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THETA-COVERS OF HAUSDORFF SPACES

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Abstract. In this article, we partially order $\theta\mathbf{COV(X)}$ and obtain the following results.

1. If (Y, f) is a θ-cover of X, then so is $(Y(s), f)$ and $(Y, f) \geq (Y(s), f)$.
2. Every semiregular θ-cover of X is a subspace of ρX.
3. Every semiregular Hausdorff extension of a θ-cover of X is contained in ρX.
4. Every minimal Hausdorff extension of a θ-cover of X is contained in ρX.

1. Introduction

All spaces are assumed to be Hausdorff unless explicitly stated otherwise. The necessary background for the material in this paper appears in [6], [7], and [8]. In particular, [6] provides the structural basis for extensions and absolutes of Hausdorff spaces. A bare minimum of notation is included here. An extension of a space X is a space Y which contains X as a dense subspace. A cover of a space X is a space Y with a perfect, irreducible continuous surjection from Y onto X. Extensions considered are either strict or simple, and the named extensions of interest are κX, σX, and μX. The emphasis of this paper is on θ-covers. A θ-cover of a space X is a space Y with a perfect, irreducible θ-continuous surjection from Y onto X, and the associated map is called a θ-covering. Let X be a space and let

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$$\theta X = \{\mathcal{U} : \mathcal{U} \text{ is an ultrafilter on } X\}.$$

For $U \in \tau(X)$, let $OU = \{\mathcal{U} \in \theta X : U \in \mathcal{U}\}$. Note that

(a) $O\emptyset = \emptyset$
(b) $OX = \theta X$
(c) $U, V \in \tau(X)$, $O(U \cap V) = OU \cap OV$.

θX with the topology generated by $\{OU : U \in \tau(X)\}$ is called the *hyperabsolute* of X.

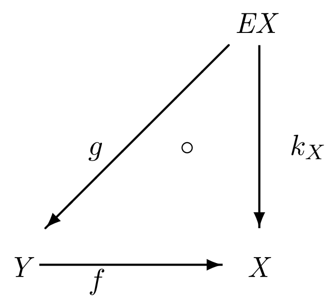
θX is homeomorphic to the Stone space of the complete Boolean algebra $\mathcal{R}O(X)$ of regular open sets of X; hence, θX is an extremally disconnected, compact Hausdorff space.

Let $EX = \{\mathcal{U} \in \theta X : ad_X\mathcal{U} \neq \emptyset\}$. EX is called the Iliadis absolute of X. The subspace EX is dense in θX (in particular, $\beta EX = \theta X$). For any open ultrafilter $\mathcal{U}$ on X, $ad_X\mathcal{U}$ is either empty or a singleton; hence, the function $k_X : EX \to X$ defined by $k_X(\mathcal{U}) = ad_X\mathcal{U}$ is well-defined.

The following properties of the Iliadis absolute (EX, k_X) are used later.

Proposition 1.1 ([6]). *Let X be a space.*

(a) *The absolute EX is extremally disconnected and Tychonoff and the surjection $k_X : EX \to X$ is a θ-covering map.*
(b) *If Y is a Hausdorff space and $f : Y \to X$ is a θ-covering map, there is a θ-covering surjection $g : EX \to Y$ such that $f \circ g = k_X$; in particular, the following diagram commutes.*



(c) *Under the hypothesis of (b), if Y is also extremally disconnected and Tychonoff, there is a homeomorphism $g : EX \to Y$ such that $f \circ g = k_X$.*

(d) *There is a homeomorphism* $h : \theta X \to E(\sigma X)$ *such that* $(k_{\sigma X} \circ h)|(\theta X \setminus EX) : \theta X \setminus EX \to \sigma X \setminus X$ *is also a homeomorphism.*
(e) *The surjection* $k_X : EX \to X$ *is continuous iff* X *is regular.*

The Banaschewski absolute, PX, has EX as the underlying set and the topology generated by the base $\{OU \cap k_X^{\leftarrow}[V] : U, V \in \tau(X)\}$. Clearly, $\tau(PX) \supseteq \tau(EX)$, and the function $\pi_X : PX \to X$ defined by $\pi_X(\mathcal{U}) = k_X(\mathcal{U})$ is continuous. The space PX is extremally disconnected and Hausdorff but not necessarily Tychonoff. Also, the surjection $\pi_X : PX \to X$ is perfect, irreducible, and continuous, and $(PX)(s) = EX$.

