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	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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THE PSEUDOCOMPACT-OPEN TOPOLOGY ON C(X)

S. KUNDU AND PRATIBHA GARG

Dedicated to Professor Charles E. Aull

ABSTRACT. This paper studies the pseudocompact-open topology on the set of all real-valued continuous functions on a Tychonoff space and compares this topology with the compact-open topology and the topology of uniform convergence. In the second half, the induced map, as well as the metrizability of this topology, is studied.

1. INTRODUCTION

The set $C(X)$ of all continuous real-valued functions on a Tychonoff space X has a number of natural topologies. The idea of topologizing $C(X)$ arose from the notion of convergence of sequences of functions. Also, continuous functions and Baire measures on Tychonoff spaces are linked by the process of integration. A number of locally convex topologies on spaces of continuous functions have been studied in order to clarify this relationship. They enable the powerful duality theory of locally convex spaces to be profitably applied to topological measure theory.

Two commonly used topologies on $C(X)$ are the compact-open topology and the topology of uniform convergence. The latter

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topology has been used for more than a century as the proper setting to study uniform convergence of sequences of functions. The compact-open topology made its appearance in 1945 in a paper by Ralph H. Fox, [19], and soon after was developed by Richard F. Arens in [1] and by Arens and James Dugundji in [2]. This topology was shown in [25] to be the proper setting to study sequences of functions which converge uniformly on compact subsets.

The compact-open topology k and the topology of uniform convergence u on $C(X)$ are equal if and only if X is compact. Because compactness is such a strong condition, there is a considerable gap between these two topologies. This gap has been especially felt in measure theory; consequently, in the last five decades, there have been quite a few topologies introduced that lie between k and u, such as the strict topology, the σ-compact-open topology, the topology of uniform convergence on σ-compact subsets, and the topology of uniform convergence on bounded subsets. See, for example, [4], [12], [20], [23], [26], [27], [28], [29], [34], [36].

The primary concern of this work is to study another natural topology, the pseudocompact-open topology on $C(X)$, in detail from the topological point of view. We denote this topology by ps and the corresponding space by $C_{ps}(X)$. Though this natural topology has occasionally been mentioned in the literature, it deserves much more attention from the researchers. In section 2, we define the pseudocompact-open topology and show that this topology can be viewed in three different ways. In section 3, we compare this topology with two well-known topologies k and u on $C(X)$ in order to have a better understanding of the pseudocompact-open topology. In section 4, we study the induced map on $C_{ps}(X)$ and in the last section, we study the submetrizability, metrizabilty, and separability of $C_{ps}(X)$.

Throughout the rest of the paper, we use the following conventions. All spaces are completely regular Hausdorff, that is, Tychonoff. If X and Y are two spaces with the same underlying set, then we use $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X. The symbols $\mathbb{R}$ and

$\mathbb{N}$ denote the space of real numbers and natural numbers, respectively. Finally, the constant zero-function in $C(X)$ is denoted by 0, more specifically by 0_X.

2. The pseudocompact-open topology on $C(X)$: different views

We recall that a space X is said to be pseudocompact if $f(X)$ is a bounded subset of $\mathbb{R}$ for each $f \in C(X)$. For any pseudocompact subset A of X and any open subset V of $\mathbb{R}$, define

$$[A, V] = \{f \in C(X) : f(A) \subseteq V\}.$$

This agrees with the usual "set-open" terminology for the compact-open topology. Now let $\mathcal{PS}(X)$ be the set of all pseudocompact subsets of X. For the pseudocompact-open topology on $C(X)$, we take as subbase, the family

$$\{[A, V] : A \in \mathcal{PS}(X), V \text{ is open in } \mathbb{R}\};$$

and we denote the corresponding space by $C_{ps}(X)$. Since the closure of a pseudocompact subset is again pseudocompact and since $\overline{f(\overline{A})} = \overline{f(A)}$ for $f \in C(X)$, we can always take closed pseudocompact subsets of X in $[A, V]$.

Now we define the topology of uniform convergence on pseudocompact sets. For each $A \in \mathcal{PS}(X)$ and $\epsilon > 0$, let

$$A_\epsilon = \{(f, g) \in C(X) \times C(X) : |f(x) - g(x)| < \epsilon \text{ for all } x \in A\}.$$

It can be easily verified that the collection $\{A_\epsilon : A \in \mathcal{PS}(X), \epsilon > 0\}$ is a base for some uniformity on $C(X)$. We denote the space $C(X)$ with the topology induced by this uniformity by $C_{ps,u}(X)$. This topology is called the topology of uniform convergence on $\mathcal{PS}(X)$. For each $f \in C(X)$, $A \in \mathcal{PS}(X)$, and $\epsilon > 0$, let

$$< f, A, \epsilon >= \{g \in C(X) : |f(x) - g(x)| < \epsilon \text{ for all } x \in A\}.$$

If $f \in C(X)$, the collection $\{< f, A, \epsilon > : A \in \mathcal{PS}(X),\ \epsilon > 0\}$ forms a neighborhood base at f in $C_{ps,u}(X)$. Actually, the collection $\{< f, A, \epsilon > : f \in C(X),\ A \in \mathcal{PS}(X),\ \epsilon > 0\}$ forms a base for the topology of uniform convergence on $\mathcal{PS}(X)$. In particular, each such set $< f, A, \epsilon >$ is open in $C_{ps,u}(X)$.

Now for each $A \in \mathcal{PS}(X)$, define the seminorm p_A on $C(X)$ by

$$p_A(f) = \sup\{|f(x)| : x \in A\}.$$

Also for each $A \in \mathcal{PS}(X)$ and $\epsilon > 0$, let

$$V_{A,\epsilon} = \{f \in C(X) : p_A(f) < \epsilon\}.$$

Let $\mathcal{V} = \{V_{A,\epsilon} : A \in \mathcal{PS}(X), \epsilon > 0\}$. It can be easily shown that for each $f \in C(X)$, $f + \mathcal{V} = \{f + V : V \in \mathcal{V}\}$ forms a neighborhood base at f. Since this topology is generated by a collection of seminorms, it is locally convex.

In the next result, we show that the three topologies on $C(X)$, defined above, are the same. But before proving this result we would like to make an observation. A closed subset of a pseudocompact set may not be pseudocompact. But a closed subdomain of a pseudocompact set is again pseudocompact. A subset A of a space X satisfying the condition $A = \overline{intA}$ is called a closed domain.

