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REFLECTION THEOREMS FOR SOME CARDINAL FUNCTIONS IN INITIALLY κ-COMPACT SPACES

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ABSTRACT. We establish that, in an initially κ-compact T_3-space X, the property of having character at most κ is discretely reflective; i.e., if X is initially κ-compact, T_3, and $\chi(\overline{D}) \leq \kappa$ for any discrete $D \subset X$, then $\chi(X) \leq \kappa$. This generalizes an analogous result of Ofelia T. Alas, Vladimir V. Tkachuk, and Richard G. Wilson in [Closures of discrete sets often reflect global properties, Topology Proc. 25, 2000, Spring, 27–44] proved for compact spaces. We also extend over the class of initially κ-compact T_3-spaces a result of Alan Dow in [An Introduction to Applications of Elementary Submodels to Topology, Topology Proc. 13, 1988, no. 1, 17–72] on reflecting uncountable character in countably compact spaces. Some corollaries on reflection of other cardinal functions are obtained.

1. Introduction

In topology, a *reflection theorem* usually has the form: If a space X has a property $\mathcal{P}$, then some *small* (in some sense) subspace of X has the property $\mathcal{P}$. However, it is usually the case that $\mathcal{P}$ is the negation of a nice property. So one might rephrase the theorem: If each small subspace of X has the given property $\mathcal{P}$, then X has it.

For example, in [1], Ofelia T. Alas, Vladimir V. Tkachuk and Richard G. Wilson obtain several results with the form: If the closure of each discrete subspace of X has a property $\mathcal{P}$, then X has $\mathcal{P}$. For Alas, Tkachuk, and Wilson, *small subspace* means it

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is the closure of a discrete subspace. To emphasize this fact, we mention the following result from [1]: *If X is a compact Hausdorff space such that, for some infinite cardinal κ, the closure of every discrete subspace of X has character $\leq \kappa$, then $\chi(X) \leq \kappa$.*

Another kind of small subspace, for a given space X, is the one formed by the subspaces of X of cardinality less than or equal to κ. For example, in [7], R. E. Hodel and J. E. Vaughan proved: *If X is a compact Hausdorff space, and $\chi(X) \geq \kappa^+$, then there exists $Y \subset X$ with $|Y| \leq \kappa^+$ such that $\chi(Y) \geq \kappa^+$.*

It is important to say that the study on reflection (and the *increasing union property*) was initiated in 1978 by M. G. Tkačhenko [11] and continued in 1980 by István Juhász [8]. However, it wasn't until 2000 that Hodel and Vaughan [7] made the first systematic study about reflection theorems in cardinal functions. It is worthwhile to say that this systematic study was made taking small subspaces as those whose cardinality is less than or equal to κ.

In this paper, we have two purposes. The first one is to establish that *if X is initially κ-compact, T_3, and $\chi(\overline{D}) \leq \kappa$ for any discrete $D \subset X$, then $\chi(X) \leq \kappa$*; (see Theorem 3.6). This generalizes an analogous result of Alas, Tkachuk, and Wilson proved for compact spaces. The second purpose is to extend over the class of initially κ-compact spaces many well-known reflection theorems which were obtained for the class of compact Hausdorff spaces; (see [7], [11]). In particular, we extend a result of Alan Dow [3] on reflecting uncountable character in countably compact spaces.

2. Notations and Definitions

We refer the reader to [6] for definitions and terminology on cardinal functions not explicitly given here. Let w, nw, psw, L, χ, and ψ denote the following standard cardinal functions: weight, netweight, point separating weight, Lindelöf degree, character, and pseudocharacter, respectively. Let ϕ be a cardinal function; the hereditary version of ϕ, denoted $h\phi$, is defined $h\phi(X) = \sup\{\phi(Y) : Y \subset X\}$. Also, it is well known that ϕ is monotone if and only if $\phi = h\phi$.

For any set X and cardinal κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with cardinality $\leq \kappa$; $[X]^{<\kappa}$ is defined analogously.

Moreover, $\overline{A}$ is the closure of A in X. The closure of a subset A of $Y \subset X$ is denoted $cl_Y(A)$.

For the reader's comfort, the formulations of the necessary results of reflection theorems for cardinal functions and initially κ-compact spaces are given here (without proof).

