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IMAGES OF RELATIVELY LOCALLY FINITE HAUSDORFF SPACES

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ABSTRACT. This paper answers some of the questions posed by A. V. Arhangel'skii and I Yu. Gordienko in [Relatively locally finite Hausdorff spaces, Questions and Answers in General Topology 12 (1994), no. 1, 15–25] concerning the representation of certain spaces as images of relatively locally finite Hausdorff spaces. In particular, it is shown that the convergent sequence is the image of a relatively locally finite Hausdorff space under a continuous open mapping. An example of a relatively locally finite Hausdorff space which is not Baire is also constructed.

1. INTRODUCTION

Topologists are always seeking connections between arbitrary spaces and nicer spaces. For example, every Tychonoff space can be densely embedded in a Hausdorff compactification and every Hausdorff space is the perfect, continuous, irreducible image of an extremally disconnected space. To this end, A. V. Arhangel'skii and I. Yu. Gordienko [1] introduce a generalization of discrete spaces, called relatively locally finite spaces, and examine conditions under which relatively locally finite Hausdorff spaces are discrete. They also pose questions about which Hausdorff spaces are images or preimages of relatively locally finite Hausdorff spaces. This paper answers a few of these questions.

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Definition 1.1. (1) A set A in a topological space X is *finitely located* if every closed in X subset of A is finite.

(2) A Hausdorff space X is *relatively locally finite at x* if x has a finitely located neighborhood in X.

(3) A Hausdorff space X is *relatively locally finite* if it is relatively locally finite at each of its points.

Clearly, every discrete Hausdorff space is relatively locally finite. In fact, in regular Hausdorff spaces, the properties relatively locally finite and discrete are equivalent. In nonregular spaces, however, the properties are not equivalent. This leads to some interesting comparisons between properties of discrete spaces and those of nonregular relatively locally finite Hausdorff spaces. For example, while every image of a discrete space under a closed continuous map is discrete, Arhangel'skii and Gordienko describe a relatively locally finite Hausdorff space and a closed continuous mapping from this space onto the convergent sequence space, which is not relatively locally finite. Because every image of a discrete space under an open continuous map is also discrete, Arhangel'skii and Gordienko posed the following set of questions: *Is it true that every Hausdorff space Y can be represented as an image of a relatively locally finite Hausdorff space X under an open continuous mapping? Is this true at least for Tychonoff spaces Y, or for metrizable spaces Y? What if Y is the simplest infinite compactum, that is, the convergent sequence?* [1]

We first answer the first two questions in the negative. Then, a relatively locally finite Hausdorff space and a continuous open surjective mapping from this space to the convergent sequence are constructed, demonstrating that the last question can be answered in the affirmative.

2. Images of relatively locally finite Hausdorff spaces

Recall that a space is *feebly compact* if every locally finite family of pairwise disjoint open subsets is finite. (See [3].) A space is *locally feebly compact* if every point has a feebly compact neighborhood.

The following is Corollary 5 in [1].

Proposition 2.1. *Every relatively locally finite Hausdorff space is locally feebly compact.*

Proposition 2.2. *Continuous open images of relatively locally finite Hausdorff spaces are locally feebly compact.*

Proof: Let X be a relatively locally finite Hausdorff space and $f : X \to Y$ be a continuous open mapping onto a Hausdorff space Y. Let $q \in X$ and $p = f(q) \in Y$.

By Proposition 2.1, q has an open neighborhood U such that $q \in U \subset V$ and V is feebly compact. $f(U)$ is an open neighborhood of p in Y and $f(U) \subset f(V)$, which is a feebly compact subspace of Y, since continuous images of feebly compact spaces are feebly compact. Thus, Y is locally feebly compact. $\square$

We can now answer the first two questions of Arhangel'skii and Gordienko.

Example 2.3. $\mathbb{Q}$ is a space which is metrizable (and thus Tychonoff and Hausdorff) but not locally feebly compact, so it is not the continuous open image of any relatively locally finite Hausdorff space.

3. A relatively locally finite preimage of the convergent sequence

While not every metrizable space can be represented as an image of a relatively locally finite Hausdorff space under a continuous open mapping, we will answer the last of the questions in the affirmative by constructing a relatively locally finite Hausdorff space X and a continuous open surjective function

$$f : X \to \{1/n : n \in \mathbb{N}\} \cup \{0\}.$$

For each $n \in \omega$, an open subspace X_n of X will be constructed and the function f will be defined so that $f(X_n) = \{1/n\}$ for $n \geq 1$ and $f(X_0) = \{0\}$.

$\omega^* = \beta\omega \setminus \omega$ is $\omega-$resolvable, so we can find ω pairwise disjoint dense subsets $\{D_n : n \geq 1\}$ of ω^*, each of cardinality $\mathfrak{c}$.