Theorem 2.1. *For any space X, the pseudocompact-open topology on $C(X)$ is same as the topology of uniform convergence on the pseudocompact subsets of X, that is, $C_{ps}(X) = C_{ps,u}(X)$. Moreover, $C_{ps}(X)$ is a Hausdorff locally convex space.*

Proof: Let $[A, V]$ be a subbasic open set in $C_{ps}(X)$ and let $f \in [A, V]$. Since $f(A)$ is compact, there exist $z_1,\ z_2, \ldots, z_n \in f(A)$ such that $f(A) \subseteq \cup_{i=1}^{n}(z_i - \epsilon_i, z_i + \epsilon_i) \subseteq \cup_{i=1}^{n}(z_i - 2\epsilon_i, z_i + 2\epsilon_i) \subseteq V$. Choose $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$. Now if $g \in < f, A, \epsilon >$ and $x \in A$, then $|g(x) - f(x)| < \epsilon$ and there exists an i such that $|f(x) - z_i| < \epsilon_i$. Hence, $|g(x) - z_i| < 2\epsilon_i$ and thus, $g(x) \in V$. So $g(A) \subseteq V$, that is, $g \in [A, V]$. But this means $< f, A, \epsilon > \subseteq [A, V]$. Now let $W = \cap_{i=1}^{k}[A_i, V_i]$ be a basic neighborhood of f in $C_{ps}(X)$. Then there exist positive real numbers $\epsilon_1,\ \epsilon_2, \ldots, \epsilon_k$ such that $f \in\ < f, A_i, \epsilon_i > \subseteq [A_i, V_i]$ for each $i = 1,\ 2, \ldots, k$. If $A = \cup_{i=1}^{k} A_i$ and $\epsilon = \min_{1 \leq i \leq k} \epsilon_i$, then $f \in\ < f, A, \epsilon > \subseteq W$. This shows $C_{ps}(X) \leq C_{ps,u}(X)$.

Now let $< f, A, \epsilon >$ be a basic neighborhood of f in $C_{ps,u}(X)$. Since $f(A)$ is compact, there exist $z_1,\ z_2, \ldots, z_n$ in $f(A)$ such that $f(A) \subseteq \cup_{i=1}^{n}(z_i - \dfrac{\epsilon}{4}, z_i + \dfrac{\epsilon}{4})$. Define $W_i = (z_i - \frac{\epsilon}{2}, z_i + \frac{\epsilon}{2})$ and $A_i = cl_A(A \cap f^{-1}(z_i - \frac{\epsilon}{4}, z_i + \frac{\epsilon}{4}))$. By the observation made above, each A_i is pseudocompact. Note that $A = \cup_{i=1}^{n} A_i$. Now we show that

$f \in \cap_{i=1}^n [A_i, W_i] \subseteq < f, A, \epsilon >$. It is clear that $f \in \cap_{i=1}^n [A_i, W_i]$. Let $g \in \cap_{i=1}^n [A_i, W_i]$ and $x \in A$. Then there exists an i such that $x \in A_i$ and consequently, $f(x) \in [z_i - \frac{\epsilon}{4}, z_i + \frac{\epsilon}{4}]$. Since $g(x) \in (z_i - \frac{\epsilon}{2}, z_i + \frac{\epsilon}{2})$, $|f(x) - g(x)| < \epsilon$. So $g \in < f, A, \epsilon >$ and consequently, $C_{ps,u}(X) \leq C_{ps}(X)$.

Note that for each $f \in C(X)$, we have $f + V_{A,\epsilon} \subseteq < f, A, \epsilon >$ and $< f, A, \frac{\epsilon}{2} > \subseteq f + V_{A,\epsilon}$ for all $A \in \mathcal{PS}(X)$. This shows that the topology of uniform convergence on the pseudocompact sets is the same as the topology generated by the collection of seminorms $\{p_A : A \in \mathcal{PS}(X)\}$. Hence, $C_{ps}(X) = C_{ps,u}(X)$ is a locally convex space. Now if f and g are two distinct functions in $C(X)$, then there exists $x \in X$ such that $f(x) \neq g(x)$. Let $2\epsilon = |f(x) - g(x)|$. Now $[\{x\}, (f(x) - \epsilon, f(x) + \epsilon)] \cap [\{x\}, (g(x) - \epsilon, g(x) + \epsilon)] = \emptyset$ and consequently, $C_{ps}(X)$ is Hausdorff. □

Corollary 2.2. *For any space X, $C_{ps}(X)$ is a Tychonoff space.*

Proof: Any uniformizable topology is completely regular. Also we could have noted that a locally convex topology is always completely regular. □

In the collection of subbasic open sets $\{[A, V] : A \in \mathcal{PS}(X), V$ is open in $\mathbb{R}\}$ in $C_{ps}(X)$, the open set V can always be taken as a bounded open interval. The precise statement follows.

Theorem 2.3. *For any space X, the collection $\{[A, V] : A \in \mathcal{PS}(X),\ V$ is a bounded open interval in $\mathbb{R}\}$ forms a subbase for $C_{ps}(X)$.*

Proof: The proof is quite similar to that of $C_{ps,u}(X) \leq C_{ps}(X)$. Let $[A, V]$ be a subbasic open set in $C_{ps}(X)$. Here A is a pseudocompact subset of X and V is open in $\mathbb{R}$. Let $f \in [A, V]$. Since $f(A)$ is compact, there exist $z_1,\ z_2, \ldots, z_n$ in $f(A)$ and positive real numbers $\epsilon_1,\ \epsilon_2, \ldots, \epsilon_n$ such that $f(A) \subseteq \cup_{i=1}^n (z_i - \epsilon_i, z_i + \epsilon_i) \subseteq \cup_{i=1}^n (z_i - 2\epsilon_i, z_i + 2\epsilon_i) \subseteq V$. Let $W_i = (z_i - 2\epsilon_i, z_i + 2\epsilon_i)$ and $A_i = cl_A(A \cap f^{-1}((z_i - \epsilon_i, z_i + \epsilon_i)))$ for each $i = 1,\ 2, \ldots, n$. Note that $A = \cup_{i=1}^n A_i$ and $f \in \cap_{i=1}^n [A_i, W_i]$. It can be easily verified that $\cap_{i=1}^n [A_i, W_i] \subseteq [A, V]$. □

We end this section with the result that $C^*(X)$ is dense in $C_{ps}(X)$. Here, $C^*(X) = \{f \in C(X) : f$ is bounded$\}$; that is,

$C^*(X)$ is the collection of all bounded real-valued continuous functions on X. But in order to prove this result, we need to make the following observation. If A is a pseudocompact subset of X, then $cl_{\beta X} A \subseteq \upsilon X$, where υX is the Hewitt realcompactification of X and $cl_{\beta X} A$ is the closure of A in βX, the Stone-Čech compactification of X.