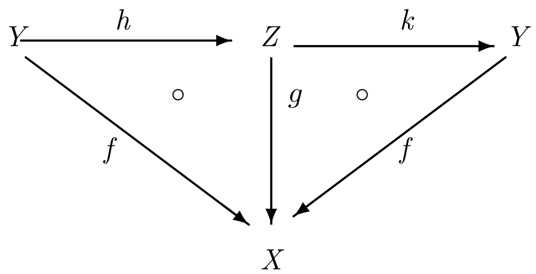
Proposition 1.2 ([9])**.** *Let* X *be a space.*

(a) *Let* $t : X \to X$ *be continuous but not the identity function. There is a closed set* $S \subset X$ *such that* $X = S \cup t^{\leftarrow}[S]$.
(b) *Let* Y *be a space,* $f : X \to Y$ *be an irreducible function such that* $f \circ t = f$. *Then* $t = id_X$.

Proposition 1.3 ([6])**.** *Let* Y *and* Z *be spaces,* $f : Y \to Z$ *and* $g : Z \to W$ *be functions.*

(a) *If* f *and* g *are closed irreducible surjections, so is* $g \circ f$.
(b) *If* f *and* g *are perfect functions, so is* $g \circ f$.
(c) *If* f *and* g *are* θ*-continuous, so is* $g \circ f$.

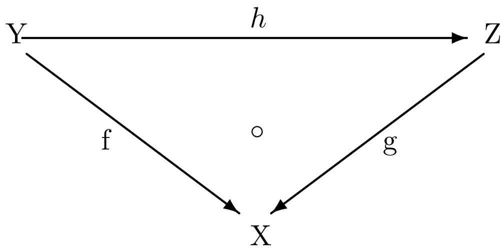
Proposition 1.4. *Let* (Y, f) *and* (Z, g) *be* θ*-covers of* X. *Suppose* $h : Y \to Z$ *is a continuous function such that* $g \circ h = f$, *and suppose* $k : Z \to Y$ *is a continuous function such that* $f \circ k = g$.



Then h *is a homeomorphism.*

Proof: Now, $g = f \circ k = g \circ (h \circ k)$. By Proposition 1.2, $h \circ k = id_Z$. Similarly, $k \circ h = id_Y$. Thus, $h^{\leftarrow} = k$, $h = k^{\leftarrow}$, and h and k are homeomorphisms. □

Define two θ-covers (Y, f) and (Z, g) of X to be *equivalent* if there is a homeomorphism $h : Y \to Z$ such that the following diagram commutes.



Let $\theta COV(X)$ denote one representative from each equivalence class. Define $\leq$ on $\theta COV(X)$ as follows: $(Z, g) \leq (Y, f)$ iff there is a continuous function $h : Y \to Z$ such that $g \circ h = f$. Clearly, $\leq$ is reflexive and transitive. By Proposition 1.4, f is antisymmetric.

Proposition 1.5. *$(\theta COV(X), \leq)$ is a partially ordered set.*

Example 1.6. (a) Any cover (Y, f) of X is also a θ-cover of X. In particular, the Banaschewski absolute (PX, π_X) is a θ-cover of X. In general, the θ-covers (PX, π_X) and (EX, k_X) are not equivalent, e.g., whenever X is Hausdorff but not regular. Now, the identity function $id : PX \to EX$ is continuous; so, $(PX, \pi_X) \geq (EX, k_X)$.

(b) (X, id_X) is a cover and, hence, is a θ-cover of X. Let $X(s)$ denote the semiregularization of X, i.e., X with the topology generated by the regular open sets; $X(s)$ is still a Hausdorff space [6]. Also, $(X(s), id_X)$ is a θ-cover of X. Note that if $(Y, c_Y) \in \theta$COV, then $(Y, c_Y) \geq (X(s), id_X)$; that is, $(X(s), id_X)$ is the smallest element of θCOV.