The reader is referred to [7] for a systematic study on reflection theorems for cardinal functions.

Definition 2.1. Let ϕ be a cardinal function and let κ be an infinite cardinal.

(1) ϕ reflects κ means: if $\phi(X) \geq \kappa$, then there exists $Y \in [X]^{\leq\kappa}$ such that $\phi(Y) \geq \kappa$.
(2) ϕ strongly reflects κ means: if $\phi(X) \geq \kappa$, then there exists $Y \in [X]^{\leq\kappa}$ such that for each $Y \subset Z \subset X$, $\phi(Z) \geq \kappa$.

For some cardinal functions it is necessary to restrict the class of spaces under consideration in order to obtain a reflection theorem. The appropriate definition in this case is that ϕ reflects κ for the class $\mathcal{C}$ if, given $X \in \mathcal{C}$ with $\phi(X) \geq \kappa$, there exists $Y \in [X]^{\leq\kappa}$ such that $\phi(X) \geq \kappa$.

The next lemma summarizes some properties about reflection (see [7]).

Lemma 2.2. *Let ϕ be a cardinal function and let $\kappa \geq \omega$.*

(1) *If ϕ strongly reflects κ, then ϕ reflects κ.*
(2) *If ϕ is monotone and ϕ reflects κ, then ϕ strongly reflects κ.*
(3) *If ϕ strongly reflects all successor cardinals, then ϕ strongly reflects all infinite cardinals. In particular, if ϕ is monotone and reflects all successor cardinals, then ϕ strongly reflects all infinite cardinals.*

Remark 2.3. Every cardinal function ϕ is defined so that $\phi(X) \geq \omega$ always holds; it easily follows that ϕ reflects ω. So in our discussion of reflection theorems, we may assume that $\kappa > \omega$.

For a detailed discussion on initially κ-compact spaces, the reader is referred to [10].

Definition 2.4. A topological space X is called initially κ-compact if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \kappa$ has a finite subcover.

Clearly, if X is initially κ-compact for some cardinal $\kappa > \omega$, then for each cardinal number $\omega \leq \theta < \kappa$, X is initially θ-compact. Moreover, if κ is a singular cardinal number and X is a topological space such that X is initially θ-compact for every cardinal number $\omega \leq \theta < \kappa$, then X is initially κ-compact (see [10, 2.3]).

Several of our results use the following key lemma (see [10, Construction 3.5]).

Lemma 2.5 (Construction)**.** *Let X be an initially κ-compact space and A any subset of X of cardinality $\leq 2^\kappa$. Then there exists an initially κ-compact subset G of X such that $A \subset G$ and $|G| \leq 2^\kappa$. In the case that X is a topological group, then G may be taken to be a subgroup of X.*

3. Main Results

Our first objective is to establish that the property $\chi(X) \leq \kappa$ is discretely reflexive within the class of initially κ-compact T_3 spaces. In other words, if X is an initially κ-compact T_3 space such that for every discrete subspace D of X, $\chi(\overline{D}) \leq \kappa$, then $\chi(X) \leq \kappa$. To see this we need the following results.

Proposition 3.1. *Let X be a T_3-space and let Y be a subspace of X. For all $p \in Y$, there is a family $\mathcal{B}$ of open neighborhoods of p in X, such that $|\mathcal{B}| \leq \psi(p, Y)$ and $\left(\bigcap\left\{\overline{B} : B \in \mathcal{B}\right\}\right) \cap Y = \{p\}$.*

Proof: Let $\gamma = \psi(p, Y)$ and let $\mathcal{U}$ be a family of open neighborhoods of p in X, such that $(\bigcap\{U : U \in \mathcal{U}\}) \cap Y = \{p\}$ and $|\mathcal{U}| = \gamma$. Since X is regular, for each $U \in \mathcal{U}$ there exists B_U, an open neighborhood of p in X, such that $\overline{B}_U \subset U$. Define $\mathcal{B} = \{B_U : U \in \mathcal{U}\}$. It is clear $|\mathcal{B}| \leq \gamma$; moreover, $\left(\bigcap\left\{\overline{B} : B \in \mathcal{B}\right\}\right) \cap Y \subset (\bigcap\{U : U \in \mathcal{U}\}) \cap Y = \{p\}$. □

The following theorem and its corollaries are generalizations of the well-known corresponding results for compact Hausdorff spaces (see [4, 3.1.3 and 3.1.5]).

