Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 355-365

MONOTONICALLY D-SPACES

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Abstract. Call X monotonically D if we can assign a closed discrete $D(\mathcal{U})$ to every open neighborhood assignment $\mathcal{U} = \{U(x) : x \in X\}$ in such a way that $X = \bigcup\{U(x) : x \in D(\mathcal{U})\}$ and $D(\mathcal{U}) \subseteq D(\mathcal{V})$ if $V(x) \subseteq U(x)$ for all x. The Michael line and countable first-countable spaces are monotonically D. The Sorgenfrey line, the unit interval, and $\omega_1 + 1$ are not.

Introduction

Monotone versions of topological properties are often interesting and useful. The most well-known such property is monotone normality; see e.g., [1], [3], [6]. Monotone paracompactness is studied in [5]. In the spring of 2004, David Lutzer visited Auburn University and presented the monotone Lindelöf property [2], which prompted the definition of monotonically D-spaces.

Recall that a family $\mathcal{U} = \{U(x) : x \in X\}$ is an *open neighborhood assignment* (ONA for short) for a topological space X if $x \in U(x)$ and $U(x)$ is open for each x. X is called a D-*space* (see [4]) if for every ONA $\mathcal{U} = \{U(x) : x \in X\}$ there is a closed discrete subset $D(\mathcal{U})$ of X such that $\{U(x) : x \in D(\mathcal{U})\}$ covers X. We call X *monotonically* D if, for each ONA $\mathcal{U}$, we can pick a closed discrete subset $D(\mathcal{U})$ with $X = \bigcup\{U(x) : x \in D(\mathcal{U})\}$ such that larger closed

2000 *Mathematics Subject Classification.* Primary 54D20, 54D30, 54F05.

Key words and phrases. generalized left separated, GLS, Michael line, monotonically D-spaces, non-Archimedean, $\omega_1 + 1$, Sorgenfrey line, unit interval.

The first author was supported by the Summer Research Award (2004) from the University of Louisiana at Lafayette and was partially supported by the National Science Fund of Bulgaria Grant MM-1105/2001.

discrete sets correspond to finer ONA's; i.e., if $\mathcal{V} = \{V(x) : x \in X\}$ is an ONA with $V(x) \subseteq U(x)$ for all x, then $D(\mathcal{V}) \supseteq D(\mathcal{U})$.

In section 1, we describe a class of monotonically D-spaces that includes all countable first-countable T_1-spaces and the Michael line, and we prove some preservation properties. In section 2, we give examples of spaces that are not monotonically D, including a countable example, $\omega_1 + 1$, the unit interval, and uncountable subsets of the Sorgenfrey line. In section 3, we define cofinally monotonically D-spaces and show that these are the same as D-spaces.

1. Some classes of monotonically D-spaces

Theorem 1.1. *If $X = \{x_n : n \in \omega\}$ is a compact, countable T_1-space, then X is monotonically D.*

Proof: For any ONA $\mathcal{U} = \{U(x_n) : n \in \omega\}$, let $k_{\mathcal{U}} = \min\{k : X = \bigcup_{n \leq k} U(x_n)\}$, and let $D(\mathcal{U}) = \{x_n : n \leq k_{\mathcal{U}}\}$. □

A topological space X is *generalized left separated* (GLS for short) [4] if there is a reflexive (not necessarily transitive) binary relation $\preceq$ on X, called a GLS relation, such that (a) every non-empty closed subset F has a $\preceq$-minimal element (where an $x \in F$ is $\preceq$-minimal for F if the conditions $y \in F$ and $y \preceq x$ imply that $y = x$), and (b) the set $\{y \in X : x \preceq y\}$ is open for each $x \in X$.

It was shown in [4] that all finite powers of the Sorgenfrey line are GLS and that every GLS-space is a D-space. It was observed that both compact Hausdorff spaces without isolated points and uncountable separable metrizable spaces are not GLS. (A countable space with the co-finite topology is a compact, first-countable, T_1, GLS-space with no isolated points.) If $\preceq$ is a GLS relation on a T_1-space X, then $\preceq$ is antisymmetric; i.e., if $x \preceq y$ and $y \preceq x$, then $x = y$ (for otherwise, the closed set $\{x, y\}$ would have no $\preceq$-minimal element).

