Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 389-402

FRACTALNESS OF SUPERCONTOURS

ANDRZEJ STAROSOLSKI

ABSTRACT. Ultrafilters and monotone sequential contours are fractal. [See Szymon Dolecki, Andrzej Starosolski, and Stephen Watson, Extension of multisequences and countable uniradial class of topologies, Comment. Math. Univ. Carolin. 44 (2003), no. 1, 165–181.] It is proved (under CH) that for each supercontour there exists a greater supercontour which is an ultrafilter. The existence (under CH) of $2^{\aleph_1}$ other fractal supercontours is shown.

1. INTRODUCTION

Let $\mathcal{F}$ be a filter on X and let $f : X \to Y$. By $f(\mathcal{F})$, we understand a filter(!) on Y for which a family $\{f(F) : F \in \mathcal{F}\}$ is a filter base. Let $\mathcal{G}$ be a filter on Y. We say that $\mathcal{F}$ is *Rudin-Keisler* greater than $\mathcal{G}$ (in symbols $\mathcal{F} \succeq \mathcal{G}$) if there is a map $f : X \to Y$ such that $f(\mathcal{F}) \supset \mathcal{G}$. If $\mathcal{F} \succeq \mathcal{G}$ and $\mathcal{G} \succeq \mathcal{F}$, then $\mathcal{F}$ and $\mathcal{G}$ are *equivalent*, and we write $\mathcal{F} \approx \mathcal{G}$. If $\mathcal{F} \succeq \mathcal{G}$ and $\mathcal{F} \not\approx \mathcal{G}$, then we write $\mathcal{F} \succ \mathcal{G}$. Of course, $\mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{F} \preceq \mathcal{G}$. Even if $X = Y$, there are filters incomparable in the set-theoretical order, comparable in the RK-order. There are also filters (on each infinite cardinal number) incomparable in the RK-order (see [11] and [2, 3.1.6]). The Rudin-Keisler order was originally introduced for ultrafilters only (see [2] and [15]). We say that families $\mathcal{F}$ and $\mathcal{G}$ of subsets of X *mesh* (in symbols $\mathcal{G} \# \mathcal{F}$) whenever $G \cap F \neq \emptyset$ for every $G \in \mathcal{G}$, $F \in \mathcal{F}$.

2000 *Mathematics Subject Classification.* 54F65, 54C99.
Key words and phrases. fractal filter, radial topologies, supercontour.
This work is a part of a Ph. D. dissertation written under the supervision of Professor Szymon Dolecki.

A topology is *prime* if at most one element is not isolated. If a prime topology τ is not discrete and $\mathcal{F}$ is the restriction to the set of isolated elements of the neighborhood filter of the non-isolated element of a prime topology τ, then we say that $\mathcal{F}$ *generates* τ. We denote such a topology by Prime($\mathcal{F}$). A topology (on X) is $\mathcal{F}$*-radial* [12] (with respect to the filter $\mathcal{F}$ on Y) if $x \in \mathrm{cl}A$ implies the existence of a map $f : Y \to X$ such that $A \in f(\mathcal{F})$ and $x \in \lim f(\mathcal{F})$.

In [1], B. Boldjiev and V. Malyhin constructed a filter $\mathcal{F}$ on ω such that each sequential topology is $\mathcal{F}$-radial. It was shown in [5, p. 8] that if $\mathcal{H}$ is a supercontour then each subsequential[1] topology is $\mathcal{H}$-radial. It was also shown in [5] that if each sequential topology is $\mathcal{H}$-radial then Prime($\mathcal{H}$) is not subsequential. Therefore, if Prime($\mathcal{H}$) is $\mathcal{H}$-radial and every sequential topology is $\mathcal{H}$-radial, then there is a non-subsequential $\mathcal{H}$-radial topology.

A filter $\mathcal{F}$ is *fractal* whenever it is Rudin-Keisler equivalent to its every restriction to sets meshing with $\mathcal{F}$. This definition was motivated by the following proposition.

Proposition 1.1 ([5])**.** *The prime topology generated by a filter $\mathcal{F}$ is $\mathcal{F}$-radial if and only if $\mathcal{F}$ is fractal.*

Thus, by the considerations above and Proposition 1.1, if $\mathcal{F}$ is a fractal supercontour then the class of $\mathcal{F}$-radial topologies contains all sequential topologies, contains all subsequential topologies, and contains some non-subsequential topology.

It is not known whether fractal supercontours exist in ZFC. It is the aim of this paper to show that there exist fractal supercontours and there exist nonfractal supercontours under CH (see Corollary 3.4 and Corollary 3.8).

A *sequential cascade* (introduced in [4]) is a tree V without infinite branches, endowed with the least element $\emptyset_V$, with the finest topology such that for all but maximal elements of V ($v \in V \setminus \max V$), the cofinite filter on the set $(v)^+$ of immediate successors of v converges to v, and for all but maximal elements of V the set $(v)^+$ is infinitely countable. For a pairwise disjoint sequence (V_n) of sequential cascades, the *confluence* $((n) \hookleftarrow V_n)$ of cascades V_n stands for a sequential cascade V such that $(\emptyset_V)^+ = \{\emptyset_{V_n} : n < \omega\}$

[1] A topology is subsequential if it is a subspace of a sequential topological space.

with preserved order on branches of V_n (for each n). The *rank* $(r(v))$ of $v \in V$ is defined inductively to be 0 if $v \in \max V$ and the least ordinal greater than the ranks of the successors of v otherwise. The rank $r(V)$ of a cascade is by definition the rank of $\emptyset_V$.

If $v \in V \setminus \max V$ and $(v)^+$ is endowed with an order of the type ω, then we denote by $(v_n)_{n\in\omega}$ the sequence of elements of $(v)^+$. The cascade V is *monotone* if for every $v \in V \setminus \max V$ the sequence $(r(v_n))$ is non-decreasing.