Theorem 3.2. *If a subspace A of a topological space X is initially κ-compact, then for every family $\mathcal{U}$ of open subsets of X with $|\mathcal{U}| \leq \kappa$ such that $A \subset \bigcup \mathcal{U}$, there exists $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $A \subset \bigcup \mathcal{V}$.*

Corollary 3.3. *Let U be an open subset of a topological space X. If a family $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ of closed subsets of X contains at least one initially $\kappa-$compact set (in particular, if X is initially $\kappa-$compact) and if $\bigcap \mathcal{F} \subset U$, then there exists $\mathcal{E} \in [\mathcal{F}]^{<\omega}$, such that $\bigcap \mathcal{E} \subset U$.*

Proof: There is no loss of generality to assume that X is initially κ-compact. Let $A = X \setminus U$ and for each $\alpha \in \kappa$, $U_\alpha = X \setminus F_\alpha$. Clearly, $A \subset \bigcup \{U_\alpha : \alpha \in \kappa\}$. Since A is closed, then A is initially κ-compact; hence, by Theorem 3.2, there exists $S \in [\kappa]^{<\omega}$, such that $A \subset \bigcup \{U_\alpha : \alpha \in S\}$; therefore, $\bigcap \{F_\alpha : \alpha \in S\} \subset U$. □

Corollary 3.4. *Let X be an initially κ-compact space. If $\mathcal{F} = \{F_\alpha : \alpha \in \kappa\}$ is a decreasing family of closed subsets of X such that $\bigcap \{F_\alpha : \alpha \in \kappa\} = \{p\}$, then $\mathcal{F}$ is a local net of p in X.*

Proof: Let U be an open neighborhood of p in X; by Corollary 3.3, there exists $S \in [\kappa]^{<\omega}$, such that $\bigcap \{F_\alpha : \alpha \in S\} \subset U$. Since $\{F_\alpha : \alpha \in \kappa\}$ is decreasing, then $F_\beta \subset \bigcap \{F_\alpha : \alpha \in S\} \subset U$ where $\beta = \max S$. Thus, $p \in F_\beta \subset U$. □

The following is a generalization of the result that establishes $\chi(X) = \psi(X)$ for X compact and Hausdorff.

Theorem 3.5 ([7]). *Let X be an initially κ-compact Hausdorff space. If $\psi(X) \leq \kappa$, then $\psi(X) = \chi(X)$.*

We are now ready to prove that the property $\chi(X) \leq \kappa$ is discretely reflexive within the class of initially κ-compact T_3-spaces. The proof follows the same pattern as the respective result from Theorem 3.5 in [1].

Theorem 3.6. *If X is an initially κ^+-compact T_3-space such that $\chi(\overline{D}) \leq \kappa$ for every discrete subspace D of X, then $\chi(X) \leq \kappa$.*

Proof: Suppose that $\chi(X) > \kappa$, then there exists $p \in X$ such that $\chi(p, X) > \kappa$. By Theorem 3.5, $\psi(p, X) \geq \kappa^+$. Construct a sequence $\{x_\alpha : \alpha \in \kappa^+\}$ of points in $X \setminus \{p\}$ and a sequence $\{\mathcal{U}_\alpha : \alpha \in \kappa^+\}$ of families of open neighborhoods of p in X such that

(1) $|\mathcal{U}_\alpha| \leq \kappa$, $\alpha < \kappa^+$;
(2) $\mathcal{U}_\beta \subset \mathcal{U}_\alpha$, if $0 \leq \beta < \alpha$;
(3) $x_\alpha \in \bigcap \{\overline{U} : U \in \mathcal{U}_\alpha\} \setminus \{p\}$, $\alpha < \kappa^+$; and
(4) $\left(\bigcap \{\overline{U} : U \in \mathcal{U}_\alpha\}\right) \cap \overline{\{x_\rho : \rho < \alpha\}} = \{p\}$, $0 < \alpha < \kappa^+$.