Theorem 1.2. *Suppose that $(X, \preceq)$ is a GLS T_1-space and, for each $x \in X$, $\mathcal{B}_x$ is a local base at x such that*

1. *$\mathcal{B}_x$ is well-ordered under reverse inclusion;*
2. *if $B \in \mathcal{B}_x$, then $B \subseteq \{y \in X : x \preceq y\}$;*

3. if $x \preceq y$ with $x \neq y$, $B \in \mathcal{B}_x$, $B' \in \mathcal{B}_y$, and $y \in B$, then $B' \subseteq B$.

Then X is monotonically D.

Proof: Given any ONA $\mathcal{U} = \{U(x) : x \in X\}$, let $U'(x)$ be the maximal (with respect to inclusion) element of $\mathcal{B}_x$ such that $x \in U'(x) \subseteq U(x)$. Let $D(\mathcal{U}) = \{z \in X : z \notin U'(y) \text{ if } y \in X \setminus \{z\}\}$. We claim that $X = \bigcup\{U'(z) : z \in D(\mathcal{U})\}$. Fix any x and let $F(x) = \{y \in X : x \in U'(y)\}$. Then $F(x)$ is closed, for if $w \in \overline{F(x)}$, then pick $y \in F(x)$ with $y \in U'(w)$. Then $w \preceq y$. If $w \neq y$, then $U'(y) \subseteq U'(w)$. Hence, $x \in U'(w)$ and $w \in F(x)$ (use that $x \in U'(y)$ since $y \in F(x)$), which completes the proof that $F(x)$ is closed. Let z be a $\preceq$-minimal element of $F(x)$. Then $z \in D(\mathcal{U})$ for if $z \in U'(y)$ for some $y \neq z$, then $y \preceq z$ and $U'(z) \subseteq U'(y)$; hence, $y \in F(x)$, contradicting $\preceq$-minimality of z in $F(x)$. Thus, every x is contained in $U'(z)$ for some $z \in D(\mathcal{U})$. The set $D(\mathcal{U})$ is closed and discrete for if $x \in U'(z)$ for some $z \in D(\mathcal{U})$, then $U'(z)$ is a neighborhood of x such that $U'(z) \cap D(\mathcal{U}) = \{z\}$ (use also that X is T_1).

Now assume that $\mathcal{V} = \{V(x) : x \in X\}$ is an ONA for X, finer than $\mathcal{U}$ (and that $V'(x)$, for each x, and $D(\mathcal{V})$ are defined, following the above procedure). The inclusion $D(\mathcal{U}) \subseteq D(\mathcal{V})$ is clear from the definition of these sets, since $V'(y) \subseteq U'(y)$ for all y. □

Corollary 1.3. (a) *Every countable, first-countable T_1-space X is monotonically D.*

(b) *Non-stationary subsets of ω_1 are monotonically D.*

Proof: (a) List X as $\{x_n : n \in \omega\}$ and let $x_n \preceq x_k \iff n \leq k$. Clearly, $\preceq$ is a GLS relation on X. For each n choose a local base $\mathcal{B}_n = \{B_i(x_n) : i \in \omega\}$ at x_n such that $B_{i+1}(x_n) \subseteq B_i(x_n) \subseteq \{x_k : n \leq k\}$ for every $i \in \omega$. Fix n. There are only finitely many $m < n$, and for each such m there are only finitely many j such that $x_n \in B_j(x_m)$. Since the intersection of these $B_j(x_m)$ is open, we may remove (by induction on $n = 1, 2, ...$) the elements of $\mathcal{B}_n$ that are not contained in this intersection to ensure that if $m < n$ and $x_n \in B_j(x_m)$ for some j, then $B_i(x_n) \subseteq B_j(x_m)$ for all i.

(b) Each non-stationary subset of ω_1 is the topological sum of countable, first-countable spaces. □

Recall that a space X is *non-Archimedean* ([7], [8]; see also [9]) if it has a base $\mathcal{B}$ of *rank 1*; i.e., if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$, then $B \subseteq B'$ or $B' \subseteq B$. Equivalently, X is non-Archimedean if it has a *tree-base*, i.e., a base which is a tree under reverse inclusion (and hence has rank 1) [10].