We denote by $v^{\uparrow}$ (for $v \in V$) the subcascade of the cascade V formed of v and all successors of v with preserved order. By (n) we denote the cofinite filter on a given infinite countable set. If $\mathbb{F} = \{\mathcal{F}_s : s \in S\}$ is a family of filters on X and if $\mathcal{G}$ is a filter on S, then the *contour of* $\{\mathcal{F}_s\}$ *along* $\mathcal{G}$ is defined by

$$\int_{\mathcal{G}} \mathbb{F} = \int_{\mathcal{G}} \mathcal{F}_s = \bigcup_{G\in\mathcal{G}} \bigcap_{s\in G} \mathcal{F}_s.$$

When considering $\int_{\mathcal{G}} \mathcal{F}_s$ in the remainder of this paper, we assume that the family $\{\mathcal{F}_s\}$ and $\mathcal{G}$ fulfill the above assumptions unless indicated otherwise.

A similar construction was used in [7], [9], and [10]. For a sequential cascade V, we define $\int V$ by induction as follows: if $r(V) = 1$, then $\int V$ is the cofinite filter on $\max V$; if $r(V) > 1$, then $\int V = \int_{(n)} \int v_n^{\uparrow}$ for $\{v_n : n \in \omega\} = \emptyset_V^+$. A filter $\mathcal{F}$ is a *sequential contour* if $\mathcal{F} = \int V$ for some monotone sequential cascade V. It was shown in [5] that if V and W are monotone sequential cascades and $\int V = \int W$, then $r(V) = r(W)$. The rank of a sequential contour $\mathcal{F}$ is the rank of a monotone sequential cascade V such that $\mathcal{F} = \int V$. (For more information about sequential contours and sequential cascades, see [4].) It was shown in [5, Proposition 4.1] that all monotone sequential contours of the same rank are equivalent.

Theorem 1.2 ([5, Corollary 3.4]). *For ordinals $\alpha < \beta < \omega_1$, each monotone sequential contour of rank β is strictly RK greater than each monotone sequential contour of rank α.*

Let $\mathcal{F}$ be a filter on ω. We denote by $\beta(\mathcal{F})$ the set of all ultrafilters that are finer than $\mathcal{F}$. If $A \subset \omega$ then $\beta(A)$ is the set of all ultrafilters that contain A. The *Stone topology* on $\beta\omega$ has

$\{\beta(A) : A \subset \omega\}$ for a base. This topology is homeomorphic to the Čech-Stone compactification of ω with the discrete topology.

A sequence of filters $(\mathcal{F}_n)$ on ω is *discrete* if there is a sequence (F_n) of pairwise disjoint sets such that $F_n \in \mathcal{F}_n$, i.e., if $\{\beta(\mathcal{F}_n) : n \in \omega\}$ is a discrete family of subsets of $\beta\omega$.

The contour operation has the following, easily provable, properties.

Lemma 1.3 ([14]). *(a) Let $\mathcal{F}$ be a filter and let $(\mathcal{F}_n)$ be a discrete sequence of filters, such that $\mathcal{F}_n \approx \mathcal{F}$ for every $n \in \omega$. If $\mathcal{H} \approx \mathcal{G}$, then $\int_{\mathcal{G}} \mathcal{F}_n \approx \int_{\mathcal{H}} \mathcal{F}_n$.*

(b) Let $\mathcal{G}$ be a free filter and let $(\mathcal{F}_n)$ be a discrete sequence of filters such that $\mathcal{F}_n \preceq \mathcal{F}_{n+1}$. Then $\int_{\mathcal{G}} \mathcal{F}_n \succeq \mathcal{F}_m$ for each $m \in \omega$. If there exists a strictly increasing (in RK-order) subsequence $(\mathcal{F}_{n_k})$, then $\int_{\mathcal{G}} \mathcal{F}_n \succ \mathcal{F}_m$.

(c) If $\mathcal{G}$ is a free filter and if $(\mathcal{F}_n)$ is a discrete sequence of filters, then $\int_{\mathcal{G}} \mathcal{F}_n \succeq \mathcal{G}$.

(d) If $\mathcal{G}$ is a non-maximal filter and if $(\mathcal{F}_n)$ is a discrete sequence of filters, then $\int_{\mathcal{G}} \mathcal{F}_n$ is not an ultrafilter.

2. Fractal filters

The *restriction* $\mathcal{F} \mid_A$ of a filter $\mathcal{F}$ (on X) to a set $A \# \mathcal{F}$ is a filter on X(!) for which $\{F \cap A : F \in \mathcal{F}\}$ constitutes a base.[2] Recall that a filter $\mathcal{F}$ (on X) is *fractal* if $\mathcal{F} \mid_A \approx \mathcal{F}$ for all $A \# \mathcal{F}$. Since $\mathcal{F} \mid_A \succeq \mathcal{F}$ for every $A \# \mathcal{F}$, $\mathcal{F}$ is fractal if for each $A \# \mathcal{F}$ there is a map $f_A : X \to X$ such that $f_A(\mathcal{F}) \supset \mathcal{F} \mid_A$.

Proposition 2.1 ([5, Proposition 6.5]). *Each ultrafilter is fractal.*

Proposition 2.2 ([5, propositions 2.2 and 3.5]). *Each monotone sequential contour is fractal.*

The following lemma, whose proof we leave to the reader, will be instrumental in a study of fractal filters.

[2] We can also understand the restriction of a filter $\mathcal{F}$ to the set A as a filter $\mathcal{F} \mid_A$ on the set A such that $G \in \mathcal{F} \mid_A$ if $G = F \cap A$ for some $F \in \mathcal{F}$. This understanding of the $\mathcal{F} \mid_A$ is different, but the filters defined in these two ways are RK-equivalent.

Lemma 2.3. *Let $\mathbb{F} = \{\mathcal{F}_s : s \in S\}$ be a discrete family of filters on a set X, let $A \subset X$, and let $\mathcal{G}$ be a free filter on S such that $A \# \int_{\mathcal{G}} \mathbb{F}$. Then $(\int_{\mathcal{G}} \mathbb{F}) |_A = \int_{\mathcal{G}|_B} \mathbb{F}_A$ where $B = \{s \in S : \mathcal{F}_s \# A\}$, $\mathbb{F}_A = \{\mathcal{F}_s |_A : s \in B\}$.*

Notice also, that for each filter $\mathcal{F}$ on ω there exists a pairwise disjoint sequence $(\mathcal{F}_n)$ of filters on ω, such that $\mathcal{F}_n \approx \mathcal{F}$ for each $n < \omega$. Indeed, let (F_n) be a pairwise disjoint sequence of infinite subsets of ω. For each $n < \omega$, let $f_n : \omega \to F_n$ be an arbitrary bijection. Then the sequence $f_n(\mathcal{F})$ fulfills the required condition.