(c) Let $\mathcal{S}$ be a finite family of pairwise disjoint, nonempty open sets of X such that if $X = \bigcup\{clU : U \in \mathcal{S}\}$. Let $\overline{\mathcal{S}} = \{\overline{U} : U \in \mathcal{S}\}$ and $\oplus\overline{\mathcal{S}}$ denote the topological sum of the subspaces of $\overline{\mathcal{S}}$. The underlying set of $\oplus\overline{\mathcal{S}}$ is $\bigcup\{\overline{U} \times \{\overline{U}\} : U \in \mathcal{S}\}$. Define $f : \oplus\overline{\mathcal{S}} \to X$ by $f(x, U) = x$ where $x \in \overline{U}$ and $U \in \mathcal{S}$. The function f is a perfect, irreducible, continuous surjection, i.e., a covering map. So, $(\oplus\overline{\mathcal{S}}, f)$ is a θ-cover of X.

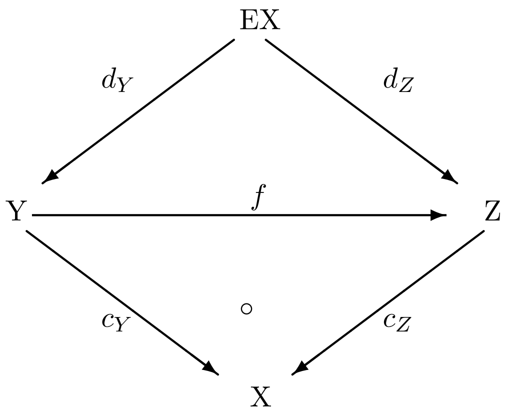
2. Properties of $\theta COV(X)$

Notation 2.1. Let Y and Z be spaces. For a function $f : Y \to Z$ and $A \subseteq Y$, the *small image* of A is $f^{\#}(A) = \{y \in Z : f^{\leftarrow}(y) \subseteq A\}$. When f is onto, it is easy to verify that $f^{\#}(A) = Z \setminus f[Y \setminus A]$ and $f^{\#}(A) \subseteq f(A)$.

Proposition 2.2 ([1], [6], [5], [10])**.** *Let Y be a space and $f : Y \to X$ be a closed, irreducible θ-continuous surjection.*

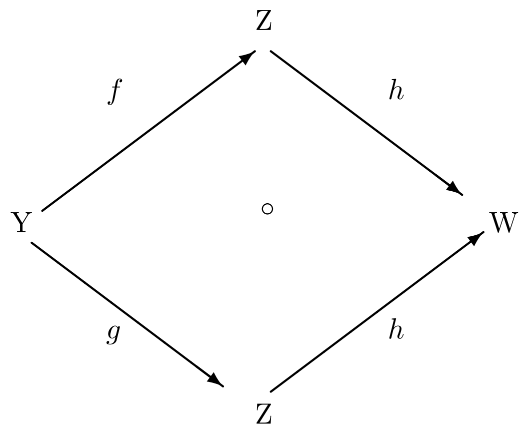
(a) *If $\emptyset \neq U \in \tau(Y)$, then $\emptyset \neq f^{\#}[U] \in \tau(X)$.*
(b) *If $U \in \tau(Y)$, then $f[cl_Y U] = cl_X f[U] = cl_X f^{\#}[U]$.*
(c) *If $U \in \tau(X)$, then $int_Y f^{\leftarrow}[cl_X U] = int_Y cl_Y f^{\leftarrow}[U]$.*
(d) *If $U \in \tau(Y)$, then $int_Y f^{\leftarrow}[cl_X f[U]] = int_Y cl_Y U$.*
(e) *Let $U \in \mathcal{R}O(Y)$, then $f^{\#}[U] \in \mathcal{R}O(X)$.*
(f) *$f^{\#} : \mathcal{R}O(Y) \to \mathcal{R}O(X)$ is an order-preserving isomorphism.*

Let $(Y, c_Y), (Z, c_Z) \in \theta COV(X)$ such that $(Y, c_Y) \geq (Z, c_Z)$. Let $f : Y \to Z$ be continuous such that $c_Z \circ f = c_Y$. By Proposition 1.2, there are θ-covering functions $d_Y : EX \to Y$ and $d_Z : EX \to Z$ such that $c_Y \circ d_Y = k_X$ and $c_Z \circ d_Z = k_Y$. That is, in the following diagram the lower diagram commutes.