Theorem 2.4. *For any space X, $C^*(X)$ is dense in $C_{ps}(X)$.*

Proof: Let $< f, A, \epsilon >$ be a basic neighborhood of f in $C_{ps}(X)$. Take a continuous extension f^υ of f from υX to $\mathbb{R}$. Since A is pseudocompact, $cl_{\beta X} A \subseteq \upsilon X$. Let f_1 be the restriction of f^υ on $cl_{\beta X} A$. Since $cl_{\beta X} A$ is compact, $f_1(cl_{\beta X} A) \subseteq [a, b]$ for some $a,\ b \in \mathbb{R}$. Now take a continuous extension f_2 of f_1 from βX to $[a, b]$. Now if g is the restriction of f_2 to X, then $g \in\ < f, A, \epsilon > \cap\ C^*(X)$. Thus, $C^*(X)$ is dense in $C_{ps}(X)$. □

3. Comparison of topologies

In this section, we compare the pseudocompact-open topology with the compact-open topology and the topology of uniform convergence. Let us recall the definitions of these latter two topologies.

Given a subset A of a space X, $f \in C(X)$ and $\epsilon > 0$, define, as before, $< f, A, \epsilon >= \{g \in C(X) : |f(x) - g(x)| < \epsilon, \forall\ x \in A\}$. If $\mathcal{K}(X)$ is the collection of all compact subsets of X, then for each $f \in C(X)$, the collection $\{< f, K, \epsilon > : K \in \mathcal{K}(X),\ \epsilon > 0\}$ forms a neighborhood base at f in the compact-open topology k on $C(X)$. When $C(X)$ is equipped with the compact-open topology k, we denote the corresponding space by $C_k(X)$. Also it can be shown that each $< f, K, \epsilon >$, where K is compact in X, is actually open in $C_k(X)$. For u, the topology of uniform convergence on $C(X)$, the collection $\{< f, X, \epsilon > : \epsilon > 0\}$ forms a neighborhood base at each $f \in C(X)$. When $C(X)$ is equipped with the topology u, the corresponding space is denoted by $C_u(X)$. But the set $< f, X, \epsilon >$ need not be open in $C_u(X)$. From the definitions of these topologies, the following result follows immediately.

Theorem 3.1. *For any space X, $C_k(X) \leq C_{ps}(X) \leq C_u(X)$.*

Now we determine when these inequalities are equalities and give examples to illustrate the differences.

Theorem 3.2. *For every space X,*

(*i*) $C_k(X) = C_{ps}(X)$ *if and only if every closed pseudocompact subset of X is compact.*

(*ii*) $C_{ps}(X) = C_u(X)$ *if and only if X is pseudocompact.*

Proof: (*i*) Note that for a subset A of X, $< f, \overline{A}, \epsilon > \subseteq < f, A, \epsilon >$. So if every closed pseudocompact subset of X is compact, then $C_{ps}(X) \leq C_k(X)$. Consequently, in this case, $C_{ps}(X) = C_k(X)$.

Conversely, suppose that $C_k(X) = C_{ps}(X)$ and let A be any closed pseudocompact subset of X. So $< 0, A, 1 >$ is open in $C_k(X)$ and consequently, there exist a compact subset K of X and $\epsilon > 0$ such that $< 0, K, \epsilon > \subseteq < 0, A, 1 >$. If possible, let $x \in A \setminus K$. Then there exists a continuous function $g : X \longrightarrow [0, 1]$ such that $g(x) = 1$ and $g(y) = 0 \ \forall \ y \in K$. Note that $g \in < 0, K, \epsilon > \setminus < 0, A, 1 >$ and we arrive at a contradiction. Hence, $A \subseteq K$ and consequently, A is compact.

(*ii*) First, suppose that X is pseudocompact. So for each $f \in C(X)$ and each $\epsilon > 0$, $< f, X, \epsilon >$ is a basic open set in $C_{ps}(X)$ and consequently, $C_u(X) = C_{ps}(X)$.

Now let $C_{ps}(X) = C_u(X)$. Since $< 0, X, 1 >$ is a basic neighborhood of the constant zero-function 0 in $C_u(X)$, there exist a pseudocompact subset A of X and $\epsilon > 0$ such that $< 0, A, \epsilon > \subseteq < 0, X, 1 >$. As before, by using the complete regularity of X, it can be shown that we must have $X = \overline{A}$. But the closure of a pseudocompact set is also pseudocompact. Hence, X is pseudocompact. $\square$

Corollary 3.3. *For any normal Hausdorff space X, $C_k(X) = C_{ps}(X)$ if and only if every closed countably compact subset of X is compact.*

Now we would like to investigate the spaces for which every closed pseudocompact subset is compact. But before further discussion on this topic, for convenience, we would like to give the following definitions.

Definition 3.4. A space X is called *isocompact* if every closed countably compact subset of X is compact. Similarly, X is called *p-isocompact* if every closed pseudocompact subset of X is compact.

Using these definitions, from the last two results we have that for every space X,

(i) $C_k(X) = C_{ps}(X)$ if and only if X is p-isocompact;
(ii) if in addition X is normal, then $C_k(X) = C_{ps}(X)$ if and only if X is isocompact.

On isocompact spaces, there has already been substantial and significant research. On this topic, one can see [11], [10], [15], [39], and [42].

Now we would like to investigate which spaces are p-isocompact. Note that a pseudocompact space is compact if it is either realcompact or paracompact. In fact, there is a link between realcompact and paracompact spaces. Both are nearly realcompact. A space X is called *nearly realcompact* if $\beta X \setminus \upsilon X$ is dense in $\beta X \setminus X$. But a nearly realcompact, pseudocompact space is compact. We would like to put a note of caution here that nearly realcompactness is not hereditary with respect to closed subsets. See Remark 1.17 in [10].

Since realcompactness and paracompactness are hereditary with respect to closed subsets, the realcompact and paracompact spaces are p-isocompact. In fact, the paracompactness can be replaced by weaker conditions, such as metacompactness and para-Lindelöfness.

A space X is called *metacompact* if every open cover of X has an open point-finite refinement. A metacompact space is also called weakly paracompact or pointwise paracompact. Brian M. Scott (in 1979) [35] and W. Stephen Watson (in 1981) [43] proved independently that every pseudocompact metacompact space is compact. Scott also noted that this result was proved independently by O. Förster also.

A space X is called *para-Lindelöf* if every open cover of X has a locally countable open refinement. Dennis K. Burke and S. W. Davis (in 1982) [14] proved that a pseudocompact para-Lindelöf space is compact.

Both metacompactness and para-Lindelöfness are hereditary with respect to closed subspaces. More precisely, if a space X is metacompact (respectively, para-Lindelöf), then every closed subset of X is also metacompact (respectively, para-Lindelöf). Hence, we obtain the following result.

Theorem 3.5. *If a space X is either metacompact or para-Lindelöf, then X is p-isocompact.*

For metacompact and para-Lindelöf spaces, one should read Burke's excellent survey paper [13].