Lemma 2.4. *For each ultrafilter $\mathcal{H}$ on ω, there exists an ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} \succ \mathcal{H}$.*

Proof: Let $\mathcal{H}$ be an ultrafilter on ω. There are $2^{\aleph_0}$ maps from ω to ω and there are $2^{2^{\aleph_0}}$ ultrafilters on ω. Therefore, there exists an ultrafilter $\mathcal{L}$ that is not RK-smaller than $\mathcal{H}$. Consider the ultrafilter $\mathcal{U} = \int_{\mathcal{L}} \mathcal{H}_n$ where $(\mathcal{H}_n)_{n \in \omega}$ is a discrete sequence of ultrafilters such that $\mathcal{H}_n \approx \mathcal{H}$ for every $n \in \omega$. By Lemma 1.3, $\mathcal{U} \succeq \mathcal{H}$ and $\mathcal{U} \succeq \mathcal{L}$. Moreover, $\mathcal{L} \not\preceq \mathcal{H}$, so that $\mathcal{U} \succ \mathcal{H}$. □

Proposition 2.5. *Let $\mathcal{G}$ be a free filter on ω. Then $\int_{\mathcal{G}} \mathbb{F}$ is fractal for each discrete family $\mathbb{F} = \{\mathcal{F}_n : n \in \omega\}$ of fractal filters on ω if and only if $\mathcal{G}$ is an ultrafilter.*

Proof: Let $\{F_n : n \in \omega\}$ be a discrete family of sets such that $F_n \in \mathcal{F}_n$. Let $A \# \int_{\mathcal{G}} \mathbb{F}$. Since $\mathcal{G}$ is an ultrafilter then either $G = \{s \in \omega : A \# \mathcal{F}_n\}$ belongs to $\mathcal{G}$, or $G^c \in \mathcal{G}$. If $G^c \in \mathcal{G}$, then $A^c \in \int_{\mathcal{G}} \mathbb{F}$ in contradiction to the assumptions, so $G \in \mathcal{G}$. For each $n \in G$ there exists a function $f_n : F_n \to F_n$ such that $f_n(\mathcal{F}_n) \supset \mathcal{F}_n |_A$. We define $f : \bigcup_{n \in G} F_n \to \bigcup_{n \in G} F_n$ by $f |_{F_n} = f_n$. By Lemma 2.3 and the monotonicity of the contour operation, $f(\int_{\mathcal{G}} \mathbb{F}) \supset \int_{\mathcal{G}} \mathbb{F} |_A$.

If $\mathcal{G}$ is not an ultrafilter then there exists a set $G \# \mathcal{G}$ such that $G^c \# \mathcal{G}$. Let $\{F_n, n \in \omega\}$ be a pairwise disjoint sequence of infinite subsets of ω. For each $n \in G^c$, let $\mathcal{F}_n$ denote the cofinite filter on F_n. By Lemma 2.4, there exists an ultrafilter $\mathcal{J} \succ \int_{\mathcal{G}|_{G^c}} \mathcal{F}_n$. For each $n \in G$, let $\mathcal{F}_n$ be a filter on F_n such that $\mathcal{F}_n \approx \mathcal{J}$. Then

$$\left(\int_{\mathcal{G}} \mathcal{F}_n\right) |_{\bigcup_{n \in G^c} F_n} = \int_{\mathcal{G}|_{G^c}} \mathcal{F}_n \prec \mathcal{J} \preceq \int_{\mathcal{G}|_G} \mathcal{F}_n = \left(\int_{\mathcal{G}} \mathcal{F}_n\right) |_{\bigcup_{n \in G} F_n} .$$

In other words, for $\mathbb{F} = \{\mathcal{F}_n : n \in \omega\}$, the filter $\int_{\mathcal{G}} \mathbb{F}$ is not fractal. □

Let $\mathcal{G}$ and $\mathcal{F}$ be filters. We define the relation "$\ll$" as $\mathcal{G} \ll \mathcal{F}$ if $\mathcal{G} \mid_A \preceq \mathcal{F}$ for each $A \# \mathcal{G}$. The relation $\ll$ is not a partial order because if $\mathcal{G}$ is not fractal then $\mathcal{G} \mid_A \not\preceq \mathcal{G}$ for some $A \# \mathcal{G}$ and so $\mathcal{G} \not\ll \mathcal{G}$. Moreover, $\mathcal{G} \ll \mathcal{G}$ if and only if $\mathcal{G}$ is fractal. The relation "$\ll$" is transitive, $\mathcal{F} \ll \mathcal{G} \Rightarrow \mathcal{F} \preceq \mathcal{G}$, and if $\mathcal{G}$ is fractal, then $\mathcal{G} \ll \mathcal{F} \Leftrightarrow \mathcal{G} \preceq \mathcal{F}$.

Proposition 2.6. *Let $\{\mathcal{F}_n : n \in \omega\}$ be a discrete family of filters on ω and let $\mathcal{G}$ be a filter on ω. If for each $B \# \mathcal{G}$ there is a function $g : \omega \to \omega$ such that $g(\mathcal{G}) \supset \mathcal{G} \mid_B$ and $\mathcal{F}_n \gg \mathcal{F}_{g(n)}$, then $\int_{\mathcal{G}} \mathcal{F}_n$ is fractal.*