We would like to show that the upper diagram commutes, i.e., $d_Z = f \circ d_Y$, but first we need a result that extends 8.4(i) in [6].

Proposition 2.3. *Let Y, Z, W be Hausdorff spaces; $f : Y \to Z$ be a θ-continuous function; $g : Y \to Z$ and $h : Z \to W$ be perfect, irreducible, θ-continuous surjections; and $h \circ f = h \circ g$, i.e., this diagram commutes. Then $f = g$.*



Proof: Assume there is a point $y \in Y$ such that $f(y) \neq g(y)$. There are open sets U, V in Z such that $f(y) \in U$, $g(y) \in V$ and $U \cap V = \emptyset$. There is an open set W in Y such that $y \in W$, $f[clW] \subseteq clU$, and $g[clW] \subseteq clV$. By Proposition 2.2(b), $clV \supseteq g[clW] = clg^{\#}[W] \supseteq g^{\#}[W]$. Hence, $int_Z cl_Z V \supseteq g^{\#}[W]$ by Proposition 2.2(a). As $g^{\#}[W] = Z \setminus g[Y \setminus W]$, $Z \setminus int_Z cl_Z V \subseteq g[Y \setminus W]$. Now, $clU \supseteq f[cl_Y W] \supseteq f[W] \supseteq f^{\#}[W] = Z \setminus f[Y \setminus W]$; hence, $Z \setminus cl_Z U \subseteq f[Y \setminus W]$. As $U \cap V = \emptyset$, $int_Z cl_Z U \cap int_Z cl_Z V = \emptyset$. In particular, $cl_Z U \cap int_Z cl_Z V = \emptyset$. So, $Z = (Z \setminus int_Z cl_Z V) \cup (Z \setminus cl_Z U)$ and $W = h[Z] = h[Z \setminus int_Z cl_Z V] \cup h[Z \setminus cl_Z U] \subseteq h \circ g[Y \setminus W] \cup h \circ f[Y \setminus W] = h \circ g[Y \setminus W]$. Since h and g are θ-covering functions, it follows by Proposition 1.5 that $h \circ g$ is a θ-covering function. As $Y \setminus W$ is a proper closed subset, $h \circ g[Y \setminus W] \neq W$ by the irreducibility property of $h \circ g$. This is a contradiction. So, we have that $f = g$. $\square$

Corollary 2.4. *Let* (Y, c_Y), $(Z, c_Z) \in \theta COV(X)$ *and* $f : Y \to Z$ *be a continuous function such that* $c_Z \circ f = c_Y$. *Let* $d_Y : EX \to Y$ *and* $d_Z : EX \to Z$ *be* θ*-covering functions such that* $c_Y \circ d_Y = c_Z \circ d_Z = k_X$. *Then* $f \circ d_Y = d_Z$.

Proof: This follows by observing that $c_Z \circ d_Z = k_X = c_Y \circ d_Y = c_Z \circ f \circ d_Y$ and applying Proposition 2.3 to $c_Z \circ d_Z = c_Z \circ (f \circ d_Y)$. □

Proposition 2.5. *If* (Y, f) *is a* θ*-cover of* X *then so is* $(Y(s), f)$ *and* $(Y, f) \geq (Y(s), f)$.