Now we would like to talk about another kind of p-isocompact space. A subset A of a space X is called *relatively pseudocompact* or *bounded* in X if every function in $C(X)$ is bounded on A. A space X is called *hyperisocompact* if every closed relatively pseudocompact subset of X is compact. Obviously, a hyperisocompact space is p-isocompact. In the literature, a hyperisocompact space is also called a *μ-space* or a *Nachbin-Shirota space (NS-space* for brevity). It is easy to show that a subset A of X is relatively pseudocompact if and only if $cl_{\beta X}A \subseteq \upsilon X$. For equivalent characterizations of relatively pseudocompact subsets, see Proposition 2.6 in [11]. Clearly, the realcompact spaces are hyperisocompact. Also the P-spaces are hyperisocompact. A space X is called a *P-space* if every G_δ-set in X is open in X. For equivalent characterizations of a P-space, see 4J in [21]. In fact, in a P-space every relatively pseudocompact set is finite; see 4K in [21]. The following example gives a P-space which is not realcompact.

Example 3.6. *Let $W(\omega_2)$ be the set of all ordinals $< \omega_2$ equipped with the order topology where ω_2 is the smallest ordinal of cardinal $\aleph_2$ and let X be the subspace of $W(\omega_2)$ obtained by deleting all limit ordinals having a countable local base. (This space appears in 9L of* [21]*.) Also, this space is a P-space, which is not realcompact. Here we have*

$$C_k(X) = C_{ps}(X) < C_u(X).$$

We continue with a few more examples which illustrate all possible inequalities between these function spaces.

Example 3.7. *For a p-isocompact space X, we have*

$$C_k(X) = C_{ps}(X) \leq C_u(X).$$

We have already seen that the family of p-isocompact spaces includes metric spaces, paracompact spaces, realcompact spaces, P-spaces, metacompact spaces, and para-Lindelöf spaces.

For a p-isocompact space X, which is not pseudocompact, we have

$$C_k(X) = C_{ps}(X) < C_u(X).$$

Example 3.8. *The space* $X = [0, \omega_1)$ *of countable ordinals* [38, Example 43] *is countably compact and collectionwise normal, but not isocompact. For this space* X*, we have*

$$C_k(X) < C_{ps}(X) = C_u(X).$$

Example 3.9. *The Dieudonné plank* D [38, Example 89] *is metacompact, but not countably paracompact, normal, or hyperisocompact. For the space* D*, we have*

$$C_k(D) = C_{ps}(D) < C_u(D).$$

Example 3.10. *The Bing-Michael space* Y [38, Example 143] *is metacompact and normal, but not paracompact. For this space* Y*, we have*

$$C_k(Y) = C_{ps}(Y) < C_u(Y).$$

Example 3.11. *Let* $X = [0, \omega_1) \oplus \mathbb{R}$ *where* $[0, \omega_1)$ *is the space mentioned in Example 3.8. For this space* X*, we have*

$$C_k(X) < C_{ps}(X) < C_u(X).$$

We end this section with the diagram of following implications:

realcompact $\Rightarrow$ hyperisocompact $\Rightarrow$ p-isocompact $\Rightarrow$ isocompact.

We note that none of the implications above can be reversed. The space $W(\omega_2)$ is hyperisocompact, but not realcompact. The Dieudonné plank D of Example 3.9 is p-isocompact, but not hyperisocompact. Finally, consider the space Ψ described in 5I of [21]. The space Ψ is pseudocompact, but not countably compact. Also, every countably compact subset of Ψ is compact (see [41]). Hence, Ψ is isocompact but not p-isocompact.

4. Induced maps

One of the most useful tools in function spaces is the following concept of induced map. If $f : X \longrightarrow Y$ is a continuous map, then the induced map of f, denoted by $f^* : C(Y) \longrightarrow C(X)$ is defined by $f^*(g) = g \circ f$ for all $g \in C(Y)$. In this work, we will study this induced map f^* when both $C(Y)$ and $C(X)$ are equipped with the pseudocompact-open topology. In order to have the first result of this section, we need the definition of an almost onto map. A map $f : X \longrightarrow Y$, where X is any nonempty set and Y is a topological space, is called *almost onto* if $f(X)$ is dense in Y.

Theorem 4.1. *Let $f : X \longrightarrow Y$ be a continuous map between two spaces X and Y. Then*

(*i*) $f^* : C_{ps}(Y) \longrightarrow C_{ps}(X)$ *is continuous;*

(*ii*) $f^* : C(Y) \longrightarrow C(X)$ *is one-to-one if and only if f is almost onto;*

(*iii*) *if $f^* : C(Y) \longrightarrow C_{ps}(X)$ is almost onto, then f is one-to-one.*

Proof: (*i*) Suppose $g \in C_{ps}(Y)$. Let $< f^*(g), A, \epsilon >$ be a basic neighborhood of $f^*(g)$ in $C_{ps}(X)$. Then $f^*(< g, f(A), \epsilon >) \subseteq < f^*(g), A, \epsilon >$ and consequently, f^* is continuous.

(*ii*) and (*iii*) See Theorem 2.2.6 in [31]. □

Now we would like to give the converse of Theorem 4.1(*iii*) with some restrictions on X and Y. To clarify these restrictions, we need the following definitions.

Definition 4.2. A space X is called an S_4*-space* if every countably compact subset of X is closed in X. Equivalently, X is an S_4-space if every non-closed subset A of X contains a sequence (x_n) which has no cluster point in A.

An S_4-space is also called *C-closed*. The class of S_4-spaces includes the sequential spaces. In particular, first countable and Fréchet spaces are S_4-space. For more details see [5] and [24].

Definition 4.3. A space X is called *weak p-isocompact* if every closed pseudocompact subset of X is countably compact.

Note that the space Ψ, though pseudocompact itself, is not weak p-isocompact. The space $[0, \omega_1)$ of countable ordinals given in Example 3.8 is not isocompact and hence, it is not p-isocompact. But since $[0, \omega_1)$ is countably compact, obviously it is weak p-isocompact. Actually, the countably compact spaces, as well as normal spaces, are weak p-isocompact spaces.

Definition 4.4. A space X is called *functionally normal* if, given any two pairs of disjoint closed sets A and B in X, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(A) \cap f(B) = \emptyset$.

Obviously, a normal space is functionally normal. But the converse need not be true. The Tychonoff plank is functionally normal, but not normal; (see [21, 8.20] and [9]). Also, a countably compact functionally normal space is normal, and every closed pseudocompact subset in a functionally normal space is C-embedded; see [6], [8], and [9]. Of course, we are assuming here spaces to be Tychonoff. Functionally normal spaces were introduced in 1951 by W. T. van Est and Hans Freudenthal in [18] and were later studied significantly by C. E. Aull.

Now we are ready to state and prove our next result.