Proof: Let (F_n) be a discrete sequence of sets such that $F_n \in \mathcal{F}_n$. Let $A \# \int_{\mathcal{G}} \mathcal{F}_n$ and let $B = \{n \in \omega : A \# \mathcal{F}_n\}$. By Lemma 2.3, $(\int_{\mathcal{G}} \mathcal{F}_n) \mid_A = \int_{\mathcal{G}|_B} \mathcal{F}_n \mid_A$. By assumption, there exists a function $g : \omega \to \omega$ such that $g(\mathcal{G}) \supset \mathcal{G} \mid_B$ and such that $\mathcal{F}_n \gg \mathcal{F}_{g(n)}$. Since $\mathcal{F}_n \gg \mathcal{F}_{g(n)}$, for each $n \in \omega$ there exists a function $f_n : F_n \to F_{g(n)}$ such that $f_n(\mathcal{F}_n) \supset \mathcal{F}_{g(n)}$. Since $(\mathcal{F}_n)$ is discrete, we can define a function f by $f \mid_{F_n} = f_n$. Then $f(\int_{\mathcal{G}} \mathcal{F}_n) \supset \int_{\mathcal{G}|_B} \mathcal{F}_n \mid_A$ $= (\int_{\mathcal{G}} \mathcal{F}_n) \mid_A$. □

Theorem 2.7. *Let $(\mathcal{F}_n)$ be a discrete sequence of filters. If there is a cofinite subset D of ω such that $\mathcal{F}_n \ll \mathcal{F}_m$ for $n < m$, where $n, m \in D$, then $\int_{(n)} \mathcal{F}_n$ is fractal.*

Proof: We can assume that $D = \omega$, because a finite number of elements does not have any influence on the contour with respect to a free filter. Let us take $\int_{(n)} \mathcal{F}_n$ as in the assumptions. Let $A \# \int_{(n)} \mathcal{F}_n$. It is natural that $(\int_{(n)} \mathcal{F}_n) \mid_A \succeq \int_{(n)} \mathcal{F}_n$.

If $A \# \int_{(n)} \mathcal{F}_n$, then by Lemma 2.3, $(\int_{(n)} \mathcal{F}_n) \mid_A = \int_{(n)|_B} (\mathcal{F}_n \mid_A)$, where $B = \{n \in \omega : \mathcal{F}_n \# A\}$.

Without loss of generality, we can assume that $1 \in B$, because there are only finitely many elements of ω less than $\min B$. We can also assume that $\mathcal{F}_1$ is fractal because $f(\int_{(n)|_B} \mathcal{F}_n) = f(\int_{(n)|_{B \setminus \{1\}}} \mathcal{F}_n)$ for each function $f : \omega \to \omega$.

We define a function $g : \omega \to \omega$ as follows: $g(i) = \max_{n \in \omega}\{n \in B : n < i\}$ for $i > 1$, and $g(1) = 1$.

The function g fulfills the assumptions of Proposition 2.6. Indeed the function g is decreasing for $n > 1$, so in view of $\mathcal{F}_n \gg \mathcal{F}_{g(n)}$, for each $H \in (n)\,|_B$, a set $\{n \in \omega : n \geq \min H\}$ belongs to (n) and $g(\{n \in \omega : n \geq \min H\}) = H$. $\square$

Theorem 2.7 shows that, for a discrete sequence $(\mathcal{F}_n)$ of filters, the fractalness of $\int_{\mathcal{G}} \mathcal{F}_n$ depends on the interrelationship between an order introduced on the set $(\mathcal{F}_n)$ by $\ll$ and the filter $\mathcal{G}$. Moreover, if a set $(\mathcal{F}_n)$ is ordered by the relation $\ll$ in a type of some monotone sequential cascade V, then $\int_{\int W} \mathcal{F}_n$ is fractal for each monotone sequential cascade W such that $\int V \subset \int W$. For details, see [14].

Let $\mathcal{F} \in \beta\omega$. Recall that $\mathcal{F}$ is a *weak P-point* if $\mathcal{F} \notin \mathrm{cl}A$ for each countable $A \subset \beta\omega \setminus \omega$, $\mathcal{F} \notin A$.

Proposition 2.8. *There are fractal filters not of the form* $\int_{\mathcal{G}} \mathcal{F}_n$, *where* $(\mathcal{F}_n)$ *is a discrete sequence of filters and* $\mathcal{G}$ *is free.*

Proof: There are weak P-points in $\beta\omega$ [15, Theorem 4.3.3]; they are fractal by Proposition 2.1, and they are not of the form $\int_{\mathcal{G}} \mathcal{F}_n$. Indeed, if x is a weak P-point and if $x = \int_{\mathcal{G}} \mathcal{F}_n$ for some discrete sequence $(\mathcal{F}_n)$ of filters and for a free filter $\mathcal{G}$, then let us take an ultrafilter $\tilde{\mathcal{G}}$ and a sequence of ultrafilters $(\tilde{\mathcal{F}}_n)$ such that $\tilde{\mathcal{G}} \supset \mathcal{G}$ and $\tilde{\mathcal{F}}_n \supset \mathcal{F}_n$. We have $\int_{\tilde{\mathcal{G}}} \tilde{\mathcal{F}}_n \supset \int_{\mathcal{G}} \mathcal{F}_n$, but $\int_{\mathcal{G}} \mathcal{F}_n$ is an ultrafilter so $\int_{\tilde{\mathcal{G}}} \tilde{\mathcal{F}}_n = \int_{\mathcal{G}} \mathcal{F}_n$, so $\int_{\mathcal{G}} \mathcal{F}_n \in \mathrm{cl}(\{\tilde{\mathcal{F}}_n\}_{n \in \omega})$. It is a contradiction because $x \notin \mathrm{cl}X$ for any countable $X \not\ni x$. $\square$

Proposition 2.6 and Theorem 2.7 require a fractal filter $\mathcal{G}$ for a discrete sequence $(\mathcal{F}_n)$ of filters to get a fractal contour $\int_{\mathcal{G}} \mathcal{F}_n$. However, there exist a non-fractal filter $\mathcal{G}$ and a discrete sequence $(\mathcal{F}_n)$ of fractal filters, such that $\int_{\mathcal{G}} \mathcal{F}_n$ is fractal. For details, see [14].

Let us recall that a filter $\mathcal{F}$ is *substantial* if the cardinality of the set of all ultrafilters containing $\mathcal{F}$ is infinite. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on the same set. The supremum $\mathcal{G} \vee \mathcal{F}$ of filters $\mathcal{F}$, $\mathcal{G}$ is the coarsest filter finer than both $\mathcal{F}$ and $\mathcal{G}$; the infimum $\mathcal{F} \wedge \mathcal{G}$ is the finest filter coarser than both $\mathcal{F}$ and $\mathcal{G}$. This notation is also used for sets via identification with its principal filter.