Proof: Since $f : Y \to X$ is a θ-continuous, perfect, irreducible surjection and $\tau(Y(s)) \subseteq \tau(Y)$, it follows that $f : Y(s) \to X$ is a perfect, irreducible surjection. Since $id_Y : Y \to Y(s)$ is continuous and a θ-homeomorphism, it is immediate that $f : Y(s) \to X$ is a θ-continuous and $(Y, f) \geq (Y(s), f)$. □

3. θ-covers and ρX

In [8], for a T_0 space X, a T_0 extension ρX of X is constructed such that $X \subseteq \rho X \subseteq \beta^+(X)$, where $\beta^+(X)$ is the upper Stone-Cech compactification T_0 extension of the embedding of X. (This is an embedding in the product space $\prod_{C^+(X)} I^+$, where I^+ has the topology generated by $\{[0, a) : 0 < a < 1\}$, and $C^+(X) = C(X, I^+)$.) $\beta^+(X)$ is defined in [3]. In [8], ρX is characterized as the space $\mathcal{OF}(X)$, the set of all open filters on X, with the topology generated by the base $\{OU : U \in \tau(X)\}$ where $OU = \{\mathcal{G} \in \mathcal{OF}(X) : U \in \mathcal{G}\}$. In this paper, we will view $\rho(X)$ in terms of this open filter characterization. By Theorem 6.4 in [6], $\rho(X)$ contains all of the strict T_0 extensions of X and no other extensions. Furthermore, $\rho(X)$ contains only one copy of each strict T_0 extension of X.

Notation 3.1. Let (Y, f) be a θ-cover of X. By Proposition 2.2, $f^{\#} : \mathcal{RO}(Y) \to \mathcal{RO}(X)$ is a Boolean algebra isomorphism. For $p \in Y$, let $\mathcal{V}_p = \{U \in \mathcal{RO}(Y) : p \in U\}$, and $\mathcal{G}_p = f^{\#}(\mathcal{V}_p) \subseteq \mathcal{RO}(X)$. Let $\widehat{\mathcal{G}}_p = \cap\{\mathcal{U} : \mathcal{U} \in \theta X$ and $\mathcal{G}_p \subseteq \mathcal{U}\}$ and $Z = \{\widehat{\mathcal{G}}_p : p \in Y\} \subseteq \mathcal{OF}(X)$. Define $\phi : Y \to Z$ by $\phi(p) = \widehat{\mathcal{G}}_p$.

We need the following fact from [6].

Proposition 3.2. [6, 2.3(k)] *Let $\mathcal{F}$ be an open filter on a space X. If $\mathcal{G} = \cap\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter and $\mathcal{U} \supseteq \mathcal{F}\}$, then $\mathcal{G} = \{T \subseteq X : T$ is open and $int_X cl_X T \in \mathcal{F}\}$.*

If $\mathcal{U}$ is an open ultrafilter on X and $U \in \tau(X)$, then by Proposition 3.2, $U \in \mathcal{U}$ iff $int_X cl_X U \in \mathcal{U}$. Let $(Y, f) \in \theta COV(X)$, $p \in Y$, and $U \in \tau(X)$. Also, by Proposition 3.2, $\widehat{\mathcal{G}_p} \in OU$ iff $\widehat{\mathcal{G}_p} \in O(int_X cl_X U)$.

Proposition 3.3. *Let $(Y, f) \in \theta COV(X)$, $V \in \mathcal{RO}(Y)$, and $p \in Y$. Then $V \in \mathcal{V}_p$ iff $f^{\#}(V) \in \widehat{\mathcal{G}_p}$.*

Proof: Suppose $V \in \mathcal{V}_p$. By definition, $f^{\#}(V) \in \mathcal{G}_p \subseteq \widehat{\mathcal{G}_p}$. Conversely, suppose $f^{\#}(V) \in \widehat{\mathcal{G}_p}$. Then $int_X cl_X f^{\#}(V) = f^{\#}(V) \in \mathcal{G}_p = f^{\#}(\mathcal{V}_p)$. So, there is a $W \in \mathcal{V}_p$ such that $f^{\#}(W) = f^{\#}(V)$. But, $f^{\#}$ is one-one; hence, $V = W \in \mathcal{V}_p$. □

Theorem 3.4. *Let $(Y, f) \in \theta COV(X)$, Z be defined as in 3.1 with the subspace topology of $\rho(X)$. Then $Y(s)$ and Z are homeomorphic.*