Corollary 1.4. *Every non-Archimedean, GLS T_1-space X (hence, every non-Archimedean, countable T_1-space) is monotonically D.*

Proof: Let $\preceq$ be a GLS relation on X and let $\mathcal{B}$ be a tree-base for X. Let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B \subseteq \{y \in X : x \preceq y\}\}$. Then conditions 1 and 2 of Theorem 1.2 are satisfied. We verify that 3 is also satisfied. If $B \in \mathcal{B}_x$, $B' \in \mathcal{B}_y$, and $y \in B$, then either $B \subseteq B'$ or $B' \subseteq B$. If $x \preceq y$ and $x \neq y$, then $y \not\preceq x$; hence, $x \notin B'$ and $B' \subseteq B$. $\square$

Corollary 1.5. (a) *Suppose X is a T_1-space and A is a closed GLS subspace which is* non-Archimedean in X*; i.e., there is a family $\mathcal{B}$ of open subsets of X which is a tree under reverse inclusion and contains a local base at x (in X) for every $x \in A$. Suppose also that $X \setminus A$ is discrete. Then X is monotonically D.*

(b) *The Michael line M is monotonically D.*

Proof: (a) If $\mathcal{B}' = \mathcal{B} \cup \{\{y\} : y \in X \setminus A\}$, then $\mathcal{B}'$ is a tree-base for X. The GLS relation $\preceq$ on A can be extended to a GLS relation $\preceq$ on X as follows: Let $x \preceq y$ if $x \in A$ and $y \in X \setminus A$, and let $x \preceq x$ if $x \in X \setminus A$. If $x \in A$, then $\{y \in X : x \preceq y\} = \{y \in A : x \preceq y\} \cup (X \setminus A)$, which is open. If $x \in X \setminus A$, then $\{y \in X : x \preceq y\} = \{x\}$, which is open. If $F \subseteq X \setminus A$, then every element of F is $\preceq$-minimal. If F is closed and $F \cap A \neq \emptyset$, then the $\preceq$-minimal elements for $F \cap A$ are also $\preceq$-minimal for F. Thus, the preceding corollary applies.

(b) The rationals are non-Archimedean in M (take $\mathcal{B}$ to be a suitable family of open intervals with irrational endpoints). $\square$

Remark 1.6. The subspace A of all non-isolated points of the space X from Example 2.2 is a converging sequence; hence, it is non-Archimedean, but it is not non-Archimedean in X.

Theorem 1.7. *(a) If X is monotonically D and F is a closed subset of X, then F is monotonically D.*

(b) If the subspace Z of all non-isolated points of a space X is discrete (in particular, is a singleton), then X is monotonically D.

(c) If X is monotonically D and $f : X \to Y$ is a continuous closed surjection, then Y is monotonically D.

(d) Suppose $Y = \bigcup_{i\in\omega} F_i = \bigcup_{i\in\omega} \operatorname{int}(F_i)$ where each F_i is a closed monotonically D-subspace of Y, and $F_i \subseteq F_{i+1}$. Then Y is monotonically D. Thus, if X is a normal monotonically D-space and Y is an open F_σ subspace, then Y is monotonically D.

Proof: (a) Let $W = X \setminus F$. For every relatively open $U \subseteq F$, let $U' = U \cup W$. Then U' is open in X and $U = U' \cap F$ (where U' is maximal with this property). If $\mathcal{U} = \{U_x : x \in F\}$ is a relatively ONA for F, let $\mathcal{U}' = \{U'_x : x \in F\} \cup \{W_x : x \in W\}$ where $W_x = W$ for each $x \in W$. Let $D_{\mathcal{U}'}$ be the closed discrete subset of X corresponding to $\mathcal{U}'$, and let $D_{\mathcal{U}} = F \cap D_{\mathcal{U}'}$. Since $W_x \cap F = \emptyset$ for each $x \in W$, $\bigcup\{U_x : x \in D_{\mathcal{U}}\} = F$. If $\mathcal{V} = \{V_x : x \in F\}$ is another relatively ONA for F such that $V_x \subseteq U_x$ for all $x \in F$, then $V'_x \subseteq U'_x$, too; hence, $D_{\mathcal{U}'} \subseteq D_{\mathcal{V}'}$ and $D_{\mathcal{U}} \subseteq D_{\mathcal{V}}$.