Index $[D_n]^\omega = \{C_n^\gamma : \gamma < \mathfrak{c}\}$.

Note that for each $\gamma < \mathfrak{c}$ and $n \geq 1$, $\left|\overline{C_n^\gamma}^{\omega^*}\right| = 2^{\mathfrak{c}}$.

Thus, we can select sets

$$E_1^\gamma \subset \overline{C_1^\gamma}^{\omega^*} \setminus \left(\bigcup\{D_n : n \geq 1\} \cup \bigcup\{E_1^\beta : \beta < \gamma\}\right)$$

such that $|E_1^\gamma| = \mathfrak{c}$ for each $\gamma < \mathfrak{c}$.

For each $n > 1$, suppose sets E_m^γ have been defined for all $\gamma < \mathfrak{c}$ and for all $m < n$, and let

$$E_n^\gamma \subset \overline{C_n^\gamma}^{\omega^*} \setminus \left(\bigcup\{D_m : m \geq 1\} \cup \bigcup\{E_m^\beta : \beta \in \mathfrak{c}, m < n\} \cup \bigcup\{E_n^\beta : \beta < \gamma\}\right)$$

so that $|E_n^\gamma| = \mathfrak{c}$ for each $\gamma \in \mathfrak{c}$.

For each $n \geq 1$, define

$$X_n = D_n \cup \bigcup\{E_n^\gamma : \gamma < \mathfrak{c}\}.$$

By the construction, $\{X_n : n \geq 1\}$ is a pairwise disjoint family of subsets of ω^*.

Let

$$X_0 = \omega^* \setminus \bigcup\{X_n : n \geq 1\},$$

and define

$$X = \bigcup\{X_n : n \in \omega\}.$$

Note that $|X_0| = 2^{\mathfrak{c}}$ since $|\bigcup\{X_n : n \geq 1\}| = \mathfrak{c}$.

The point set of X is the same as that of ω^*, but the topology τ on X will be defined as follows:

The sets X_n, where $n \geq 1$, are defined to be clopen in X, and the sets D_n, where $n \geq 1$, are defined to be open in X_n, and thus in X. This implies that for all $n \geq 1$, $\bigcup\{X_m : m \geq n\}$ and $\bigcup\{D_m : m \geq n\}$ are open subspaces of X.

Furthermore, $U \in \tau(X)$ if

(1) $p \in X_n \cap U$ for some $n \geq 1$ implies that there is a $V \in \tau(\omega^*)$ such that $p \in V$ and $V \cap D_n \subseteq U$, and
(2) $p \in X_0 \cap U$ implies that there is a $V \in \tau(\omega^*)$ and an $n \in \omega$ such that $p \in V$ and $V \cap (\bigcup\{D_m : m \geq n\}) \subseteq U$.

It is straightforward to verify that this defines a Hausdorff topology on X that is strictly stronger than $\tau(\omega^*)$. To prove that X is relatively locally finite, we use the following lemma.

Lemma 3.1. *(1) If $A \subset D_m$ for some $m \geq 1$, and $p \in X_m \setminus D_m$, then $p \in \overline{A}^X$ if and only if $p \in \overline{A}^{X_m}$ if and only if $p \in \overline{A}^{\omega^*}$.*

(2) If $p \in X_0$, $A \subset \bigcup\{D_m : m \geq 1\}$ is infinite, and $A \cap D_m$ is finite for all m, then $p \in \overline{A}^X$ if and only if $p \in \overline{A}^{\omega^*}$.

Proof: First, note that in both parts of the lemma, one direction follows from the fact that the topology on X is stronger than the topology on ω^*, so $\overline{A}^X \subseteq \overline{A}^{\omega^*}$ for all $A \subseteq X$.

(1) Suppose $A \subset D_m$ for some $m \geq 1$, and $p \in X_m \setminus D_m$.

Suppose $p \in \overline{A}^{\omega^*}$. Let $p \in U \in \tau(X)$ (if and only if $p \in U \cap X_m \in \tau(X_m)$). There is a $V \in \tau(\omega^*)$ such that $p \in V$ and $V \cap D_m \subset U$ (if and only if $V \cap D_m \subset U \cap X_m$). We then have:

$$\emptyset \neq V \cap A = V \cap A \cap D_m \subset U \cap A.$$

Thus, $p \in \overline{A}^X$.

(2) Suppose $p \in X_0$, $A \subset \bigcup\{D_m : m \geq 1\}$ is infinite, and $A \cap D_m$ is finite for all $m \in \omega$.