Theorem 4.5. *Let X be a weak p-isocompact space and Y be a functionally normal S_4-space. If $f : X \longrightarrow Y$ is a one-to-one continuous map, then $f^* : C(Y) \longrightarrow C_{ps}(X)$ is almost onto.*

Proof: We need to show that $f^*(C(Y))$ is dense in $C_{ps}(X)$. Let $g \in C(X)$ and $< g, A, \epsilon >$ be a basic neighborhood of g in $C_{ps}(X)$ where A is a closed pseudocompact subset of X and $\epsilon > 0$. Since X is weak p-isocompact, A is countably compact in X. Now since f is one-to-one and Y is an S_4-space, $f|_A : A \longrightarrow f(A)$ is a homeomorphism. Consequently, $g \circ (f|_A)^{-1} : f(A) \longrightarrow \mathbb{R}$ is continuous. Note that $f(A)$ is a closed countably compact set in Y which is functionally normal. So $f(A)$ is C-embedded in Y and hence, there exists $h \in C(Y)$ such that $h|_{f(A)} = g \circ (f|_A)^{-1}$. Let $\phi = h \circ f$. So $\phi = f^*(h) \in f^*(C(Y))$ and $\phi = g$ on A. Therefore, $\phi \in < g, A, \epsilon > \cap f^*(C(Y))$ and consequently, f^* is almost onto. $\square$

Our next goal is to find out when f^* is an embedding. We need the following definition of a p-covering map. A continuous map $f : X \longrightarrow Y$ is called *p-covering* if, given any pseudocompact subset A in Y, there exists a pseudocompact subset C in X such that $A \subseteq \overline{f(C)}$.

Theorem 4.6. *Let $f : X \longrightarrow Y$ be a continuous map between two spaces X and Y. If $f^* : C_{ps}(Y) \longrightarrow C_{ps}(X)$ is an embedding, then f is a p-covering map.*

Proof: With A a pseudocompact subset of Y, $f^*(< 0_Y, A, 1 >)$ is an open neighborhood of the zero function 0_X in $f^*(C_{ps}(Y))$. Choose a pseudocompact subset C of X and an $\epsilon > 0$ such that $0_X \in < 0_X, C, \epsilon > \cap f^*(C_{ps}(Y)) \subseteq f^*(< 0_Y, A, 1 >)$. We claim that $A \subseteq \overline{f(C)}$. If possible, let $y \in A \setminus \overline{f(C)}$. So there exists a continuous function $g : Y \longrightarrow [0, 1]$ such that $g(y) = 1$ and $g(\overline{f(C)}) = 0$. Since $g(f(C)) = 0$, $f^*(g) \in < 0_X, C, \epsilon > \cap f^*(C_{ps}(Y)) \subseteq f^*(< 0_Y, A, 1 >$

). Since f^* is injective, $g \in < 0_Y, A, 1 >$. But $y \in A$ implies $|g(y)| < 1$. We arrive at a contradiction and hence, $A \subseteq \overline{f(C)}$. Consequently, f is a p-covering map. $\square$

For the converse, we have the following result.

Theorem 4.7. *Suppose that every pseudocompact subset of Y is closed. If a continuous map $f : X \longrightarrow Y$ is p-covering, then $f^* : C_{ps}(Y) \longrightarrow C_{ps}(X)$ is an embedding.*

Proof: Since for each $a \in X$, there exists a pseudocompact subset C of X such that $\{a\} \subseteq f(C)$, f is onto. Hence, by Theorem 4.1(ii), f^* is one-to-one. We need to show that $f^* : C_{ps}(Y) \longrightarrow f^*(C_{ps}(X))$ is an open map. Let $< g, A, \epsilon >$ be a basic open set in $C_{ps}(Y)$ where A is pseudocompact in Y and $\epsilon > 0$. Let $h \in f^*(< g, A, \epsilon >)$. So there exists $h_1 \in < g, A, \epsilon >$ such that $f^*(h_1) = h$. Since $< g, A, \epsilon >$ is open in $C_{ps}(Y)$, there exists a pseudocompact set B in Y and $\delta > 0$ such that $< h_1, B, \delta > \subseteq < g, A, \epsilon >$. Since f is p-covering, there exists a pseudocompact set C in X such that $B \subseteq f(C)$.

Now we claim that $< h, C, \delta > \cap f^*(C_{ps}(Y)) \subseteq f^*(< h_1, B, \delta >)$. Choose $l \in C(Y)$ such that $f^*(l) \in < h, C, \delta > \cap f^*(C_{ps}(Y))$. Since $B \subseteq f(C)$, for each $b \in B$, there exists $c \in C$ such that $b = f(c)$. Since $f^*(l) \in < h, C, \delta >$, $|l(b) - h_1(b)| = |l(f(c)) - h_1(f(c))| = |f^*(l)(c) - f^*(h_1)(c)| = |f^*(l)(c) - h(c)| < \delta$. So $l \in < h_1, B, \delta >$, that is, $f^*(l) \in f^*(< h_1, B, \delta >)$. Hence, $< h, C, \delta > \cap f^*(C_{ps}(Y)) \subseteq f^*(< h_1, B, \delta >) \subseteq f^*(< g, A, \epsilon >)$ and consequently, $f^*(< g, A, \epsilon >)$ is open in $f^*(C_{ps}(Y))$. $\square$

Remark 4.8. For the spaces in which the pseudocompact subsets are closed, one should read the excellent paper [16].

Another kind of useful map on function spaces is the "sum function." Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces. If $\oplus X_\alpha$ denotes their topological sum, then the sum function s is defined by $s : C(\oplus X_\alpha) \longrightarrow \Pi\{C(X_\alpha) : \alpha \in \Lambda\}$ where $s(f) =< f|_{X_\alpha} >$ for each $f \in C(\oplus X_\alpha)$. Note for any topological space Y, a map $f : \oplus X_\alpha \longrightarrow Y$ is continuous if and only if $f|_{X_\alpha}$ is continuous for each $\alpha \in \Lambda$.

In order to prove the last result of this section, we need the following lemma.

Lemma 4.9. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces and let A be a subset of $\oplus_{\alpha\in\Lambda} X_\alpha$. Then A is pseudocompact in $\oplus_{\alpha\in\Lambda} X_\alpha$, if and only if $A \cap X_\alpha = \emptyset$ for all but finitely many $\alpha \in \Lambda$ and every non-empty intersection $A \cap X_\alpha$ is pseudocompact in X_α.*

Theorem 4.10. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces. Then the sum function $s : C_{ps}(\oplus X_\alpha) \longrightarrow \Pi\{C_{ps}(X_\alpha) : \alpha \in \Lambda\}$ is a homeomorphism.*

Proof: Define $t : \Pi\{C(X_\alpha) : \alpha \in \Lambda\} \longrightarrow C(\oplus X_\alpha)$ by $t(< g_\alpha >_{\alpha\in\Lambda}) = g$ where $g|_{X_\alpha} = g_\alpha$.