(b) For every ONA $\mathcal{U} = \{U(x) : x \in X\}$, the set $D_{\mathcal{U}} = Z \cup (X \setminus \bigcup\{U(x) : x \in Z\})$ works (use also that Z must be closed).

(c) Suppose that $\mathcal{V} = \{V_y : y \in Y\}$ is an ONA for Y. For each $y \in Y$ and $x \in f^{-1}(y)$, let $U_x = f^{-1}(V_y)$. Then $\mathcal{U} = \{U_x : x \in X\}$ is an ONA for X, and if $D_{\mathcal{U}}$ is the corresponding closed discrete subset of X, then $D_{\mathcal{V}} = f(D_{\mathcal{U}})$ works.

(d) Let $H_0 = F_0$ and, for each $i \geq 1$, let $H_i = F_i \setminus \operatorname{int}_Y(F_{i-1})$. Each H_i is closed in F_i, hence monotonically D, and $Y = \bigcup_{i\in\omega} H_i$. If D_i is closed discrete in H_i, then $\bigcup_{i\in\omega} D_i$ is closed discrete in Y, since the family $\{H_i : i \in \omega\}$ is locally finite in Y. Given an ONA $\mathcal{U} = \{U_y : y \in Y\}$ for Y, let D_i be the closed discrete subspace of H_i corresponding to $\{U_y \cap H_i : y \in H_i\}$, and let $D_{\mathcal{U}} = \bigcup_{i\in\omega} D_i$. $\square$

Remark 1.8. In relation to part (b), if the subspace Z of all non-isolated points of a space X is a D-space, then X is a D-space. Here we cannot replace D by monotonically D, for if X is the space from Example 2.2, then the subspace of all non-isolated points is monotonically D, but X is not. Although discrete spaces are GLS, part (b) cannot be deduced from Corollary 1.5(a), since even if Z is a singleton, it need not be non-Archimedean in X.

F_σ-subspaces of monotonically D T_1-spaces need not be monotonically D. Let X be countable but not monotonically D (see

Example 2.2), and take $X \cup \{p\}$ where p is a new point whose neighborhoods are co-finite. Clearly, X is F_σ in $X \cup \{p\}$, which is monotonically D by Theorem 1.1. We do not know if monotonically D is hereditary, or F_σ-hereditary, in the class of Hausdorff spaces.

2. Spaces that are not monotonically D

Example 2.1. $\omega_1 + 1$ *is not monotonically D (although it is D).*

Proof: For each $\alpha < \omega_1$, let $\mathcal{U}^\alpha = \{U^\alpha(\delta) : \delta \leq \omega_1\}$ where $U^\alpha(\delta) = [0, \delta]$ if $\delta < \omega_1$, and $U^\alpha(\omega_1) = (\alpha, \omega_1]$. Then $U^\alpha(\delta)$ is a neighborhood of δ for all $\delta \leq \omega_1$, and hence each $\mathcal{U}^\alpha$, $\alpha < \omega_1$, is an ONA. Assume ω_1+1 were monotonically D, and let D^α be the closed discrete set corresponding to $\mathcal{U}^\alpha$. Since $\alpha \notin (\alpha, \omega_1] = U^\alpha(\omega_1)$, it follows that $\alpha \in U^\alpha(\delta(\alpha)) = [0, \delta(\alpha)]$ for some $\delta(\alpha) \in D^\alpha$ with $\alpha \leq \delta(\alpha) < \omega_1$. Define $\alpha_0 = 0$ and $\alpha_{n+1} = \delta(\alpha_n) + 1$, and let $\beta = \sup_{n<\omega} \alpha_n$. Since $\mathcal{U}^\beta$ is a finer ONA than each of the $\mathcal{U}^{\alpha_n}$, D^β must contain all D^{α_n}; hence, all $\delta(\alpha_n)$, $n < \omega$, a contradiction. $\square$

Example 2.2. *There is a countable space X that is not monotonically D (although it is D, since every σ-compact T_1-space is D).*