Suppose $p \in \overline{A}^{\omega^*}$. Let $p \in U \in \tau(X)$.

$$U \supseteq \{p\} \cup \left(V \cap \bigcup\{D_m : m \geq n\}\right)$$

for some $V \in \tau(\omega^*)$ and $n \geq 1$. $V \cap A$ is infinite, because $p \in \overline{A}^{\omega^*}$, so $V \cap A \cap D_m \neq \emptyset$ for infinitely many m, and in particular for infinitely many $m \geq n$. Thus, $U \cap A \neq \emptyset$, and $p \in \overline{A}^X$. □

Theorem 3.2. *X is relatively locally finite.*

Proof: First, suppose $p \in X_m$ for some $m \geq 1$. An open set in X containing p is $U = V \cap (D_m \cup \{p\})$ where $p \in V \in \tau(\omega^*)$. Let $S \in [U]^\omega$. Without loss of generality, $S \in [V \cap D_m]^\omega$. So $S = C_m^\gamma$ for some γ, and by Lemma 3.1,

$$(X_m \setminus D_m) \cap \overline{S}^X = (X_m \setminus D_m) \cap \overline{S}^{\beta\omega} \supseteq E_m^\gamma,$$

so $\left|(X_m \setminus D_m) \cap \overline{S}^X\right| \geq \mathfrak{c}$. Thus, S is not closed in X, and X is relatively locally finite at p. (By Lemma 3.1, S is also not closed in X_m; thus, X_m is relatively locally finite at p.)

Now suppose $p \in X_0$. An open set in X containing p is

$$W = V \cap \left(\{p\} \cup \bigcup\{D_m : m \geq n\}\right),$$

where $p \in V \in \tau(\omega^*)$. If $S \in [W]^\omega$, then $S \setminus \{p\} \in [\bigcup\{D_n : n \geq 1\}]^\omega$.

If $S \cap D_m$ is infinite for some $m \geq 1$, then S is not closed in X by the preceding argument. Suppose otherwise. Then $S \cap D_m \neq \emptyset$

for infinitely many m. S is an infinite subset of ω^*, so $\left|\overline{S}^{\omega^*}\right| = 2^{\mathfrak{c}}$. Since

$$\left|\bigcup\{X_m \mid m \geq 1\}\right| = \mathfrak{c}\,,$$

it follows that

$$\left|X_0 \cap \overline{S}^{\beta\omega}\right| = 2^{\mathfrak{c}}.$$

By Lemma 3.1,

$$\left|X_0 \cap \overline{S}^{X}\right| = 2^{\mathfrak{c}},$$

and thus that S is not closed in X, so X is relatively locally finite at p. □

Define a function $f : X \to Y = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ by:

$$f(x) = \begin{cases} 1/n, & \text{if } x \in X_n,\ n \geq 1; \\ 0, & \text{if } x \in X_0. \end{cases}$$

Theorem 3.3. *$f : X \to Y$ is a continuous open surjection.*

Proof: By definition, f is onto.

Clearly, f is continuous at all points of X_n for $n \geq 1$.

Let $p \in X_0$. $T_n = \{0\} \cup \{1/m : m \geq n\}$ is a basic open neighborhood of $f(p) = 0$ in Y. $X_0 \cup \bigcup\{X_m : m \geq n\}$ is a basic open neighborhood of p in X and $f\left(X_0 \cup \bigcup\{X_m : m \geq n\}\right) = T_n$, so f is continuous at p.

Let V be an open set in X. If $V \cap X_0 = \emptyset$, then $f(V) \subseteq \{1/n : n \geq 1\}$, so $f(V)$ is open in Y. On the other hand, suppose there is a point $p \in V \cap X_0$.

There is a set $W \in \tau(\omega^*)$ and an $n \in \omega$ such that $p \in W$ and

$$W \cap \bigcup\{D_m : m \geq n\} \subseteq V.$$

Since the sets D_m are all dense in ω^*, $W \cap D_m \neq \emptyset$ for all m. In particular, $V \cap D_m \neq \emptyset$ for all $m \geq n$. Hence,

$$f(V) \supset \{0\} \cup \{1/m : m \geq n\},$$

and $f(V)$ is open in Y. □

Every topological space is the image of a discrete space (and thus a relatively locally finite space) under a continuous mapping. However, no nondiscrete space is the image of a discrete space under a closed continuous mapping or an open continuous mapping. In both cases, the situation differs for relatively locally finite Hausdorff spaces: A space which is not locally feebly compact cannot be represented as the continuous open image of any relatively locally finite space. This paper provides an example of a relatively locally finite Hausdorff space whose open continuous image is the non-relatively locally finite convergent sequence, and Arhangel'skii and Gordienko provided the example of the Katětov extension of the countable discrete space, which is a relatively locally finite Hausdorff space whose closed continuous image is the convergent sequence. This raises the question as to whether the convergent sequence is the closed and open continuous image of some relatively locally finite Hausdorff space. More generally, a characterization of those spaces which can be represented as images of relatively locally finite Hausdorff spaces under open (or closed) continuous mappings is needed.