Lemma 2.9. *Let* $(\mathcal{F}_n)$ *be a sequence of substantial filters. There exists a discrete sequence* (F_k) *of sets and an increasing sequence* (n_k) *such that* $F_k \# \mathcal{F}_{n_k}$ *and* $\mathcal{F}_{n_k}\,|_{F_k}$ *is a substantial filter for each* $k \in \omega$.

Proof: First notice that if $\mathcal{F}$ is a substantial filter (on ω) and $F \in \mathcal{F}$, then there exists a set $H \subset F$ such that $H\#\mathcal{F}$, $H^c\#\mathcal{F}$, and filters $\mathcal{F}\mid_H$ and $\mathcal{F}\mid_{H^c}$ are substantial. Indeed, if such a set does not exist, then for each subset H_1 of F such that $H_1\#\mathcal{F}$ and $H_1^c\#\mathcal{F}$, one of the filters $\mathcal{F}\mid_{H_1}$ or $\mathcal{F}\mid_{H_1^c}$ is not substantial (let us say $\mathcal{F}\mid_{H_1}$) and thus the second ($\mathcal{F}\mid_{H_1^c}$) is substantial. We take a filter $\mathcal{F}\mid_{H_1^c}$ and a set H_1^c and continue our procedure. Since we assume that a set H does not exist, then we can continue this procedure for each $n < \omega$. But notice that $\bigcup\limits_{n<\omega} H_{2n}$ and $\bigcup\limits_{n<\omega} H_{2n+1}$ fulfill the claim.

Now we will need the following statement (*): If A is a set and if $(\mathcal{F}_n)$ is a sequence of substantial filters such that for each $n < \omega$, $(A)^c \in \mathcal{F}_n$, then there exists a set $B \subset A^c$ and there exists a subsequence $(\mathcal{F}_{n_k})$ of a sequence $(\mathcal{F}_n)$ such that $B\#\mathcal{F}_{n_1}$, and $\mathcal{F}_1\mid_B$ is substantial, and for each $k \in \omega$, $\mathcal{F}_{n_k}\#(B \cup A)^c$ and $\mathcal{F}_{n_k}\mid_{(B\cup A)^c}$ is substantial.

Let $G \subset A^c$ be a set such that $\mathcal{F}_1\mid_G$ and $\mathcal{F}_1\mid_{G^c}$ are substantial. At least one of the following possibilities holds.

1) There exists a subsequence $(\mathcal{F}_{n_k})$ such that $G\#\mathcal{F}_{n_k}$ and $\mathcal{F}_{n_k}\mid_G$ is substantial.

2) There exists a subsequence $(\mathcal{F}_{n_k})$ such that $G^c\#\mathcal{F}_{n_k}$ and $\mathcal{F}_{n_k}\mid_{G^c}$ is substantial.

Indeed, if 1) is not true, then let $T = \{n < \omega$: 1^* G do not mesh $\mathcal{F}_n$ or 2^* ($G\#\mathcal{F}_n$ and $\mathcal{F}_n\mid_G$ is not substantial)$\}$. By assumption, T is cofinite so infinite. Let T_i $(i = 1, 2)$ be the set of all $n < \omega$ such that i^* holds. So $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T$, and thus, at least one of those sets is infinite. If T_1 is infinite, then $G^c \in \mathcal{F}_n$ for each $n \in T_1$ and so a sequence $(\mathcal{F}_n)_{n\in T_1}$ satisfies 2). If a set T_2 is infinite, then for each $n \in T_2$ there is $G^c\#\mathcal{F}_n$, and $\mathcal{F}_n\mid_{G^c}$ is substantial. Then a sequence $(\mathcal{F}_n)_{n\in T_2}$ satisfies 2).

If 1) holds, then let $B_1 = G$; if 2) holds, then let $B_1 = G^c$, and define $B = B_1 \cap A^c$.

To prove the lemma, we will prove by induction the following statement (**): Let $(F_i)_{i=1,..,n}$ be a pairwise disjoint n-sequence of sets and let $(\mathcal{F}_i)_{i=1,...,n}$ be an n-sequence of substantial filters such that $F_i \in \mathcal{F}_i$ for each $i \le n$. If $(\mathcal{F}_k)_{n<k<\omega}$ is a sequence of substantial filters such that $(\bigcup_{i=1}^n F_i)^c \in \mathcal{F}_k$ for each $n < k < \omega$, then there exist a set $F_{n+1} \subset (\bigcup_{i=1}^n F_i)^c$ and a subsequence $(k_j)_{j\in\omega}$ such that $k_j = j$ for $1 \le j \le n+1$, $F_{n+1}\#\mathcal{F}_{n+1}$, and $\mathcal{F}_{n+1}\mid_{F_{n+1}}$

is substantial and for each $n+1 < j < \omega$, $(\bigcup_{i=1}^{n+1} F_i)^c \# \mathcal{F}_{n_k}$, and $\mathcal{F}_{n_k} |_{(\bigcup_{i=1}^{n+1} F_i)^c}$ is substantial.

For $n = 1$, the claim is true by (*). If the claim is true for each $n < m$, then let us define $A = \bigcup_{i=1}^{m-1} F_i$. Thus by (*), the claim is true. Let $F_0 = \emptyset$; then by (*) and (**), the sequence (F_n) fulfills the claim of the lemma. □

Proposition 2.10 ([3]). *Let $\mathcal{F}$ be a monotone sequential contour and let $f : \omega \to \omega$. Then there exists a monotone sequential contour $\mathcal{G}$ such that $\mathcal{G} \supset f(\mathcal{F})$ and $r(\mathcal{G}) \leq r(\mathcal{F})$.*

Proof: We will prove by induction that under our assumptions there exists a monotone sequential contour $\mathcal{G}$ such that $\mathcal{G} \supset f(\mathcal{F})$, and $r(\mathcal{G}) = 0$ or $r(\mathcal{G}) = r(\mathcal{F})$. Let $\mathcal{F}$ be a monotone sequential contour; let $f : \omega \to \omega$; and if $r(\mathcal{F}) \geq 1$, then let $V = (n) \looparrowleft V_n$ be the monotone sequential cascade such that $\int V = \mathcal{F}$.