Proof: Let X be a subset (but not a subspace) of the plane consisting of the origin $(0,0)$, all points of the sequence $T = \{(\frac{1}{n}, 0) : n \geq 1\}$, and, for each $n \geq 1$, all points of the sequence $S(n) = \{(\frac{1}{n}, \frac{1}{m}) : m \geq 1\}$. Let $T(n)$ be the tail of T determined by n (i.e., $T(n) = \{(\frac{1}{n'}, 0) : n' \geq n\}$) and $S(n,m)$ be the tail $\{(\frac{1}{n}, \frac{1}{m'}) : m' \geq m\}$ of $S(n)$ determined by m (thus, $S(n) = S(n,1)$). By definition, all points $(\frac{1}{n}, \frac{1}{m})$ are isolated. The basic neighborhoods of $(\frac{1}{n}, 0)$ are of the form $S(n,m) \cup \{(\frac{1}{n}, 0)\} = \overline{S(n,m)}$, $m \geq 1$. The basic neighborhoods of $(0,0)$ are of the form $\{(0,0)\} \cup T(n) \cup (\cup_{n' \geq n} S(n', m_{n'}))$, where $n \geq 1$, and $m_{n'} \geq 1$ for each $n' \geq n$. In other words, basic neighborhoods of $(0,0)$ contain a tail $T(n)$ for some n, together with a tail $S(n', m_{n'})$ for each $(\frac{1}{n'}, 0) \in T(n)$.

For the remainder of the proof, we will consider only ONA's that assign $\overline{S(n)} = S(n) \cup \{(\frac{1}{n}, 0)\}$ to $(\frac{1}{n}, 0)$ for each $n \geq 1$ and assign $\{(\frac{1}{n}, \frac{1}{m})\}$ to $(\frac{1}{n}, \frac{1}{m})$. In order to specify such an ONA, we need only to specify the neighborhood assigned to $(0,0)$. Assume X were monotonically D. If U (with subscripts) is a neighborhood assigned to $(0,0)$, then $\mathcal{U}$ and D (with subscripts) will denote the resulting ONA and the corresponding closed discrete set, respectively.

Let $U_1 = \{(0,0)\} \cup (\cup_{n'>1} \overline{S(n')})$ and $U_{1,m} = U_1 \cup \overline{S(1,m+1)}$ for each m. Then $U_1 \subseteq U_{1,m}$; hence, $\mathcal{U}_1$ is finer than $\mathcal{U}_{1,m}$ and $D_1 \supseteq D_{1,m}$ for all m. Since D_1 is closed and discrete, there is an m_1 such that $S(1,m_1) \cap D_1 = \emptyset$. Then $S(1,m_1) \cap D_{1,m_1} = \emptyset$. Since $(1, \frac{1}{m_1}) \notin U_{1,m_1}$, there are only two sets in $\mathcal{U}_{1,m_1}$ that contain $(1, \frac{1}{m_1})$, namely $\{(1, \frac{1}{m_1})\}$ and $\overline{S(1)}$ (assigned to $(1,0)$). Since $(1, \frac{1}{m_1}) \in S(1,m_1)$, it follows that $(1, \frac{1}{m_1}) \notin D_{1,m_1}$; hence, $(1,0) \in D_{1,m_1}$. By induction, assume that $n > 1$ and $m_{n'}$ have been defined for all $n' < n$. Let $U_n = \{(0,0)\} \cup (\cup_{n'>n} \overline{S(n')}) \cup (\cup_{n'<n} \overline{S(n', m_{n'}+1)})$ and $U_{n,m} = U_n \cup \overline{S(n,m+1)}$ for each m. Then $D_n \supseteq D_{n,m}$ for all m, and an argument similar to the above shows there is an m_n such that $(\frac{1}{n}, 0) \in D_{n,m_n}$.

Let $U = \{(0,0)\} \cup (\cup_{n \geq 1} \overline{S(n, m_n+1)})$. Then $U \subseteq U_{n,m_n}$ for all n. Hence, the corresponding closed discrete D must contain all D_{n,m_n}. It follows that $\{(\frac{1}{n}, 0) : n \geq 1\} \subseteq D$, a contradiction. $\square$

Part (b) of the following theorem shows that non-Archimedean spaces need not be monotonically D.

