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IRREDUCIBILITY OF PRODUCT SPACES WITH FINITELY MANY POINTS REMOVED

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Abstract. We prove an induction step that can be used to show that in certain cases the removal of finitely many points from a product space produces an irreducible space. For example, we show that whenever γ is less than $\aleph_\omega$, removing finitely many points from the product of γ many first countable compact spaces gives an irreducible space. This result answers questions asked privately by Alexander Arhangel'skii.

If $\mathcal{O}$ is a collection of open sets corresponding to some topological space, we call $\mathcal{O}'$ an *open refinement* of $\mathcal{O}$ if every member of $\mathcal{O}'$ is an open set and a subset of a member of $\mathcal{O}$. A collection $\mathcal{S}$ of sets is said to be *minimal* if for each $S \in \mathcal{S}$, there is an $x \in S$ which is not in any other member of $\mathcal{S}$. A topological space is said to be *irreducible* if every open cover of the space has a minimal open refinement covering the space (see [1], [2], [3], [4]). In this note, we prove an induction step which can be used to show in certain cases that the removal of finitely many points from a product space yields an irreducible space. This induction step applies to regular limits, and the most immediate question left open is how to get past singular ones. One larger project, as we understand it, is to collect examples of noncompact irreducible spaces, though we cannot claim expertise on this topic.

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A *cube* in a product space $X = \prod_{a\in A} X_a$ is a set of the form

$$I^A_f = \{x \in X \mid \forall a \in dom(f)\ x(a) \in f(a)\}$$

for some finite function f with domain contained in A such that each $f(a)$ is an open subset of X_a. We call the domain of f the *support* of I^A_f. We say that a subspace Y of a product space X is *cube-irreducible* if every cover of Y by open subsets of X has a minimal open refinement covering Y and consisting of cubes. Note that every compact subspace of a product space is cube-irreducible.

Following [5], we say that a topological space X is (γ, ∞)-*compact* if every open cover of X has a subcover of cardinality less than γ.

Theorem 1 below gives an argument that applies to the removal of one point from a product space, and Theorem 2 gives a slightly generalized argument that handles the removal of finitely many points. One could naturally ask about the removal of infinite sets satisfying certain conditions.

Note that condition (1) in Theorem 1 and Theorem 2 are satisfied when each X_α is compact and first countable, since then the removal of finitely many points from X_α produces a Lindelöf space, and the product of a Lindelöf space with a compact space is Lindelöf.

Theorem 1. *Let κ be an uncountable regular cardinal. For each $\alpha < \kappa$, let X_α be a topological space and let e_α be a point in X_α. For each $A \subset \kappa$, let $X(A)$ denote $\prod_{\alpha\in A} X_\alpha$, and let $e(A)$ denote the element of $X(A)$ whose α-th coordinate is e_α for each $\alpha \in A$. Suppose that the following hold:*

(1) *for all $\alpha < \kappa$, $(X_\alpha \setminus \{e_\alpha\}) \times X(\kappa \setminus (\alpha+1))$ is (κ, ∞)-compact;*
(2) *for all $\beta < \gamma < \kappa$, $X([\beta, \gamma)) \setminus \{e([\beta, \gamma))\}$ is irreducible.*

Then $X(\kappa) \setminus \{e(\kappa)\}$ is irreducible. Furthermore, if for all $\beta < \gamma < \kappa$,

$$X([\beta, \gamma)) \setminus \{e([\beta, \gamma))\}$$

is cube-irreducible, then so is $X(\kappa) \setminus \{e(\kappa)\}$.

Proof: Since every open subset of $X(\kappa) \setminus \{e(\kappa)\}$ is a union of cubes in $X(\kappa)$ not containing $e(\kappa)$, it suffices to consider the case where $\mathcal{O}$ is a set of cubes in $X(\kappa)$ not containing $e(\kappa)$ which covers $X(\kappa) \setminus \{e(\kappa)\}$. It follows that for every $I^\kappa_f \in \mathcal{O}$, $e_\alpha \notin f(\alpha)$ for some $\alpha \in dom(f)$. For each $\alpha < \kappa$, let $\mathcal{E}_\alpha$ be a subset of $\mathcal{O}$ of cardinality less than κ covering the set of $z \in X(\kappa)$ for which α is the least $\beta < \kappa$ such that $z(\beta) \neq e_\beta$. We may assume that whenever f is a function defining a cube in $\mathcal{E}_\alpha$ and $\beta \in dom(f) \cap \alpha$, then $e_\beta \in f(\beta)$. Since for each $\alpha < \kappa$ the union of the supports of the members of $\mathcal{E}_\alpha$ is a bounded subset of κ, and since κ is regular, we may fix a club subset C of κ with $0 \in C$ such that for each $\beta \in C$ and each $\alpha < \beta$, the support of each member of $\mathcal{E}_\alpha$ is contained in β. For each $\beta \in C$, let Y_β be the set of $z \in X(\kappa)$ for which the least α such that $z(\alpha) \neq e_\alpha$ is in the interval $[\beta, c(\beta))$, where $c(\beta)$ is defined to be the least element of C above β.

