UX+ \leftrightarrow -X,$$

$$OX- \leftrightarrow X + I, \quad UX- \leftrightarrow -X - I, \quad X \in \mathbb{Z}.$$

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[1] An independently derived list of quandles with up to six elements, obtained by Yamada, is included in [3].

To allow for multi-component links, each component is ended with a "0".

Example 5. The multicomponent Gauss code

$$U1 + O2 - O3 - U2-, \quad O1 + U3-$$

is represented in our `Maple` notation as the vector

$$[-1, 2 + I, 3 + I, -2 - I, 0, 1, -3 - I, 0].$$

Our quandle-difference computing algorithm then takes a Gauss code and first runs it through `gfix`, which detects the situation described above and does Reidemeister I moves as necessary until the code has an upper quandle presentation matrix with no conflicting entries. The program `gauss2pres` then reads off the upper quandle presentation matrix from the Gauss code. Our main program, `qdiff`, uses `homcount` from `quandles-maple.txt` to compute the number of homomorphisms from the upper quandle of the matrix returned by `gfix` into the specified target quandle. Then, the program simply multiplies each entry in the code by -1 to switch the direction of the arrows and cycles through to put the -1 or $-1 - i$ entry in the first position, then repeats the procedure to get the lower quandle presentation matrix and its counting invariant. Finally, the program reports the difference between these two numbers. We also include `q2chom` which computes the 2-component quandle counting invariant.

In order to get some feeling for the effectiveness of quandle difference invariants at detecting non-classicality of virtual knots, we generated lists of all non-evenly intersticed single-component Gauss codes with three and four crossings, removing codes which reduce by an obvious type I or type II move, using `rglist`. The non-evenly intersticed condition guarantees non-planarity of the corresponding virtual knot, which may be classical or non-classical (including pseudoclassical). There are 172 such 3-crossing Gauss codes; however, QD_T does not detect non-classicality in any of these 3-crossing codes for the six smallest connected quandles. There are 17040 such 4-crossing codes. We then computed the number of codes in which non-classicality was detected using some connected quandles of order up to six. The results are collected in Table 1.

It is not clear what percentage of the Gauss codes with $QD_T = 0$ are classical (i.e., diagrams of the unknot, trefoil, or figure eight),

Target quandle T	Number detected/ 17040 (4-crossing codes)
$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$	3060
$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$	1350
$\begin{bmatrix} 1 & 3 & 4 & 5 & 2 \\ 3 & 2 & 5 & 1 & 4 \\ 4 & 5 & 3 & 2 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$	492
$\begin{bmatrix} 1 & 4 & 5 & 3 & 2 \\ 3 & 2 & 4 & 5 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 1 & 2 & 4 & 3 \\ 4 & 3 & 1 & 2 & 5 \end{bmatrix}$	72
$\begin{bmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$	426
$\begin{bmatrix} 1 & 4 & 5 & 2 & 3 & 1 \\ 4 & 2 & 6 & 1 & 2 & 3 \\ 5 & 6 & 3 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 & 6 & 5 \\ 3 & 5 & 1 & 6 & 5 & 4 \\ 6 & 3 & 2 & 5 & 4 & 6 \end{bmatrix}$	3060

TABLE 1. Number of non-evenly intersticed single-component Gauss codes in which qdiff detects non-classicality

what percentage are pseudoclassical, and what percentage may
have non-classicality detected by larger finite quandles, nor is it
clear how many distinct virtual knots are represented. However,
eliminating the codes which have trivial values for the quandle
counting invariants for both upper and lower quandles for all six
listed quandles, a step which eliminates the classical and pseudo-
classical codes from the lists, gives a list of 16 3-crossing codes and
4140 4-crossing codes. Of these 4-crossing codes, 3570, or 86%, have
non-classicality detected by at least one of the six listed quandle
difference invariants.

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