Theorem 2.3. *None of the following spaces is monotonically D:*
(a) *the closed unit interval* $I = [0,1]$*,*
(b) *the Cantor set* C*,*
(c) *any uncountable Polish space* P*,*
(d) *any compact Hausdorff space* Y *with no isolated points.*

Proof: Part (b) follows from (a) (and Theorem 1.7(c)) since I is the closed continuous image of C. Part (c) follows from (b) (and Theorem 1.7(a)) since every uncountable Polish space P contains a closed copy of C. For part (d), observe that Y contains a compact subspace K which can be mapped by a quotient (and closed) map onto the Cantor set C. We prove (a).

Assume I were monotonically D and define the ONA $\mathcal{U}_1 = \{I_x : x \in I\}$ where $I_x = I$ for each x. Let F_1 be the corresponding closed discrete set. Since I is compact, F_1 must be finite. For each $x \in F_1$, let U_x be a small neighborhood of x so that the Lebesgue measure of the union of these U_x is less than $\frac{1}{2}$, i.e., $m(\bigcup\{U_x : x \in F_1\}) < \frac{1}{2}$. Let $\mathcal{U}_2 = \{I_x : x \in I \setminus F_1\} \cup \{U_x : x \in F_1\}$, and let F_2 be the corresponding closed and discrete, hence finite, set. Since $\mathcal{U}_2$ is finer than $\mathcal{U}_1$, $F_1 \subseteq F_2$. The set F_2 is strictly larger than F_1 since the neighborhoods assigned to points in F_1 only cover a set of measure less than $\frac{1}{2}$. For each $x \in F_2 \setminus F_1$, fix a neighborhood U_x so that $m(\bigcup\{U_x : x \in F_2 \setminus F_1\}) < \frac{1}{4}$. If $\mathcal{U}_3 = \{I_x : x \in I \setminus F_2\} \cup \{U_x : x \in F_2\}$ and F_3 is the corresponding finite set, then F_3 is strictly larger than F_2 (since the neighborhoods assigned to points in F_1, together with the neighborhoods assigned to points in $F_2 \setminus F_1$, can only cover a set of measure less than $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$). By induction, if $\mathcal{U}_n$ and F_n are defined, fix a neighborhood U_x for each $x \in F_n \setminus F_{n-1}$ such that $m(\bigcup\{U_x : x \in F_n \setminus F_{n-1}\}) < \frac{1}{2^n}$. If $\mathcal{U}_{n+1} = \{I_x : x \in I \setminus F_n\} \cup \{U_x : x \in F_n\}$, then the corresponding closed, discrete (and hence finite) set F_{n+1} is strictly larger than F_n, since the neighborhoods assigned to points in F_n can only cover a set of measure less than $1 - \frac{1}{2^n}$. Let $F' = \bigcup\{F_n : n = 1, 2, ...\}$ and $\mathcal{U} = \{I_x : x \in I \setminus F'\} \cup \{U_x : x \in F'\}$, and let F be the closed discrete set that corresponds to $\mathcal{U}$. Since $\mathcal{U}$ is finer than $\mathcal{U}_n$ for all n, F must contain each F_n. Since the F_n are strictly increasing, F must be infinite, a contradiction.[1] $\square$

We do not know if there is an uncountable subset of the real line (possibly a Lusin set) that is monotonically D.

Theorem 2.4. *A subspace X of the Sorgenfrey line is monotonically D if and only if X is countable.*

Proof: If X is countable, then apply Corollary 1.3(a).

If X is uncountable, then we may assume that X has no isolated points (otherwise, replace X with the closed set of all condensation points, i.e., points which have only uncountable neighborhoods). For each $y, t \in X$ with $y \leq t$, and for each $x \in X$, let $U_{y,t}(x) = (-\infty, t)$ if $x < y$, and let $U_{y,t}(x) = [x, \infty)$ if $x \geq y$. Then $\mathcal{U}_{y,t} = \{U_{y,t}(x) : x \in X\}$ is an ONA for X. If X were monotonically D, let $D_{y,t}$ denote the corresponding closed discrete set. Since $\mathcal{U}_{y,y}$ is finer than $\mathcal{U}_{y,t}$, $D_{y,y} \supseteq D_{y,t}$ for all $t \geq y$. There must be $t_y > y$ such that $y \in D_{y,t_y}$; to see this, assume the contrary and observe that

[1]Added in proof. Without a reference to measure theory, one can show that any countably compact, uncountable T_1-space in which points are G_δ is not monotonically D. In part (d), it is enough to assume that the compact Hausdorff space Y is not scattered.