Then for each $\beta \in C$, $\mathcal{O}_\beta = \bigcup\{\mathcal{E}_\alpha \mid \alpha \in [\beta, c(\beta))\}$ is a covering of Y_β and $\{I^{[\beta,c(\beta))}_{f\restriction[\beta,c(\beta))} \mid I^\kappa_f \in \mathcal{O}_\beta\}$ is a covering of $X([\beta, c(\beta))) \setminus \{e([\beta, c(\beta)))\}$, which can be refined to a minimal open covering $\mathcal{U}_\beta$ of $X([\beta, c(\beta))) \setminus \{e([\beta, c(\beta)))\}$. For each open set O in each $\mathcal{U}_\beta$, pick a function f such that $I^\kappa_f \in \mathcal{O}_\beta$ and $O \subset I^{[\beta,c(\beta))}_{f\restriction[\beta,c(\beta))}$ and let $F(O)$ be the set of $z \in X(\kappa)$ such that $z\restriction[\beta, c(\beta)) \in O$ and $z \in I^\kappa_f$. Note that for such an f, $e_\alpha \in f(\alpha)$ for every $\alpha \in dom(f) \cap \beta$, and also that if O is a cube then so is $F(O)$.

Then $\mathcal{O}^* = \{F(O) : O \in \bigcup\{\mathcal{U}_\beta \mid \beta \in C\}\}$ is an open refinement of $\mathcal{O}$. It remains to see that $\mathcal{O}^*$ covers $X(\kappa) \setminus \{e(\kappa)\}$ and is minimal.

To see that $\mathcal{O}^*$ covers $X(\kappa) \setminus \{e(\kappa)\}$, fix $z \in X(\kappa) \setminus \{e(\kappa)\}$ and let β be the least $\gamma \in C$ such that $z(\alpha) \neq e_\alpha$ for some α in the interval $[\gamma, c(\gamma))$. Then $z\restriction[\beta, c(\beta))$ is in some member O of $\mathcal{U}_\beta$, and since $z(\alpha) = e_\alpha$ for all $\alpha < \beta$, z is in $F(O)$.

To see that $\mathcal{O}^*$ is minimal, fix $\beta < \kappa$ and $O \in \mathcal{U}_\beta$. Since $\mathcal{U}_\beta$ is a minimal cover of $X([\beta, c(\beta))) \setminus \{e([\beta, c(\beta)))\}$, there exists a $z_0 \in X([\beta, c(\beta))) \setminus \{e([\beta, c(\beta)))\}$ which is not an element of any member of $\mathcal{U}_\beta$ other than O. Let $z \in X(\kappa)$ be the extension of z_0 taking value e_α at each α outside of $[\beta, c(\beta))$. Then z is not a member of any $F(O')$ for any $O' \neq O$ in $\mathcal{U}_\beta$, and for each $\beta' \in \kappa\setminus\{\beta\}$ and each $O' \in \mathcal{U}_{\beta'}$, z is not in $F(O')$, since every element of O' takes value other than e_α at some point $\alpha \in [\beta', c(\beta'))$. □

If s is a function with domain an ordinal, we let $l(s)$ denote the domain (equivalently, the length) of s.

Theorem 2. *Let κ be an uncountable regular cardinal. For each $\alpha < \kappa$, let X_α be a Hausdorff topological space. For each $A \subset \kappa$, let $X(A)$ denote $\prod_{\alpha\in A} X_\alpha$. Let $E = \{e_i : i < n\}$ be a subset of $X(\kappa)$ and for each $A \subset \kappa$, let $E(A)$ denote $\{e_i\restriction A \mid i < n\}$.*

Let Σ *denote* $\{e_i \restriction \gamma \mid i < n \wedge \gamma < \kappa\}$*, and for each* $s \in \Sigma$*, let* $N(s)$ *denote the set of* $x \in X_{l(s)}$ *such that* $s^\frown\langle x\rangle \in \Sigma$*. For each* $s \in \Sigma$ *and each* $\gamma \in (l(s), \kappa)$*, let* $M(s, \gamma)$ *denote*

$$\{x \restriction [l(s), \gamma)) \mid x \in X(\gamma) \cap \Sigma \wedge s \subset x\}.$$

Suppose that the following hold:

(1) *for all* $s \in \Sigma$*,*

$$(X_{l(s)} \setminus N(s)) \times X(\kappa \setminus (l(s) + 1))$$

is (κ, ∞)*-compact;*

(2) *for all* $s \in \Sigma$ *and all* $\gamma \in (l(s), \kappa)$*,* $X([l(s), \gamma)) \setminus M(s, \gamma)$ *is irreducible.*

Then $X(\kappa) \setminus E$ *is irreducible. Furthermore, if for all* $s \in \Sigma$ *and all* $\gamma \in [l(s), \kappa)$*,* $X([l(s), \gamma)) \setminus M(s, \gamma)$ *is cube-irreducible, then so is* $X(\kappa) \setminus E$*.*

Proof: Fix an open covering $\mathcal{O}$ of $X(\kappa) \setminus E$. Since every open subset of $X(\kappa) \setminus E$ is a union of cubes disjoint from E, it suffices to consider the case where $\mathcal{O}$ is a set of cubes disjoint from E covering $X(\kappa) \setminus E$. Then for every $I^\kappa_f \in \mathcal{O}$ and each $e \in E$, there is a $\beta \in dom(f)$ such that $e(\beta) \notin f(\beta)$. Let D be a finite subset of κ such that for each pair e_i, e_j of distinct elements of E, there is an $\alpha \in D$ such that $e_i(\alpha) \neq e_j(\alpha)$. For each α in D, fix disjoint open subsets $O(i, \alpha)$ $(i < n)$ of X_α containing $e_i(\alpha)$ but no distinct $e_j(\alpha)$, $j < n$. We may assume, furthermore, that for each function f with $I^\kappa_f \in \mathcal{O}$, $D \subset dom(f)$ and, for each $\alpha \in D$ and $i < n$, if $e_i(\alpha) \in f(\alpha)$, then $f(\alpha) \subset O(i, \alpha)$.

For each $s \in \Sigma$, let $\mathcal{V}_s$ be a subset of $\mathcal{O}$ of cardinality less than κ covering the set of $z \in X(\kappa)$ such that $s \subset z$ and for which $l(s)$ is the least $\beta < \kappa$ such that $z \restriction \beta \notin \Sigma$. (Note that this set, and thus $\mathcal{V}_s$, might be empty.) We may assume that whenever f is a function defining a cube in $\mathcal{V}_s$ and $\beta \in dom(f) \cap l(s)$, $s(\beta) \in f(\beta)$. For each $\alpha < \kappa$, the union of the supports of the members of all the $\mathcal{V}_s$ with $l(s) < \alpha$ is a bounded subset of κ, and so we may fix a club subset C of κ satisfying the following:

- $0 \in C$;
- $C \cap ((\max(D) + 1) \setminus \{0\}) = \emptyset$;
- each nonzero element of C is a limit ordinal;

- for each $\beta \in C$ and each $s \in \Sigma$ with $l(s) < \beta$, the support of each member of $\mathcal{V}_s$ is contained in β.

For each $s \in \Sigma$ with $l(s) \in C$, let Y_s be the set of $z \in X(\kappa)$ such that $s \subset z$, and for which the least α such that $z{\restriction}\alpha \notin \Sigma$ is in the interval $(l(s), c(l(s)))$, where for each $\beta < \omega_1$, $c(\beta)$ is defined to be the least element of C above β.

Then for each $s \in \Sigma$ with $l(s) \in C$,

$$\mathcal{O}_s = \bigcup\{\mathcal{V}_t \mid s \subset t \wedge t \in \Sigma \wedge l(t) \in [l(s), c(l(s)))\}$$

is a covering of Y_s and

$$\{I^{[l(s),c(l(s)))}_{f\restriction[l(s),c(l(s)))} \mid I^{\kappa}_f \in \mathcal{O}_s\}$$

is a covering of $X([l(s), c(l(s))))\setminus M(s, c(l(s)))$, which can be refined to a minimal open covering $\mathcal{U}_s$ of $X([l(s), c(l(s)))) \setminus M(s, c(l(s)))$. For each open set O in each $\mathcal{U}_s$, pick a function f such that $I^{\kappa}_f \in \mathcal{O}_s$ and $O \subset I^{[l(s),c(l(s)))}_{f\restriction[l(s),c(l(s)))}$ and let $F(O)$ be the set of $z \in X(\kappa)$ such that $z{\restriction}[l(s), c(l(s))) \in O$ and $z \in I^{\kappa}_f$. Note that for such an f, $s(\alpha) \in f(\alpha)$ for every $\alpha \in dom(f) \cap l(s)$. Note also that if O is a cube then so is $F(O)$.