Proof: The function ϕ as defined in 3.1 is one-one and onto. Let W be open in X. Then in Z, $OW = O(intclW)$. Thus, there is a regular open set T in Y such that $f^{\#}(T) = intclW$ and $\phi^{\leftarrow}[OW] = \phi^{\leftarrow}[O(intclW)] = \phi^{\leftarrow}[O(f^{\#}(T))] = \phi^{\leftarrow}[\{\widehat{\mathcal{G}_p} : f^{\#}(T) \in \widehat{\mathcal{G}_p}\}] = \{p \in Y : T \in \mathcal{V}_p\} = T$. Also, $\phi[T] = O(f^{\#}(T)) \cap Z$. Thus, $\phi : Y(s) \to Z$ is a homeomorphism. □

Corollary 3.5. *Every semiregular, θ-cover of X is a subspace of ρX.*

Now, we consider the special cases of the θ-covers (X, id_X) and $(X(s), id_X)$ of X. For $p \in X$, let $\mathcal{W}_p = \{U \in \tau(X) : p \in U\}$, $\mathcal{V}_p = \{U \in \mathcal{RO}(X) : p \in U\}$, and $\widehat{\mathcal{V}_p} = \bigcap\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter and $\mathcal{V}_p \subseteq \mathcal{U}\}$. Then $T = \{\mathcal{W}_p : p \in X\}$ and $R = \{\widehat{\mathcal{V}_p} : p \in X\}$ are subspaces of $\mathcal{OF}(X)$.

Proposition 3.6. *The space T is homeomorphic to X and R is homeomorphic to $X(s)$.*

Proof: Let $\phi : X \to T : p \longmapsto \mathcal{W}_p$ and $\psi : X(s) \to R : p \longmapsto \widehat{\mathcal{V}_p}$. Clearly, ϕ and ψ are bijections. If $U \in \tau(X)$, $\phi^{\leftarrow}[OU] =$

$\{p \in X : U \in \mathcal{W}_p\} = U$. Also, $\phi[U] = OU \cap T$. So, ϕ is a homeomorphism. For $p \in X$ and $U \in \tau(X)$, by Proposition 3.2, $\widehat{\mathcal{V}}_p \in OU$ iff $\widehat{\mathcal{V}}_p \in O(int_X cl_X)$. Let $V \in \tau(X)$. $\psi^{\leftarrow}[OV \cap R] = \psi^{\leftarrow}[O(int_X cl_X V) \cap R] = \{p \in X : int_X cl_X V \in \mathcal{W}_p\} = int_X cl_X V$. Also, $\psi[int_X cl_X V] = OV \cap R$. Thus, ϕ is a homeomorphism. □

Next, we wish to link H-closed extensions of θ-covers of X and ρX.

Notation 3.7. Let X be a space, (Y, f) a θ-cover of X, and Z an Hausdorff extension of Y. For $p \in Z$, let $\mathcal{V}_p = \{U \cap Y : p \in U \in \mathcal{RO}(Z)\}$ and for $V \in \tau(X)$, $o_Z(V) = \{p \in Z : V \supseteq W$ for some $W \in \mathcal{V}_p\}$. Let $\mathcal{G}_p = f^{\#}(\mathcal{V}_p)$ and $\widehat{\mathcal{G}}_p = \cap\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter and $\mathcal{U} \supseteq \mathcal{G}_p\}$. Let $T = \{\widehat{\mathcal{G}}_p : p \in Z\}$ and $\phi : Z \to T : p \longmapsto \widehat{\mathcal{G}}_p$. The proof of Proposition 3.3 also shows that ϕ is a bijection. Furthermore, we show that ϕ is always continuous.