If $r(\mathcal{F}) = 0$, then the claim is obvious. If the claim is true for all $\alpha < \alpha_0$, then consider a sequence $(\mathcal{F}_n)$ of monotone sequential contours, such that $\mathcal{F}_n \supset f(\int V_n)$ and either $r(\mathcal{F}_n) = 0$ or $r(\mathcal{F}_n) = r(\int V_n)$. Such filters exist by inductive assumptions.

If there exists a subsequence $(n_k)_{k\in\omega}$ such that $r(\mathcal{F}_{n_k}) = 0$, then either there exists a stable subsubsequence (n_{k_m}) or $f(\int_{(k)} \mathcal{F}_{n_k})$ is a free filter. If there exists a stable subsubsequence (n_{k_m}), then $f(\int_{(m)} \mathcal{F}_{n_{k_m}}) \subset \mathcal{C}_0$ where $\mathcal{C}_0$ is a principal filter of $f(\int \mathcal{F}_{n_{k_1}})$. If $f(\int_{(k)} \mathcal{F}_{n_k})$ is a free filter, then $f(\int_{(m)} \mathcal{F}_{n_{k_m}}) \subset \mathcal{C}_1$ where $\mathcal{C}_1$ is a cofinite filter on the set $\{\mathcal{F}_{n_k}\}$, so it is enough to take any monotone sequential contour $\mathcal{C}_{r(\mathcal{F})}$ of rank $r(\mathcal{F})$.

If there exists a subsequence $(n_k)_{k\in\omega}$ such that $r(\mathcal{F}_{n_k}) > 0$, then by Lemma 2.9, there exist a pairwise disjoint sequence $(G_{n_{k_m}})$ of sets and an increasing subsequence n_{k_m} of a sequence n_k such that $G_{n_{k_m}} \# \mathcal{F}_{n_{k_m}}$. Then $(\mathcal{F}_{n_{k_m}} |_{G_{n_{k_m}}})$ is a discrete sequence of contours and $\int_{(m)} \mathcal{F}_{n_{k_m}} |_{G_{n_{k_m}}} \supset f(\int V)$. □

Proposition 2.11. *Let $\mathcal{G}$ be a free filter and let $(\mathcal{F}_n)$ be a discrete sequence of non-maximal filters. Then $\int_{\mathcal{G}} \mathcal{F}_n$ is not an ultrafilter.*

If $\mathcal{G}$ is an ultrafilter and $(\mathcal{F}_n)$ is a discrete sequence of sequential filters, then $\int_{\mathcal{G}} \mathcal{F}_n$ is not a sequential contour.

Proof: Let (F_n) be a discrete sequence of infinite countable sets, such that $F_n \in \mathcal{F}_n$. For each $n \in \omega$ there exist sets F_n^1, F_n^2

such that $F_n^i \# \mathcal{F}_n$, $F_n \supset F_n^i$ for $i = 1, 2$, and $F_n^1 \cap F_n^2 = \emptyset$. We have $\bigcup_{n\in\omega} F_n^1 \# \int_{\mathcal{G}} \mathcal{F}_n$ and $\bigcup_{n\in\omega} F_n^2 \# \int_{\mathcal{G}} \mathcal{F}_n$. The sets $\bigcup_{n\in\omega} F_n^1$ and $\bigcup_{n\in\omega} F_n^2$ are disjoint, so $\int_{\mathcal{G}} \mathcal{F}_n$ is not an ultrafilter.

By Lemma 1.3 c), there is a map f such that $f(\int_{\mathcal{G}} \mathcal{F}_n) \supset \mathcal{G}$. On the other hand, by Proposition 2.10, for every sequential contour $\mathcal{F}$ and each map f, there exists a (monotone) sequential contour $\mathcal{H} \supset f(\mathcal{F})$. Hence, if $\int_{\mathcal{G}} \mathcal{F}_n$ were a sequential contour, then there would be a sequential contour $\mathcal{H}$ finer than the ultrafilter $\mathcal{G}$, which is impossible. $\square$

In a similar way for a discrete sequence $\mathcal{F}_n$ of RK-equivalent ultrafilters, $\int_{(n)} \mathcal{F}_n$ is fractal and is neither an ultrafilter nor a sequential contour.

It is worth mentioning that the behavior of fractal filters under maps is difficult to anticipate. Images and preimages of fractal filters need not be fractal. Each class of RK-equivalent filters contains some nonfractal filter; moreover, there exist classes of RK-equivalent filters which do not contain any fractal filter. For details, see [14].

3. Supercontours

Recall that a *supercontour* (defined in [5]) is a filter of the form $\bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ is a monotone sequential contour of rank α, and $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for $\alpha < \beta < \omega_1$. Hence, each supercontour is a supremum of an increasing ω_1 sequence of fractal filters.

When considering supercontour $\bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ in the remainder of this paper, we assume that the family $\{\mathcal{F}_\alpha\}$ fulfills the assumptions above unless indicated otherwise. Recall also, that we still do not know if there exist fractal supercontours and if there exist non-fractal supercontours in ZFC. Results under CH are presented below.

Theorem 3.1 ([5, Theorem 4.6])**.** *There are $2^{\aleph_1}$ disjoint supercontours.*[3]

Theorem 3.2. *(CH) For each supercontour there exists a finer supercontour which is an ultrafilter.*

[3]The proof of [5, Theorem 4.6] that there are $2^{\aleph_1}$ supercontours shows also that there are $2^{\aleph_1}$ disjoint supercontours.

Proof: Let $\mathcal{S} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ be a supercontour, where $\mathcal{F}_\alpha$ is a monotone sequential contour of rank α and $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for $\alpha < \beta$. Under CH there exists an ω_1-sequence $\{A_\alpha, A_\alpha^c\}$ of pairs of sets, such that each subset of ω belongs to exactly one pair of this sequence. Indeed, under CH, there are exactly $\aleph_1$ subsets of each countable infinite set, so there are exactly $\aleph_1$ pairs described above, and so we can order them in type of ω_1.