Then $\mathcal{O}^* = \{F(O) : O \in \bigcup\{\mathcal{U}_s \mid s \in \Sigma \wedge l(s) \in C\}\}$ is an open refinement of $\mathcal{O}$. It remains to see that $\mathcal{O}^*$ covers $X(\kappa) \setminus E$ and is minimal.

To see that $\mathcal{O}^*$ covers $X(\kappa)\setminus E$, fix $z \in X(\kappa)\setminus E$ and let β be the least $\gamma \in C$ such that $z{\restriction}\alpha \notin \Sigma$ for some α in the interval $(\gamma, c(\gamma))$. Then $z{\restriction}\beta \in \Sigma$, $z{\restriction}[\beta, c(\beta))$ is in some member O of $\mathcal{U}_{z\restriction\beta}$, and z is in $F(O)$, since for each $\alpha < \beta$ and each $I^{\kappa}_f \in \mathcal{O}_{z\restriction\beta}$, $z(\alpha) \in f(\alpha)$.

To see that $\mathcal{O}^*$ is minimal, fix $s \in \Sigma$ with $l(s) \in C$ and fix $O \in \mathcal{U}_s$. Since $\mathcal{U}_s$ is a minimal cover of $X([l(s), c(l(s))))\setminus M(s, c(l(s)))$, there exists a $z_0 \in X([l(s), c(l(s))))\setminus M(s, c(l(s)))$ which is not an element of any member of $\mathcal{U}_s$ other than O. Fix $i < n$ as follows. If $l(s) = 0$, let i be the unique $j < n$ such that $z_0(\alpha) \in O(j, \alpha)$ for all $\alpha \in D$, if such a j exists, and if no such j exists let $i = 0$. If $l(s) > 0$, let i be the unique j such that $s \subset e_j$. Let $z = z_0 \cup (e_i{\restriction}(\kappa \setminus [l(s), c(l(s)))$. Then $z \in X(\kappa) \setminus E$, and z is not a member of any $F(O')$ for any $O' \neq O$ in $\mathcal{U}_s$.

Suppose that $l(s) \neq 0$ and fix $s' \in \Sigma$ such that $l(s') \in C$ and $s' \restriction D \neq s \restriction D$. Fix $\alpha \in D$ such that $s(\alpha) \neq s'(\alpha)$. Since $s(\alpha) \notin f'(\alpha)$ for all functions f' with $I^{\kappa}_{f'} \in \mathcal{O}_{s'}$, z is not in $F(O')$ for any $O' \in \mathcal{U}_{s'}$.

Now, suppose that $s' \in \Sigma$, $l(s') \in C \setminus \{l(s)\}$ and $s' \restriction D = s \restriction D$ or $l(s) = 0$. Then s' is an initial segment of s or vice-versa. First, suppose that s' is an initial segment of e_i (which subsumes the subcase $s' \subset s$). Then $e_i \restriction [l(s'), c(l(s'))) \notin O'$ for any $O' \in \mathcal{U}_{s'}$, and so $z \notin F(O')$, since $[l(s'), c(l(s'))) \subset (\kappa \setminus [l(s), c(l(s))))$.

Finally, if s' is not an initial segment of e_i (in this case $l(s) = 0$, though we don't use this fact), then s' is an initial segment of some e_j and for some $\alpha \in D$, $z(\alpha)$ is not in $O(j, \alpha)$. Then every $F(O')$ for some $O' \in \mathcal{U}_{s'}$ is a subset of $I^{\kappa}_{f'}$ for a function f' with the property that $O(j, \alpha) \subset f'(\alpha)$, and so $z \notin F(O')$. □

Remark 3. One could weaken condition (2) in the statement of Theorem 2 to say (a) that for all $\gamma < \kappa$, $X([0, \gamma)) \setminus M(\emptyset, \gamma)$ is irreducible, and (b) that there exists a $\rho < \kappa$ such that for all $s \in \Sigma$ of length greater than ρ and for all $\gamma \in (l(s), \kappa)$, $X([l(s), \gamma)) \setminus M(s, \gamma)$ is irreducible. Then in the proof, make the least nonzero element of C greater than ρ and $\max(D)$. One has in this case that $M(s, \gamma)$ is a singleton for each nonempty $s \in \Sigma$ with $l(s) \in C$.

Naturally, one would like to improve the proofs of Theorem 1 and Theorem 2 to get past singular limits. Another natural issue is to find base cases for the induction steps proved above. The following fact follows easily from standard arguments.

Theorem 4. *The removal of finitely many points from a countable product of first countable compact spaces produces a cube-irreducible space.*

Adding this to Theorem 2, one gets the following.

Corollary 5. *The removal of finitely many points from a product of less than $\aleph_\omega$ many first countable compact spaces produces a cube-irreducible space.*

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