Since $s \circ t$ and $t \circ s$ are the identity maps on $\Pi\{C(X_\alpha) : \alpha \in \Lambda\}$ and $C(\oplus X_\alpha)$, respectively, s is a bijection and $s^{-1} = t$. Now we claim that both s and s^{-1} are continuous.

In order to avoid confusion, let $[A, V]_\oplus = \{f \in C(\oplus X_\alpha) : f(A) \subseteq V\}$ where A is a pseudocompact subset of $\oplus X_\alpha$ and V is open in $\mathbb{R}$, and let $[A, V]_\alpha = \{f \in C(X_\alpha) : f(A) \subseteq V\}$ where A is a pseudocompact subset of X_α and V is open in $\mathbb{R}$. Let $p_\alpha : \Pi\{C_{ps}(X_\alpha) : \alpha \in \Lambda\} \longrightarrow C_{ps}(X_\alpha)$ be the α-th projection. Let $p^{-1}_{\alpha_1}([A_1, V_1]_{\alpha_1}) \cap \cdots \cap p^{-1}_{\alpha_n}([A_n, V_n]_{\alpha_n})$ be a basic open set in $\Pi\{C_{ps}(X_\alpha) : \alpha \in \Lambda\}$ where A_i is pseudocompact in X_{α_i} and each V_i is open in $\mathbb{R}$, $1 \leq i \leq n$. Then $s^{-1}(p^{-1}_{\alpha_1}([A_1, V_1]_{\alpha_1}) \cap \cdots \cap p^{-1}_{\alpha_n}([A_n, V_n]_{\alpha_n})) = s^{-1}p^{-1}_{\alpha_1}([A_1, V_1]_{\alpha_1}) \cap \cdots \cap s^{-1}p^{-1}_{\alpha_n}([A_n, V_n]_{\alpha_n}) = [A_1, V_1]_\oplus \cap \cdots \cap [A_n, V_n]_\oplus$. So s is continuous.

Let $[A, V]_\oplus$ be a subbasic open set in $C_{ps}(\oplus X_\alpha)$ where A is pseudocompact in $\oplus X_\alpha$ and V is open in $\mathbb{R}$. Since A is pseudocompact in $\oplus X_\alpha$, by the previous lemma, $A \cap X_\alpha = \emptyset$ for all but finitely many α. Suppose $A \cap X_{\alpha_i} \neq \emptyset$, $1 \leq i \leq n$. Let $A_i = A \cap X_{\alpha_i}$. Each A_i is pseudocompact in X_{α_i}, $1 \leq i \leq n$.

Note $s([A, V]_\oplus) = p^{-1}_{\alpha_1}([A_1, V_1]_{\alpha_1}) \cap \cdots \cap p^{-1}_{\alpha_n}([A_n, V_n]_{\alpha_n})$. So s is an open map and consequently, a homeomorphism. □

5. Additional properties : metrizability and separability

Here we study the metrizability and separability of $C_{ps}(X)$. But in order to study the metrizability of $C_{ps}(X)$ in a broader perspective, first we show that a number of properties of $C_{ps}(X)$ are equivalent to submetrizability. We begin with the definition of submetrizability and some immediate consequences of this property.

Definition 5.1. A completely regular Hausdorff space (X, τ) is called *submetrizable* if X admits a weaker metrizable topology, equivalently if there exists a continuous injection $f : (X, \tau) \longrightarrow (Y, d)$ where (Y, d) is a metric space.

Remark 5.2. (i) If a space X has a G_δ-diagonal, that is, if the set $\{(x, x) : x \in X\}$ is a G_δ-set in the product space $X \times X$, then every point in X is a G_δ-set. Note that every metrizable space has a zero-set diagonal. Consequently, every submetrizable space has also a zero-set-diagonal.

(ii) Every pseudocompact set in a submetrizable space is a G_δ-set. In particular, all compact subsets, countably compact subsets, and the singletons are G_δ-sets in a submetrizable space. A space X is called an E_0*-space* if every point in the space is a G_δ-set. So the submetrizable spaces are E_0-spaces.

For more information on E_0-spaces, see [7], and for submetrizable spaces, see [22].

For our next result, we need the following definitions.

Definition 5.3. A completely regular Hausdorff space X is called σ*-pseudocompact* if there exists a sequence $\{A_n\}$ of pseudocompact sets in X such that $X = \cup_{n=1}^{\infty} A_n$. A space X is said to be *almost* σ*-pseudocompact* if it has a dense σ-pseudocompact subset.

Theorem 5.4. *For any space X, the following are equivalent.*

(a) $C_{ps}(X)$ *is submetrizable.*
(b) *Every pseudocompact subset of* $C_{ps}(X)$ *is a* G_δ*-set in* $C_{ps}(X)$.
(c) *Every countably compact subset of* $C_{ps}(X)$ *is a* G_δ*-set in* $C_{ps}(X)$.
(d) *Every compact subset of* $C_{ps}(X)$ *is a* G_δ*-set in* $C_{ps}(X)$.
(e) $C_{ps}(X)$ *is an* E_0*-space.*
(f) X *is almost* σ*-pseudocompact.*
(g) $C_{ps}(X)$ *has a zero-set-diagonal.*
(h) $C_{ps}(X)$ *has a* G_δ*-diagonal.*

Proof: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$ are all immediate.

$(e) \Rightarrow (f)$. If $C_{ps}(X)$ is an E_0-space, then the constant zero-function 0 defined on X is a G_δ-set. Let $\{0\} = \cap_{n=1}^{\infty} < 0, A_n, \epsilon_n >$ where each A_n is pseudocompact in X and $\epsilon_n > 0$. We claim that $X = \overline{\cup_{n=1}^{\infty} A_n}$.

Suppose that $x_0 \in X \setminus \overline{\cup_{n=1}^{\infty} A_n}$. So there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ for all $x \in \overline{\cup_{n=1}^{\infty} A_n}$ and $f(x_0) = 1$. Since $f(x) = 0$ for all $x \in A_n$, $f \in < 0, A_n, \epsilon_n >$ for all n and hence, $f \in \cap_{n=1}^{\infty} < 0, A_n, \epsilon_n > = \{0\}$. This means $f(x) = 0$ for all $x \in X$. But $f(x_0) = 1$. Because of this contradiction, we conclude that X is almost σ-pseudocompact.