4. Relationship to Baire spaces

The following question was posed to the author by Dave Lutzer: *Are relatively locally finite Hausdorff spaces Baire spaces?* Recall that a space is Baire if the intersection of any countable family of dense open sets is also dense. While the example described in the previous section is indeed a Baire space, we also construct a relatively locally finite Hausdorff space which is not Baire.

Theorem 4.1. *X is a Baire space.*

Proof: Let $\{U_n : n \in \mathbb{N}\}$ be a countable family of dense open sets in the relatively locally finite space X constructed in the previous section. Each $U_n \supseteq \bigcup\{V_n \cap D_m : m \geq 1\}$, where V_n is a dense open set in ω^* and $\{D_m : m \in \mathbb{N}\}$ are dense open sets in ω^* as defined earlier. Note that since ω^* is a Baire space, $\cap\{V_n : n \in \mathbb{N}\}$ is dense in ω^*.

To see that $\bigcap\{U_n : n \in \mathbb{N}\}$ is dense in X, let $W \in \tau(X)$. If $W \cap X_0 \neq \emptyset$, then there exists a set $O \in \tau(\omega^*)$ and an $N \in \mathbb{N}$ such that $O \cap \bigcup\{D_m : m \geq N\} \subset W$. Thus,

$$W \cap \bigcap\{U_n : n \in \mathbb{N}\} \supseteq \bigcup\{O \cap D_m \cap \bigcap V_n : m \geq N\}.$$

Each intersection in the union is nonempty because $O \cap D_m$ is a nonempty open set in ω^*.

If $W \cap X_N \neq \emptyset$ for some $N > 0$, then there exists a set $O \in \tau(\omega^*)$ such that $O \cap D_N \subset W$. Thus,

$$W \cap \bigcap \{U_n : n \in \mathbb{N}\} \supseteq O \cap D_N \cap \bigcap V_n \neq \emptyset. \qquad \square$$

This shows that X is a Baire space, which motivates a search for a relatively locally finite space which is *not* Baire.

Let $\mathbb{I}$ denote the usual unit interval and let $E\mathbb{I}$ be the absolute of $\mathbb{I}$. The absolute consists of the convergent ultrafilters on the algebra of regular open sets of $\mathbb{I}$ [3]. We need two facts about $E\mathbb{I}$.

Proposition 4.2. (1) *$E\mathbb{I}$ is separable (with D denoting a separable dense subset of $E\mathbb{I}$).*

(2) *Every infinite closed subset of $E\mathbb{I}$ is uncountable.*

Proof: (1) Let $k : E\mathbb{I} \to \mathbb{I}$ be the absolute map (see [3]) that is perfect, irreducible, $\theta-$continuous, and onto. Let Q be a countable, dense subset of $\mathbb{I}$ and for each $r \in Q$, select a point $d_r \in k^{\leftarrow}(r)$, and let $D = \{d_r : r \in Q\}$. D is a countable subset of $E\mathbb{I}$. To see that D is dense, let U be a nonempty open subset of $E\mathbb{I}$. Then $k^{\#}[U] = \{x \in \mathbb{I} : k^{\leftarrow}(x) \subseteq U\}$ is a nonempty open subset of $\mathbb{I}$ and so contains a point $r \in Q$. Now $d_r \in k^{\leftarrow}(r) \subseteq U$ and $U \cap D \neq \emptyset$.

(2) $E\mathbb{I}$ is a compact, extremally disconnected space, so every infinite closed subset has cardinality $|\beta\omega|$. (See [2, 6.2.G(b)].) $\square$

Let Y denote $E\mathbb{I}$ with this topology: A subset $U \subseteq E\mathbb{I}$ is open in Y if for each $p \in U$, there is an open set $V \in \tau(E\mathbb{I})$ such that $p \in V$ and $V \cap D \subseteq U$.

Theorem 4.3. *Y is a relatively locally finite Hausdorff space which is not Baire.*

Proof: D is a countable dense open subset of Y. Since $\{D \backslash \{d\} : d \in D\}$ is a countable family of dense, open sets whose intersection is empty, Y is not Baire.

Let $p \in Y$ and let $U = \{p\} \cup (V \cap D)$, where $V \in \tau(E\mathbb{I})$, be a basic open neighborhood of p in Y. Let $C \subset U$ be closed in Y. By an argument similar to the proof of Lemma 3.1, $C \cap D = F \cap D$ where F is closed in $E\mathbb{I}$, so $C \setminus \{p\}$ is closed in $E\mathbb{I}$. Since $C \subset D$, C is at most countable. By Proposition 4.2, infinite closed subsets of $E\mathbb{I}$ are uncountable, so C must be finite. Thus, Y is relatively locally finite. $\square$

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