Proposition 3.8. *Let Z be a Hausdorff extension of a θ-cover (Y, f) of X. The function $\phi : Z \to T$ defined in* 3.7 *is a continuous bijection.*

Proof: ϕ is one-one and onto. To show that ϕ is continuous, let U be open in X. Then $OU \cap T = O(int_X cl_X U) \cap T$ and $\phi^{\leftarrow}[O(int_X cl_X U)] = \{p \in Z : \widehat{\mathcal{G}}_p \in O(int_X cl_X U)\} = \{p \in Z : int_X cl_X(U) \in \widehat{\mathcal{G}}_p\}$. There is a regular open V in Y such that $f^{\#}(V) = int_X cl_X U$. Hence, $\phi^{\leftarrow}[O(int_X cl_X U)] = \{p \in Z : f^{\#}(V) \in \widehat{\mathcal{G}}_p\} = \{p \in Z : f^{\#}(V) \in \mathcal{G}_p\} = \{p \in Z : f^{\#}(V) \in f^{\#}(\mathcal{V}_p)\} = \{p \in Z : V \in \mathcal{V}_p\} = o_Z V$. Additionally, $\phi[o_Z V] = O(int_Z cl_Z U) \cap T = OU \cap T$. □

Proposition 3.9. ([6, 7.5(h)(1)]) If Z is a semiregular extension of X, then $\{o_Z U : U \in \mathcal{RO}(X)\}$ is a base for Z.

Proposition 3.10. *Let Z be an extension of a θ-cover (Y, f) of X and $\phi : Z \to T$, as defined in* 3.7. *Then ϕ is a homeomorphism if Z is semiregular.*

Proof: If Z is a semiregular extension of Y, then $\{o_Z U : U \in \mathcal{RO}(Y)\}$ is an open base for Z. □

Corollary 3.11. *Every semiregular Hausdorff extension of a θ-cover of X is contained in ρX.*

An immediate consequence of Corollary 3.11 is the next result.

Corollary 3.12. *Every minimal Hausdorff extension of a θ-cover of X is contained in ρX.*

4. Examples

Example 4.1. The Iliadis absolute (EX, k_X) is a semiregular θ-cover of X. By Corollary 3.11, EX is contained in ρX.

Definition 4.2. For an open filter $\mathcal{F}$ on X, $\mathcal{F}_s = \langle \mathcal{F} \cap \mathcal{R}O\ (X)\rangle$.

Construction 4.3. Let H be an H-closed extension of a Hausdorff space X. For $U \in \tau(X)$, define $\lambda(U) = U \cup \{y \in H \backslash X : U \in (O^y)_s\}$. Note that $\lambda(\emptyset) = \emptyset$, $\lambda(X) = H$, and for $U, V \in \tau(X)$, $\lambda(U \cap V) = \lambda(U) \cap \lambda(V)$. So, $\{\lambda(U) : U \in \tau(X)\}$ is a base for a topology on H. H with this topology is denoted as H_0.

Proposition 4.4. *Let H be an H-closed extension of X. Then H_0 is a strict H-closed extension of X.*

Example 4.5. By Corollary 3.11, any minimal Hausdorff extension of EX is contained in ρX. So, βEX and every Hausdorff compactification of EX is contained in ρX. But $\beta EX \equiv_{EX} \theta X \equiv_{EX} \mu EX$ [6, 6.6(e)(1) and 7B(3)]). By [6, 7.5(h)(4)], $\mu EX \setminus EX$ is homeomorphic to $\sigma EX \setminus EX$ and by 4.3, $\sigma EX \setminus EX$ and $\sigma X \setminus X$ are homeomorphic. But by [6, 7.7(e)], there is an order isomorphism between $\mathcal{H}^\theta(X)$ and $\{P : P$ is a partition of $\sigma X \setminus X$ into nonempty compact sets $\}$. Since $\sigma EX \setminus EX$ and $\sigma X \setminus X$ are homeomorphic by 4.3, there is an order isomorphism between $\mathcal{H}^\theta(EX)$ and $\mathcal{H}^\theta(X)$. Also, by [6, 7.7(e)], there is an order isomorphism between $\mathcal{H}^\theta(EX)$ and the set $\mathcal{M}(EX)$ of minimal Hausdorff extensions of EX. So, there is a parallel analogue between the result that every minimal Hausdorff extension of EX is contained in $\rho X$and the result that every H-closed extension H_0 of X, where H_0 is defined in Proposition 4.4 for an arbitrary H-closed extension H of X, is contained in ρX. The obvious parallel between $\mathcal{M}(EX)$ and $\mathcal{M}(X)$ might not work in this case for if X is not semiregular, $\mathcal{M}(X) = \emptyset$.