Let $0<\alpha<\kappa^+$ and suppose that for each $\beta\in\alpha$, x_β and $\mathcal{U}_\beta$ have been constructed such that (1) – (4) holds. The set $Y_\alpha=\{x_\beta:\beta\in\alpha\}$ is discrete. Indeed, for each $\gamma<\alpha$, we have $x_\gamma\notin\overline{\{x_\beta:\beta<\gamma\}}$ by the properties (2) and (4). Applying (4) once more, we can observe that $x_\gamma\notin\bigcap\left\{\overline{U}:U\in\mathcal{U}_{\gamma+1}\right\}\supset\overline{\{x_\beta:\gamma<\beta<\alpha\}}$; thus, $x_\gamma\notin\overline{\{x_\beta:\beta<\gamma\}}\cup\overline{\{x_\beta:\gamma<\beta<\alpha\}}$. As a consequence, $\chi\left(p,\{p\}\cup\overline{Y}_\alpha\right)\le\kappa$. By Proposition 3.1, there exists a collection $\mathcal{U}_\alpha$ of open neighborhoods of p in X, with $\bigcup\{\mathcal{U}_\beta:\beta\in\alpha\}\subset\mathcal{U}_\alpha$ and $|\mathcal{U}_\alpha|\le\kappa$, such that $\left(\bigcap\left\{\overline{U}:U\in\mathcal{U}_\alpha\right\}\right)\cap\overline{\{x_\rho:\rho<\alpha\}}=\{p\}$. Finally, since $\psi(p,X)>\kappa$, we can choose $x_\alpha\in\bigcap\left\{\overline{U}:U\in\mathcal{U}_\alpha\right\}\setminus\{p\}$. This completes the construction.

Let $D=\{x_\alpha:\alpha\in\kappa^+\}$ and let $Y_\alpha=\{x_\beta:\beta<\alpha\}$ for each $\alpha<\kappa^+$. It is clear D is a discrete subset of X; hence, $\chi(\overline{D})\le\kappa$. Now, by (4), for each $\alpha\in\kappa^+$,

$$\begin{aligned}(a)\ \overline{Y}_\alpha\cap\overline{D\setminus Y_\alpha}&=\overline{\{x_\beta:\beta<\alpha\}}\cap\overline{\{x_\beta:\alpha\le\beta<\kappa^+\}}\\ &\subset\overline{\{x_\beta:\beta<\alpha\}}\cap\left(\bigcap\left\{\overline{U}:U\in\mathcal{U}_\alpha\right\}\right)=\{p\}.\end{aligned}$$

On the other hand, the collection $\left\{\overline{D\setminus Y_\alpha}:\alpha\in\kappa^+\right\}$ is a decreasing sequence of at most κ^+ closed nonempty subsets of X; hence, as X is initially κ^+-compact, $F=\bigcap\left\{\overline{D\setminus Y_\alpha}:\alpha\in\kappa^+\right\}$ is nonempty. Moreover, if $y\in F$, then $y\in\overline{D}=cl_{\overline{D}}(D)$ and since $t(\overline{D})\le\chi(\overline{D})\le\kappa$, there exists $C\in[D]^{\le\kappa}$ such that $y\in cl_{\overline{D}}(D)$. By regularity of κ^+, there exists $\alpha\in\kappa^+$ such that $C\subset Y_\alpha$; therefore, $y\in cl_{\overline{D}}(C)\subset cl_{\overline{D}}(Y_\alpha)\subset\overline{Y}_\alpha$. Hence, $y\in\overline{Y}_\alpha\cap\overline{D\setminus Y_\alpha}\subset\{p\}$; thus, $F=\{p\}$, and hence the collection $\left\{\overline{D\setminus Y_\alpha}:\alpha\in\kappa^+\right\}$ is a local net of p in X (Corollary 3.4).