$(f) \Rightarrow (a)$. Let $\cup\{A_n : n \in \mathbb{N}\}$ be a dense subset of X where each A_n is pseudocompact. Let $S = \oplus\{A_n : n \in \mathbb{N}\}$ be the topological sum of the A_n and let $\phi : S \longrightarrow X$ be the natural map. Then the induced map $\phi^* : C_{ps}(X) \longrightarrow C_{ps}(S)$ defined by $\phi^*(f) = f \circ \phi$ is continuous. Since ϕ is almost onto, by Theorem 4.1(ii), ϕ^* is one-to-one. By Theorem 4.10, $C_{ps}(\oplus\{A_n : n \in \mathbb{N}\})$ is homeomorphic to $\Pi\{C_{ps}(A_n) : n \in \mathbb{N}\}$. But each $C_{ps}(A_n)$ is (completely) metrizable, since the supremum metric generates the pseudocompact-open topology whenever the domain is pseudocompact; see Theorem 3.2(ii). Since $C_{ps}(S)$ is metrizable and ϕ^* is a continuous injection, $C_{ps}(X)$ is submetrizable.

By Remark 5.2(i), $(a) \Rightarrow (g) \Rightarrow (h) \Rightarrow (e)$. □

Theorem 5.5. *Suppose that X is almost σ-pseudocompact. If $\mathcal{K}$ is a subset of $C_{ps}(X)$, then the following are equivalent.*

- (a) *$\mathcal{K}$ is compact.*
- (b) *$\mathcal{K}$ is sequentially compact.*
- (c) *$\mathcal{K}$ is countably compact.*
- (d) *$\mathcal{K}$ is pseudocompact.*

Proof: $(b) \Rightarrow (c) \Rightarrow (d)$ are all immediate. By Theorem 5.4, $C_{ps}(X)$ is submetrizable and so also is $\mathcal{K}$. A pseudocompact submetrizable is metrizable and hence compact. But in a metrizable space, all these kinds of compactness coincide. Hence, $(a) \Rightarrow (b)$ and $(d) \Rightarrow (a)$. □

Our next goal is to show that there are several topological properties which are equivalent to the metrizability of $C_{ps}(X)$. So we first define these topological properties.

Definition 5.6. A subset S of a space X is said to have *countable character* if there is a sequence $\{W_n : n \in \mathbb{N}\}$ of open subsets in X such that $S \subseteq W_n$ for each n, and if W is any open set containing S, then $W_n \subseteq W$ for some n.

A space X is said to be of *(pointwise) countable type* if each (point) compact set is contained in a compact set having countable character.

A space X is a *q-space* if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. Another property stronger than being a q-space is that of being an *M-space*, which can be characterized as a space that can be mapped onto a metric space by a quasi-perfect map (a continuous closed map in which inverse images of points are countably compact).

Both a space of pointwise countable type and an M-space are q-spaces. We note that a metrizable space is of countable type.

For more details on the properties discussed above, see [3], [32], [33], and [37].

Now we are ready to relate the metrizability of $C_{ps}(X)$ with the several topological properties discussed above.

Theorem 5.7. *For any space X, the following are equivalent.*

(a) *$C_{ps}(X)$ is metrizable.*
(b) *$C_{ps}(X)$ is of first countable.*
(c) *$C_{ps}(X)$ is of countable type.*
(d) *$C_{ps}(X)$ is of pointwise countable type.*
(e) *$C_{ps}(X)$ has a dense subspace of pointwise countable type.*
(f) *$C_{ps}(X)$ is an M-space.*
(g) *$C_{ps}(X)$ is a q-space.*
(h) *X is hemipseudocompact; that is, there exists a sequence of pseudocompact sets $\{A_n : n \in \mathbb{N}\}$ in X such that for any pseudocompact subset A of X, $A \subseteq A_n$ holds for some n.*

Proof: From the earlier discussions, we have $(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (g)$, $(a) \Rightarrow (f) \Rightarrow (g)$, and $(a) \Rightarrow (b) \Rightarrow (g)$.

$(d) \Leftrightarrow (e)$. It can be easily verified that if D is a dense subset of a space X and A is a compact subset of D, then A has countable character in D if and only if A is of countable character in X. Now since $C_{ps}(X)$ is a locally convex space, it is homogeneous. If we combine this fact with the previous observation, we have $(d) \Leftrightarrow (e)$.

$(g) \Rightarrow (h)$. Suppose that $C_{ps}(X)$ is a q-space. Hence, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the zero-function

0 in $C_{ps}(X)$ such that if $f_n \in U_n$ for each n, then $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $C_{ps}(X)$. Now for each n, there exists a closed pseudocompact subset A_n of X and $\epsilon_n > 0$ such that $0 \in < 0, A_n, \epsilon_n > \subseteq U_n$.

Let A be a pseudocompact subset of X. If possible, suppose that A is not a subset of A_n for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A \setminus A_n$. So for each $n \in \mathbb{N}$, there exists a continuous function $f_n : X \longrightarrow [0,1]$ such that $f_n(a_n) = n$ and $f_n(x) = 0$ for all $x \in A_n$. It is clear that $f_n \in < 0, A_n, \epsilon_n >$. But the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not have a cluster point in $C_{ps}(X)$. If possible, suppose that this sequence has a cluster point f in $C_{ps}(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_{n_k} \in < f, A, 1 >$. So for all $k \in \mathbb{N}$, $f(a_{n_k}) > f_{n_k}(a_{n_k}) - 1 = n_k - 1 \geq k$. But this means that f is unbounded on the pseudocompact set A. So the sequence $\{f_n\}_{n\in\mathbb{N}}$ cannot have a cluster point in $C_{ps}(X)$ and consequently, $C_{ps}(X)$ fails to be a q-space. Hence, X must be hemipseudocompact.

$(h) \Rightarrow (a)$. Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable; (see page 119 in [40]). Now the locally convex topology on $C(X)$ generated by the countable family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$ is metrizable and weaker than the pseudocompact-open topology. However, since for each pseudocompact set A in X, there exists A_n such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms $\{p_A : A \in \mathcal{PS}(X)\}$, that is, the pseudocompact-open topology, is weaker than the topology generated by the family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$. Hence, $C_{ps}(X)$ is metrizable. □

We conclude this paper with the following result on the separability of $C_{ps}(X)$.

Theorem 5.8. *For any space X, the following are equivalent.*

- (a) *$C_{ps}(X)$ is separable.*
- (b) *$C_p(X)$ is separable where p denotes the point-open topology on $C(X)$.*
- (c) *$C_k(X)$ is separable.*
- (d) *X has a weaker separable metrizable topology.*

Proof: First, note by Corollary 4.2.2 in [31] that (b), (c), and (d) are equivalent. Also, since $C_p(X) \leq C_{ps}(X)$, $(a) \Rightarrow (b)$.

$(d) \Rightarrow (a)$. If X has a weaker separable metrizable topology, then X is realcompact; (see page 219 in [17]). Hence, X is p-isocompact. Consequently, $C_{ps}(X) = C_k(X)$. Since $(d) \Rightarrow (c)$, $C_{ps}(X)$ is separable. □

References

[1] Richard F. Arens, *A topology for spaces of transformations,* Ann. of Math. (2) **47** (1946), 480–495.