$D_{y,t} \cap [y,t] \neq \emptyset$ (in order for t to be covered). If $d_t \in D_{y,t} \cap [y,t]$, then (assuming $y < d_t$ for all $t > y$) it follows that y is a limit point of $\{d_t : t > y\} \subseteq D_{y,y}$, a contradiction. For each y, fix $t_y > y$ such that $y \in D_{y,t_y}$. Then there is some n such that the set $Y = \{y : t_y - y > \frac{1}{n}\}$ is uncountable. For each $x \in X$, let $V(x) = [x, x + \frac{1}{n})$. If $y \in Y$ and $x < y$, then $x + \frac{1}{n} < t_y$ and hence, $V(x) \subseteq (-\infty, t_y) = U_{y,t_y}(x)$. If $y \in Y$ and $x \geq y$, then $V(x) \subseteq [x, \infty) = U_{y,t_y}(x)$. Thus, the ONA $\mathcal{V} = \{V(x) : x \in X\}$ is finer than $\mathcal{U}_{y,t_y}$ for all $y \in Y$, and hence, the closed discrete set D that corresponds to $\mathcal{V}$ must contain D_{y,t_y} for all $y \in Y$; in particular, D must contain y for all $y \in Y$, a contradiction. □

3. Monotone characterizations of D-spaces

Given a space X, let $\mathcal{N} = \{\mathcal{U} : \mathcal{U}$ is an ONA for $X\}$. If $\mathcal{U}, \mathcal{V} \in \mathcal{N}$, then the notation $\mathcal{V} \leq \mathcal{U}$ means that $V(x) \subseteq U(x)$ for every $x \in X$; we say that $\mathcal{V}$ is finer and $\mathcal{U}$ is coarser.

A mapping D with domain $\mathcal{N}$ is a D-*operator* if, for all $\mathcal{U} \in \mathcal{N}$,

(1) $D(\mathcal{U})$ is a closed and discrete subset of X, and

(2) $X = \bigcup\{U(x) : x \in D(\mathcal{U})\}$.

D is called *monotone* if, in addition, $\mathcal{V} \leq \mathcal{U}$ implies $D(\mathcal{U}) \subseteq D(\mathcal{V})$.

A family $\mathcal{K} \subseteq \mathcal{N}$ of ONA's is called *cofinal* in $\mathcal{N}$ if for every ONA $\mathcal{W} \in \mathcal{N}$ there is a finer ONA $\mathcal{U} \leq \mathcal{W}$ with $\mathcal{U} \in \mathcal{K}$.

Call a space X *cofinally monotonically* D if there exists a cofinal $\mathcal{K} \subseteq \mathcal{N}$ and a *partial* D-operator – defined on $\mathcal{K}$ only, and satisfying (1) and (2) above for all $\mathcal{U} \in \mathcal{K}$ – that is monotone on $\mathcal{K}$. Every monotonically D-space is cofinally monotonically D (with $\mathcal{K} = \mathcal{N}$).

Theorem 3.1. *X is cofinally monotonically D if and only if it is D.*

Proof: If X is cofinally monotonically D, let D be a partial D-operator with domain $\mathcal{K}$ for some cofinal $\mathcal{K} \subseteq \mathcal{N}$. (For this part, we do not need D to be monotone on $\mathcal{K}$). For any $\mathcal{W} \in \mathcal{N} \setminus \mathcal{K}$, pick a $\mathcal{U} \in \mathcal{K}$ finer than $\mathcal{W}$ and define $D(\mathcal{W}) = D(\mathcal{U})$ to get a D-operator with domain $\mathcal{N}$. (It follows that a space X is cofinally monotonically D if and only if there exists a D-operator D on $\mathcal{N}$ and a cofinal $\mathcal{K}$ such that the restriction $D|_{\mathcal{K}}$ is monotone.)