Finally, as $\psi\left(p,\overline{D}\right)\le\kappa$, there exists a family $\mathcal{U}=\{U_\beta:\beta\in\kappa\}$ of open neighborhoods of p in X, such that $(\bigcap\mathcal{U})\cap\overline{D}=\{p\}$. For each $\beta\in\kappa$, choose $\alpha(\beta)\in\kappa^+$, such that $\overline{D\setminus Y_{\alpha(\beta)}}\subset U_\beta$. Then, for each $\alpha\in\kappa^+\setminus\sup\{\alpha(\beta):\beta\in\kappa\}$, $x_\alpha\in(\bigcap\mathcal{U})\cap\overline{D}=\{p\}$. Indeed, if $\alpha\in\kappa^+\setminus\{\alpha(\beta):\beta<\kappa\}$, then $x_\alpha\notin Y_{\alpha(\beta)}$, for all $\beta<\kappa$. Hence, $x_\alpha\in\bigcap\{\overline{D\setminus Y_{\alpha(\beta)}}\}\cap\overline{D}\subset\mathcal{U}\cap\overline{D}=\{p\}$, which is a contradiction. Therefore, $\chi(X)\le\kappa$. $\square$

Corollary 3.7 ([1]). *Let X be a compact space such that, for some infinite cardinal κ, the closure of every discrete subspace of X has character $\leq \kappa$. Then $\chi(X) \leq \kappa$.*

Now we turn our attention to the second objective of this paper. The following result is due to Vaughan [12] (stated for the case of countable tightness in [3]).

Theorem 3.8. *If X is initially κ^+-compact, T_3, and $t(X) \geq \kappa^+$, then there exists $Y \in [X]^{\leq \kappa^+}$ such that $t(Y) \geq \kappa^+$. In the class of compact T_2-spaces, tightness reflects every cardinal.*

Theorems 3.6 and 3.8 motivate the following natural question:

Question 3.9. *Let X be an initially κ^+-compact Hausdorff with $\phi(X) \geq \kappa^+$, where $\phi \in \{\chi, \psi, psw, nw\}$. Is it true that there exists $Y \in [X]^{\leq \kappa^+}$ such that $\phi(Y) \geq \kappa^+$?*

We will give partial results for this question showing similar results to Theorem 3.8, for the monotone cardinal functions χ, ψ, psw, and nw.

Our first result shows that t can be replaced by χ in the Theorem 3.8. The corresponding proof for χ is just a slight modification of the one given in [8].

Theorem 3.10 (Dow [3], for countable character). *If X is initially κ^+-compact, T_3, and $\chi(X) \geq \kappa^+$, then there exists $Y \in [X]^{\leq \kappa^+}$ such that $\chi(Y) \geq \kappa^+$.*

Proof: Since $t(X) \leq \chi(X)$, by Theorem 3.8, we may assume that $t(X) \leq \kappa$. Now, as $\chi(X) \geq \kappa^+$, then there exists $p \in X$ such that $\chi(p, X) \geq \kappa^+$.

We assume in what follows that $S \in [X]^{\leq \kappa}$ and $p \in S$ imply $\chi(p, S) < \kappa^+$. (If there is an $S \in [X]^{\leq \kappa}$ such that $p \in S$ and $\chi(p, S) \geq \kappa^+$, then there is nothing to do.)

Construct a sequence $\{x_\alpha : \alpha < \kappa^+\}$ of points of X and one $\{\mathcal{B}_\alpha : \alpha < \kappa^+\}$ formed by collections of open neighborhoods of p in X such that $p = x_0$, $\mathcal{B}_0 = \emptyset$, and for each $0 < \alpha < \kappa^+$,

(1) $|\mathcal{B}_\alpha| < \kappa^+$;
(2) $\bigcup\{\mathcal{B}_\beta : \beta < \alpha\} \subset \mathcal{B}_\alpha$;
(3) $x_\alpha \in \bigcap \left\{\overline{B} : B \in \mathcal{B}_\alpha\right\} \setminus \{p\}$; and
(4) $(\bigcap\{\overline{B} : B \in \mathcal{B}_\alpha\}) \cap \overline{S}_\alpha = \{p\}$, where $S_\alpha = \{x_\beta : \beta < \alpha\}$.

Let $0 < \alpha < \kappa^+$, and suppose that for each $\beta \in \alpha$, x_β and $\mathcal{B}_\beta$ have been constructed such that (1) – (4) holds. It is clear $\chi(p, \overline{S}_\alpha) = \chi(p, S_\alpha) \leq \kappa$. By Proposition 3.1, there exists a collection $\mathcal{B}_\alpha$ of open neighborhoods of p in X, such that (1), (2), and (4) hold. Now as $\psi(p, X) > \kappa$, we can choose $x_\alpha \in \bigcap \{\overline{B} : B \in \mathcal{B}_\alpha\} \setminus \{p\}$, which completes the construction.