Example 4.6. Since $(X(s), id_X)$ is a θ-cover of X, by Corollary 3.11, every minimal Hausdorff extension of $X(s)$ is contained in ρX. Now, there is an order isomorphism between $\mathcal{H}^\theta(X)$ (or $\mathcal{H}_0(X) = \{H_0 : H \in \mathcal{H}(X)\}$) and $\mathcal{M}(X(s))$ by 1.15. Combining this result with that of (b), there is an order isomorphism between $\mathcal{M}(EX)$ and $\mathcal{M}(X(s))$. This parallel between ρX containing elements of $\mathcal{M}(EX)$ and $\mathcal{M}(X)$ is obtained without requiring that X be semiregular.

Example 4.7. Let $\mathcal{B}$ be an open base for X. For $U \in \mathcal{B}(X)$, let $U^0 = U$ and $U^1 = X \setminus cl_X U$. For $t \in \prod_{\mathcal{B}} \mathbf{2}$, let $T_{\mathcal{B}}X = \{U^{t(U)} : U \in \mathcal{B}\}$. Let $cT_{\mathcal{B}}X = \{t \in \prod_{\mathcal{B}} \mathbf{2} : T_{\mathcal{B}}X$ has the finite intersection property$\}$ and $T_{\mathcal{B}}X = \{t \in cT_{\mathcal{B}}X : ad_X \mathcal{B}^t \neq \emptyset\}$. Jack Porter [4] has shown that $cT_{\mathcal{B}}X$ is a zero-dimensional Hausdorff compactification of $T_{\mathcal{B}}X$. The function $l_{\mathcal{B}} : T_{\mathcal{B}}X \to X$ defined by $l_{\mathcal{B}}(t) = ad_X \mathcal{B}^t$ is a θ-covering map. So, $(T_{\mathcal{B}}X, l_{\mathcal{B}})$ is a θ-cover of X. As $cT_{\mathcal{B}}X$ is a Hausdorff compactification of $T_{\mathcal{B}}X$, $cT_{\mathcal{B}}X$ is contained in ρX. Porter [4] shows that every zero-dimensional Hausdorff compactification of every Hausdorff θ-cover of X can be obtained using this method.

Example 4.8. Recall ([6] and [2]) a relation $\rho \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ is a θ-proximity if for A, B, $C \in \mathcal{P}(X)$,

1 $\emptyset \not\rho X$;
2 $A \rho B$ implies $B \rho A$;
3 $A \not\rho B$ and $A \not\rho C$ iff $A \not\rho (B \cup C)$;
4 $A \not\rho B$ implies there is a $U \in \mathcal{RO}(X)$ such that $B \not\rho U$ and $A \not\rho X \setminus clU$;
5 $\{x\} \rho A$ iff $\mathcal{N}_x$ meets $\mathcal{N}_A$.

An open filter $\mathcal{F}$ on X is a θ-filter if for each $U \in \mathcal{F}$, there is $V \in \mathcal{F}$ such that $V \not\rho X \setminus clU$. Let $b_\theta X_\theta = \{\mathcal{U} : \mathcal{U}$ is a maximal θ-filter on $X\}$. For $U \in \tau(X)$, let $OU = \{\mathcal{U} \in b_\theta X_\theta : U \in \mathcal{U}\}$. $\{OU : U \in \tau(X)\}$ is a base for a compact Hausdorff topology on $b_\theta X_\theta$. Define $\pi_\theta : X_\theta \to X$ by $\pi_\theta(\mathcal{U}) = ad(\mathcal{U})$; π_θ is a θ-covering map. Thus, (X_θ, π_θ) is a θ-cover of X. Since $b_\theta X_\theta$ is a Hausdorff compactification of X_θ, $b_\theta X_\theta$ is contained in ρX. V. Fedorcuk [2] shows that every Hausdorff compactification of every Tychonoff cover of X can be generated by a θ- proximity on X.

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