Let $\mathcal{F}_\alpha^0 = \mathcal{F}_\alpha$.

If $\mathcal{S}$ is not an ultrafilter, then let α_1 be the least ordinal α, such that $A_\alpha \# \mathcal{S}$ and $A_\alpha^c \# \mathcal{S}$. Without loss of generality we can assume that $\alpha_1 > 1$. We define sequences $\{\mathcal{F}_\alpha^\beta\}_{\alpha<\omega_1}$ for $\beta \leq \alpha_1$ as follows:

$\mathcal{F}_\alpha^\beta = \mathcal{F}_\alpha$ for $\beta < \alpha_1$;
$\mathcal{F}_\alpha^{\alpha_1} = \mathcal{F}_\alpha$ for $\alpha < \alpha_1$;
$\mathcal{F}_\alpha^{\alpha_1} = \mathcal{F}_\alpha |_{A_{\alpha_1}}$ for $\alpha \geq \alpha_1$.

We define $\mathcal{S}^\beta = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha^\beta$.

If $\mathcal{S}^{\alpha_\gamma}$ is defined for each $\gamma < \gamma_0$ and γ_0 is not a limit ordinal, then either $\mathcal{S}^{\alpha_{\gamma_0 - 1}}$ is an ultrafilter or $\mathcal{S}^{\alpha_{\gamma_0 - 1}}$ is not an ultrafilter. In the first case, the proof is complete. In the second case, let α_{γ_0} be the least ordinal α such that $A_\alpha \# \mathcal{S}^{\alpha_{\gamma_0 - 1}}$ and $A_\alpha^c \# \mathcal{S}^{\alpha_{\gamma_0 - 1}}$.

We define $\mathcal{F}_\alpha^\beta = \mathcal{F}_\alpha^{\alpha_{\gamma_0 - 1}}$ for $\alpha_{\gamma_0 - 1} < \beta < \alpha_{\gamma_0}$;
$\mathcal{F}_\alpha^{\alpha_{\gamma_0}} = \mathcal{F}_\alpha^{\alpha_{\gamma_0 - 1}}$ for $\alpha < \alpha_{\gamma_0}$;
$\mathcal{F}_\alpha^{\alpha_{\gamma_0}} = \mathcal{F}_\alpha^{\alpha_{\gamma_0 - 1}} |_{A_{\alpha_{\gamma_0}}}$ for $\alpha \geq \alpha_{\gamma_0}$.

If $\mathcal{S}^{\alpha_\gamma}$ is defined for each $\gamma < \gamma_0$ and γ_0 is a limit ordinal, then let us choose any increasing sequence δ_n of ordinals such that $\lim_{n\in\omega} \delta_n = \gamma_0$.

By Lemma 2.9, there exists a subsequence $(\delta_{n_k})_{k\in\omega}$ of the sequence $(\delta_n)_{n\in\omega}$ and there exists a discrete sequence $(B_k)_{k\in\omega}$ of sets, such that $B_k \# \mathcal{S}^{\delta_{n_k}}$.

We define $\mathcal{G}_\alpha^{\alpha_{\gamma_0}} = \mathcal{F}_\alpha^{\alpha_\gamma}$ if $\alpha < \alpha_{\gamma_0}$ and $\alpha_\gamma < \alpha < \alpha_{\gamma+1}$;

$\mathcal{G}_\alpha^{\alpha_{\gamma_0}} = \int_{(k)} (\mathcal{F}_{\alpha-1}^{\delta_{n_k}} |_{B_k})$ if α has a predecessor and $\alpha \geq \alpha_{\gamma_0}$;
$\mathcal{G}_\alpha^{\alpha_{\gamma_0}} = \int_{(k)} (\mathcal{F}_{\eta_k}^{\delta_{n_k}} |_{B_k})$ if $\alpha \geq \alpha_{\gamma_0}$ and α is a limit number, where $(\eta_n)_{n\in\omega}$ is an increasing sequence of ordinals such that $\lim_{n\in\omega} \eta_n = \alpha$.

If $\bigcup_{\theta<\omega_1} \mathcal{G}_\theta^{\alpha_{\gamma_0}}$ is an ultrafilter, then we define $\mathcal{F}_\alpha^{\alpha_{\gamma_0}} = \mathcal{G}_\alpha^{\alpha_{\gamma_0}}$ for $\alpha < \omega_1$; otherwise, (if $\bigcup_{\theta<\omega_1} \mathcal{G}_\theta^{\alpha_{\gamma_0}}$ is not an ultrafilter), we define

$\mathcal{F}_\alpha^{\alpha_{\gamma_0}} = \mathcal{G}_\alpha^{\alpha_{\gamma_0}}$ for $\alpha < \alpha_{\gamma_0}$;

$\mathcal{F}_\alpha^{\alpha_{\gamma_0}} = \mathcal{G}_\alpha^{\alpha_{\gamma_0}} |_{A_{\alpha_{\gamma_0}}}$, for $\alpha_{\gamma_0} \leq \alpha < \omega_1$, where α_{γ_0} is the least ordinal α such that $A_\alpha \# \bigcup_{\theta<\omega_1} \mathcal{G}_\theta^{\alpha_{\gamma_0}}$ and $A_\alpha^c \# \bigcup_{\theta<\omega_1} \mathcal{G}_\theta^{\alpha_{\gamma_0}}$.

One can see that $\mathcal{F}_\alpha^\beta$ is a monotone sequential contour of rank α, and $\mathcal{F}_\alpha^\beta \subset \mathcal{F}_\alpha^\gamma$ for $\beta < \gamma$, and $\mathcal{F}_\alpha^\beta \subset \mathcal{F}_\gamma^\beta$ for $\alpha < \gamma$. Also, for each $\alpha < \omega_1$, either $A_\alpha \in \mathcal{S}^\alpha$ or $A_\alpha^c \in \mathcal{S}^\alpha$ by construction of $\mathcal{S}^\alpha$.