Let $Y = \{x_\alpha : \alpha < \kappa^+\}$ and $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa^+\}$. As $t(X) \leq \kappa$, then $\overline{Y} = \bigcup\{\overline{S}_\alpha : \alpha < \kappa^+\}$.

It is not difficult to check that $\bigcap\{B : B \in \mathcal{B}\} \cap \overline{Y} = \bigcap\{\overline{B} : B \in \mathcal{B}\} \cap \overline{Y} = \{p\}$. Thus, $\psi(p, \overline{Y}) \leq \kappa^+$.

Claim. $\psi(p, \overline{Y}) = \kappa^+$. Indeed, if $\psi(p, \overline{Y}) \leq \kappa$, then there exists a collection $\mathcal{U} = \{U_\beta : \beta < \kappa\}$ of open neighborhoods of p in X such that $\{p\} = (\bigcap \mathcal{U}) \cap \overline{Y}$. Note that for each $\beta < \kappa$, the collection $\{(\overline{B} \cap \overline{Y}) \setminus U_\beta : B \in \mathcal{B}\}$ is a family of at most κ^+ closed subsets of $\overline{Y}$ such that $\bigcap\{(\overline{B} \cap \overline{Y}) \setminus U_\beta : B \in \mathcal{B}\} = \emptyset$; since $\overline{Y}$ is initially κ^+-compact, there exists $\mathcal{V}_\beta \in [\mathcal{B}]^{<\omega}$ such that $\bigcap\{(\overline{B} \cap \overline{Y}) \setminus U_\beta : B \in \mathcal{V}_\beta\} = \emptyset$. Hence, $\bigcup\{(\overline{B} \cap \overline{Y}) : B \in \mathcal{V}_\beta\} \subset U_\beta$, for each $\beta < \kappa$. Let $\mathcal{V} = \bigcup\{\mathcal{V}_\beta : \beta < \kappa\}$. Since $|\mathcal{V}| \leq \kappa$, by regularity of κ^+, there exists $\alpha_0 \in \kappa^+$ such that $\mathcal{V} \subset \mathcal{B}_{\alpha_0}$. Note $\{p\} = \bigcap\{U_\beta : \beta < \kappa\} \cap \overline{Y} \supset \bigcap\{\overline{B} \cap \overline{Y} : B \in \mathcal{V}\} \supset \bigcap\{\overline{B} \cap \overline{Y} : B \in \mathcal{B}_{\alpha_0}\} \ni x_{\alpha_0}$, which is a contradiction. Thus, $\psi(p, \overline{Y}) = \kappa^+$.

Finally, since $\chi(p, Y) = \chi(p, \overline{Y}) \geq \psi(p, \overline{Y}) = \kappa^+$, we have $\chi(Y) \geq \kappa^+$. □

Corollary 3.11 ([7])**.** *For the class of compact Hausdorff spaces, χ reflects all infinite cardinals.*

In [7], Hodel and Vaughan proved that under Generalized Continuum Hypothesis (GCH), ψ reflects all infinite cardinals for the class of compact Hausdorff spaces. This result is now a consequence of the following corollary.

Corollary 3.12. *Assume* GCH. *If X is initially κ^+-compact, T_3, and $\psi(X) \geq \kappa^+$, then there exists $Y \in [X]^{\leq \kappa^+}$ such that $\psi(Y) \geq \kappa^+$.*

It was also proved in [7] that, under GCH, *psw* reflects all infinite cardinals for the class of compact Hausdorff spaces. We will extend this result to the class of initially κ-compact spaces. To see this, we need a couple of results. The first one is a generalization of

$w(X) = psw(X)$ for X compact and Hausdorff. The second is the remarkable reflection theorem for weight which was shown for T_3 spaces by Tkačhenko [11] and later, for any space (no separation axiom is needed), by A. Hajnal and Juhász [5].