Conversely, if X is a D-space, let $\mathcal{K}$ be the class of ONA's $\mathcal{U} = \{U(x) : x \in X\}$ for which there is a closed discrete $D(\mathcal{U})$ such that

1. $X = \bigcup\{U(x) : x \in D(\mathcal{U})\}$,

2. $U(x) \cap D(\mathcal{U}) = \{x\}$ if $x \in D(\mathcal{U})$, and
3. $U(y) \cap D(\mathcal{U}) = \emptyset$ if $y \notin D(\mathcal{U})$.

If $\mathcal{U} \in \mathcal{K}$, then $D(\mathcal{U})$ is uniquely determined by the above three conditions. We cannot remove an x from $D(\mathcal{U})$ since x is only covered by $U(x)$. If $y \notin D(\mathcal{U})$, we cannot add y to $D(\mathcal{U})$ since that would violate condition 2 for some $x \in D(\mathcal{U})$ for which $y \in U(x)$.

Claim 1. $\mathcal{K}$ is cofinal in the class of all ONA's.

Proof of Claim. If $\mathcal{W}$ is any ONA, pick a closed discrete D such that $X = \bigcup\{W(x) : x \in D\}$. If $x \in D$, let $U(x) = W(x) \setminus (D \setminus \{x\})$. If $y \notin D$, let $U(y) = W(y) \setminus D$. Then $\mathcal{U} \leq \mathcal{W}$, and $\mathcal{U}$ is in $\mathcal{K}$ with $D(\mathcal{U}) = D$.

Claim 2. If $\mathcal{U}, \mathcal{V} \in \mathcal{K}$, with $\mathcal{V} \leq \mathcal{U}$, then $D(\mathcal{V}) \supseteq D(\mathcal{U})$.

Proof of Claim. Suppose $x \in D(\mathcal{U})$. Then $x \notin \bigcup\{U(t) : t \in X \setminus \{x\}\}$; hence, $x \notin \bigcup\{V(t) : t \in X \setminus \{x\}\}$. It follows that $x \in D(\mathcal{V})$. □

Here is another characterization of D-spaces which makes the D-space property look like a monotone property.

Proposition 3.2. *The following are equivalent for a T_1-space X.*

(a) *X is not a D-space;*

(b) there is an operator N that assigns an open set $N(C)$ to every closed discrete subset C of X such that C is contained in $N(C)$ which is a proper subset of X, and $N(C) \subseteq N(D)$ if $C \subseteq D$.

Proof: (b) $\implies$ (a). Form the ONA defined by $U(x) = N(\{x\})$ for all x. If C is any closed discrete set, then $\bigcup\{U(x) : x \in C\} \subseteq N(C)$, which is a proper subset of X. Thus, $\{U(x) : x \in C\}$ is not a cover.

(a) $\implies$ (b). Fix an ONA $\{U(x) : x \in X\}$ such that there is no closed discrete C for which $\bigcup\{U(x) : x \in C\} = X$. Define $N(C) = \bigcup\{U(x) : x \in C\}$ for every closed discrete C. □

Finally, we define two less restrictive modifications of monotonically D spaces and give examples, leaving the proofs to the reader.

Call X *countably monotonically* D if, for every ONA $\mathcal{U}$, one can fix a countable family $\mathcal{C}(\mathcal{U})$ of closed discrete sets such that $X = \bigcup\{U(x) : x \in D\}$ for all $D \in \mathcal{C}(\mathcal{U})$, and if $\mathcal{V} \leq \mathcal{U}$, then for each $D \in \mathcal{C}(\mathcal{U})$ there is a $D' \in \mathcal{C}(\mathcal{V})$ with $D \subseteq D'$. One can show that $\omega_1 + 1$ and X from Example 2.2 are countably monotonically D.

Call X *monotonically** D if there is a D-operator D such that if $\mathcal{V} \leq \mathcal{U}$, then $\bigcup\{U(x) : x \in D(\mathcal{U}) \cap D(\mathcal{V})\} = X$. The unit interval, $\omega_1 + 1$, and X from Example 2.2 are not monotonically* D.

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