[2] Richard Arens and James Dugundji, *Topologies for function spaces,* Pacific J. Math. **1** (1951), 5–31.

[3] A. V. Arhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces,* Soviet Math. Dokl. **4** (1963), 1051–1055.

[4] ______, *On linear homeomorphisms of function spaces,* Soviet Math. Dokl. **25** (1982), 852–855.

[5] C. E. Aull, *Sequences in topological spaces,* Prace Mat. **11** (1968) 329–336.

[6] ______, *Notes on separation by continuous functions,* Nederl. Akad. Wetensch. Proc. Ser. A 72 = Indag. Math. **31** (1969), 458–461.

[7] ______, *Some base axioms for topology involving enumerability,* General Topology and its Relations to Modern Analysis and Algebra, III. (Proc. Conf., Kanpur, 1968). Prague: Academia, 1971. 55–61.

[8] ______, *On C- and C^*-embeddings,* Nederl. Akad. Wetensch. Proc. Ser. A 78 = Indag. Math. **37** (1975), 26–33.

[9] ______, *Absolute C-embedding in functionally normal spaces and related spaces,* Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978). Colloquia Mathematica Societatis János Bolyai, 23. Amsterdam-New York: North-Holland, 1980. 129–136.

[10] Robert L. Blair and Eric K. van Douwen, *Nearly realcompact spaces,* Topology Appl. **47** (1992), no. 3, 209–221.

[11] Robert L. Blair and Mary Anne Swardson, *Spaces with an Oz Stone-Čech compactification,* Topology Appl. **36** (1990), no. 1, 73–92.

[12] R. Creighton Buck, *Bounded continuous functions on a locally compact space,* Michigan Math. J. **5** (1958), 95–104.

[13] Dennis K. Burke, *Covering properties.* Kunen and Vaughan. 347–422.

[14] D. K. Burke and S. W. Davis, *Pseudocompact para-Lindelöf spaces are compact,* Abstracts Amer. Math. Soc. **3** (1982), 213.

[15] S. W. Davis, *A cushioning-type weak covering property,* Pacific J. Math. **80** (1979), no. 2, 359–370.

[16] Alan Dow, Jack R. Porter, R. M. Stephenson, Jr., and R. Grant Woods, *Spaces whose pseudocompact subspaces are closed subsets,* Appl. Gen. Topol. **5** (2004), no. 2, 243–264.

[17] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[18] W. T. van Est and Hans Freudenthal, *Trennung durch stetige Funktionen in topologischen Räumen,* Nederl. Akad. Wetensch. Proc. Ser. A 54 = Indag. Math. **13** (1951), 359–368.

[19] Ralph H. Fox, *On topologies for function spaces,* Bull. Amer. Math. Soc. **51** (1945), 429–432.

[20] Robin Giles, *A generalization of the strict topology,* Trans. Amer. Math. Soc. **161** (1971), 467–474.

[21] Leonard Gillman and Meyer Jerison, *Rings of Continuous Functions.* The University Series in Higher Mathematics. Princeton, New Jersey: D. Van Nostrand, Co., Inc., 1960.

[22] Gary Gruenhage, *Generalized metric spaces.* Kunen and Vaughan. 423–501.

[23] Denny Gulick, *The σ-compact-open topology and its relatives,* Math. Scand. **30** (1972), 159–176.

[24] Mohammad Ismail and Peter Nyikos, *On spaces in which countably compact sets are closed, and hereditary properties,* Topology Appl. **11** no. 3, (1980), 281–292.

[25] James R. Jackson, *Comparison of topologies on function spaces,* Proc. Amer. Math. Soc. **3** (1952), 156–158.

[26] S. Kundu, *$C_b(X)$ revisited: induced map and submetrizability,* Quaest. Math. **27** (2004), no. 1, 47–56

[27] S. Kundu and R. A. McCoy, *Topologies between compact and uniform convergence on function spaces,* Int. J. Math. Math. Sci. **16** (1993), no. 1, 101–109.

[28] S. Kundu, R. A. McCoy, and A. B. Raha, *Topologies between compact and uniform convergence on function spaces, II,* Real Anal. Exchange **18** (1992/93), no.1, 176–189.

[29] S. Kundu and A. B. Raha, *The bounded-open topology and its relatives,* Rend. Istit. Mat. Univ. Trieste **27** (1995), no. 1-2, 61–77 (1996).

[30] Kenneth Kunen and Jerry E. Vaughan, eds. *Handbook of Set-Theoretic Topology.* Amsterdam: North-Holland Publishing Co., 1984.

[31] R. A. McCoy and I. Ntantu, *Topological Properties of Spaces of Continuous Functions.* Lecture Notes in Mathematics, 1315. Berlin: Springer-Verlag, 1988.

[32] E. Michael, *$\aleph_0$-spaces,* J. Math. Mech. **15** (1966), 983–1002.

[33] Kiiti Morita, *Products of normal spaces with metric spaces,* Math. Ann. **154** (1964), 365–382.

[34] Jean Schemets, *Espaces de Fonctions Continues.* Lecture Notes in Mathematics, 519. Berlin-New York: Springer-Verlag, 1976.

[35] Brian M. Scott, *Pseudocompact, metacompact spaces are compact,* Topology Proc. **4** (1979), no. 2, 577–587 (1980).

[36] F. Dennis Sentilles, *Bounded continuous functions on a completely regular space,* Trans. Amer. Math. Soc. **168** (1972), 311–336.

[37] Frank Siwiec, *Generalizations of the first axiom of countability,* Rocky Mountain J. Math. **5** (1975), 1–60.

[38] Lynn Arthur Steen and J. Arthur Seebach, Jr., *Counterexamples in Topology.* 2nd ed. New York: Springer-Verlag, 1978.

[39] Mary Anne Swardson and Paul J. Szeptycki, *When X^* is a P'-space,* Canad. Math. Bull. **39** (1996), 476–485.

[40] Angus Ellis Taylor and David C. Lay, *Introduction to Functional Analysis.* 2nd ed. New York: John Wiley & Sons, 1980.

[41] Aaron R. Todd, *Pseudocompact sets, absolutely Warner bounded sets and continuous function spaces,* Arch. Math. (Basel) **56** (1991), no. 5, 474–481.

[42] Jerry E. Vaughan, *Countably compact and sequentially compact spaces.* Kunen and Vaughan 569–602.

[43] W. Stephen Watson, *Pseudocompact metacompact spaces are compact,* Proc. Amer. Math. Soc. **81** (1981), no. 1, 151–152.

(Kundu) Department of Mathematics; Indian Institute of Technology Delhi; New Delhi 110016, India.
E-mail address: skundu@maths.iitd.ernet.in

(Garg) Department of Mathematics; Indian Institute of Technology Delhi; New Delhi 110016, India.
E-mail address: pratibhagarg@gmail.com