Lemma 3.13 ([7]). *If X is initially κ^+-compact, T_2, and $psw(X) \leq \kappa$, then $psw(X) = w(X)$.*

Theorem 3.14. *If X is a topological space and $w(X) \geq \kappa$, then there exists $Y \in [X]^{\leq\kappa}$ such that $w(Y) \geq \kappa$.*

Theorem 3.15. *Assume* GCH. *If X is initially κ-compact, T_2, and $psw(X) \geq \kappa^+$, then there exists $Y \in [X]^{\leq\kappa^+}$ such that $psw(Y) \geq \kappa^+$.*

Proof: Since $psw(X) \geq \kappa^+$ and $psw(X) \leq w(X)$, then $w(X) \geq \kappa^+$; hence, by Theorem 3.14, there exists $Y \in [X]^{\leq\kappa^+}$ such that $w(Y) \geq \kappa^+$. Now, since X is initially κ-compact and we are assuming GCH, by Lemma 3.13, we suppose, without loss of generality, that Y is initially κ-compact. Hence, by Lemma 3.13, $psw(Y) \geq \kappa^+$. □

Corollary 3.16. *Assume* GCH. *For the class of compact Hausdorff spaces, psw reflects all infinite cardinals.*

Corollary 3.17. *Assume* GCH. *If X is initially κ-compact, T_2, and $psw(Y) < \kappa^+$ for any $Y \in [X]^{\leq\kappa^+}$, then X is compact.*

Proposition 3.18. *If X is initially κ-compact, T_2, and $nw(X) \geq \kappa^+$, then there exists $Y \in [X]^{\leq\kappa^+}$ such that $nw(Y) \geq \kappa^+$*

Proof: Since $nw(X) \geq hL(X)$ and hL reflects all infinite cardinals (see [7, Theorem 2.1]), we may assume that $hL(X) < \kappa^+$. Then $L(X) \leq \kappa$; hence, X is compact. Thus, by Theorem 1.5 in [11], there exists $Y \in [X]^{\kappa^+}$ such that $nw(Y) \geq \kappa^+$. □

Corollary 3.19. *If X is initially κ-compact, T_2, and $nw(Y) < \kappa^+$ for any $Y \in [X]^{\leq\kappa^+}$, then X is compact.*

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References

[1] Ofelia T. Alas, Vladimir V. Tkachuk, and Richard G. Wilson, *Closures of Discrete Sets Often Reflect Global Properties,* Topology Proc. **25** (2000), Spring, 27–44.

[2] Stavros Christodoulou, *Initially $\kappa-$compact spaces for large κ,* Comment. Math. Univ. Carolin. **40** (1999), no. 2, 319–325.

[3] Alan Dow, *An Introduction to Applications of Elementary Submodels to Topology,* Topology Proc. **13** (1988), no. 1, 17–72.

[4] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. Mathematical Monographs, Vol. 60. Warsaw: PWN—Polish Scientific Publishers, 1977.

[5] A. Hajnal and I. Juhász, *Having a small weight is determined by the small subspaces,* Proc. Amer. Math. Soc. **79** (1980), no. 4, 657–658.

[6] R. E. Hodel, *Cardinal Functions I.* Kunen and Vaughan 1–61.

[7] R. E. Hodel and J. E. Vaughan, *Reflection theorems for cardinal functions,* Topology Appl. **100** (2000), no. 1, 47–66.

[8] István Juhász, *Cardinal Functions in Topology–Ten Years Later.* Mathematical Centre Tracts, 123. Amsterdam: Mathematish Centrum, 1980.

[9] Kenneth Kunen and Jerry E. Vaughan, eds. *Handbook of Set-Theoretic Topology.* Amsterdam: North-Holland, 1984.

[10] R. M. Stephenson, *Initially κ-Compact and Related Spaces.* Kunen and Vaughan 603–632.

[11] M. G. Tkačhenko, *Chains and cardinals,* Soviet Math. Dokl. **19** (1978), 382–385.

[12] Jerry E. Vaughan, *An introduction to reflection theorems for cardinal functions,* Proceedings of the Second Galway Topology Colloquium at Oxford (1998). (electronic). North Bay, Ontario, Canada: Topology Atlas, 1998. (http://at.yorku.ca/p/p/a/d/25.htm).

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