If $\mathcal{S}^{\alpha_0}$ is an ultrafilter for some $\alpha_0 < \omega_1$, then $\tilde{\mathcal{S}} = \mathcal{S}^{\alpha_0}$; if $\mathcal{S}^{\alpha_0}$ is not an ultrafilter for each $\alpha_0 < \omega_1$, then we define $\tilde{\mathcal{S}} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha^\alpha$. □

By Theorem 3.1 and Theorem 3.2, there is

Corollary 3.3. *(CH) There are* $2^{\aleph_1}$ *maximal supercontours on* ω*.*

By Corollary 3.3 and Proposition 2.1, we have

Corollary 3.4. *(CH) There are* $2^{\aleph_1}$ *fractal supercontours on* ω*.*

Proposition 3.5 ([5, p. 12])**.** *If* $(\mathcal{F}_n)$ *is a discrete sequence of supercontours, then* $\int_{(n)} \mathcal{F}_n$ *is a supercontour.*

As we can see, under CH, using Theorem 3.1 and Lemma 1.3(d), there are supercontours not of the form $\int_{(n)} \mathcal{F}_n$, for a discrete sequence $(\mathcal{F}_n)$ of filters.

Lemma 3.6. *Let* $(\mathcal{F}_n)$ *be a sequence of filters, let* $(\mathcal{G}_\alpha)_{\alpha<\omega_1}$ *be an increasing* ω_1*-sequence of monotone sequential contours, and let* $\mathcal{G}$ *be the supercontour,* $\mathcal{G} = \bigcup_{\alpha<\omega_1} \mathcal{G}_\alpha$*. Then* $\int_{\mathcal{G}} \mathcal{F}_n = \bigcup_{\alpha<\omega_1} \int_{\mathcal{G}_\alpha} \mathcal{F}_n$*.*

Proof: If $U \in \int_{\mathcal{G}} \mathcal{F}_n$, then there exists $G \in \mathcal{G}$ such that $U \in \mathcal{F}_n$ for all $n \in G$. But $G \in \mathcal{G}_\alpha$ for some $\alpha < \omega_1$ so that $U \in \int_{\mathcal{G}_\alpha} \mathcal{F}_n$ and therefore, $G \in \bigcup_{\alpha<\omega_1} \int_{\mathcal{G}_\alpha} \mathcal{F}_n$.

If $U \in \bigcup_{\alpha<\omega_1} \int_{\mathcal{G}_\alpha} \mathcal{F}_n$, then there exists $\alpha < \omega_1$ such that $U \in \int_{\mathcal{G}_\alpha} \mathcal{F}_n$, so there exists a set $G \in \mathcal{G}_\alpha$ such that $U \in \mathcal{F}_n$ for all $n \in G$. But $G \in \mathcal{G}$ and thus, $U \in \int_{\mathcal{G}} \mathcal{F}_n$. □

Theorem 3.7. *(CH) There exist* $2^{\aleph_1}$ *fractal non-maximal supercontours.*

Proof: Let (F_n) be a sequence of countable infinite disjoint sets; let $(\mathcal{F}_n)$ be a sequence of sequential filters such that $F_n \in \mathcal{F}_n$; and let $\mathcal{G} = \bigcup_{\alpha<\omega_1} \mathcal{G}_\alpha$ be an ultrafilter-supercontour (such exists by

Theorem 3.2) where $(\mathcal{G}_\alpha)$ is an increasing ω_1-sequence of monotone sequential contours such that $r(\mathcal{G}_\alpha) = \alpha$. Consider $\int_{\mathcal{G}} \mathcal{F}_n$. By Proposition 2.11, $\int_{\mathcal{G}} \mathcal{F}_n$ is neither an ultrafilter nor a sequential contour, but in view of Proposition 2.4, $\int_{\mathcal{G}} \mathcal{F}_n$ is fractal. We have $r(\int_{\mathcal{G}_\alpha} \mathcal{F}_n) = 1 + \alpha$, because $r(\mathcal{F}_n) = 1$. The sequence $(\int_{\mathcal{G}_\alpha} \mathcal{F}_n)_\alpha$ is increasing, because $\mathcal{G}_{\alpha_1} \subset \mathcal{G}_{\alpha_2}$ for $\alpha_1 < \alpha_2$, and a contour operation is monotone. By Lemma 3.6 and the fact that a sequential contour of rank 1 on $\bigcup_{n\in\omega} F_n$ is a subset of each sequential contour, $\int_{\mathcal{G}} \mathcal{F}_n$ is a supercontour.

Now let us take an ultrafilter-supercontour $\mathcal{H}$, such that $\mathcal{H} \neq \mathcal{G}$. There is a set H, such that $H \in \mathcal{H}$ and $H^c \in \mathcal{G}$. Then $\int_{\mathcal{H}} \mathcal{F}_n$ is neither an ultrafilter nor a sequential contour, but it is a fractal supercontour.

We have $\bigcup_{n\in H} F_n \in \int_{\mathcal{H}} \mathcal{F}_n$, $\bigcup_{n\in H^c} F_n \in \int_{\mathcal{G}} \mathcal{F}_n$, but $\bigcup_{n\in H} F_n \cap \bigcup_{n\in H^c} F_n = \emptyset$, so $\int_{\mathcal{H}} \mathcal{F}_n \neq \int_{\mathcal{G}} \mathcal{F}_n$. Therefore, for distinct ultrafilters-supercontours, the contours of a family $\{\mathcal{F}_n\}$ with respect to these ultrafilters are distinct. In view of Corollary 3.3, there exists $2^{\aleph_1}$ fractal supercontours which are not ultrafilters. $\square$

The infimum of two monotone sequential contours of the same rank is a monotone sequential contour of the same rank, and so the infimum of two supercontours is a supercontour.

Corollary 3.8. *($2^{\aleph_0} < 2^{\aleph_1}$) There are $2^{\aleph_1}$ non-fractal supercontours.*

Proof: By Theorem 3.1, there exist $2^{\aleph_1}$ supercontours on ω. There are $2^{\aleph_0}$ maps $\omega \to \omega$, so there are $2^{\aleph_1}$ classes of RK-equivalent supercontours. Now let us take two non-equivalent supercontours, $\mathcal{F}$ and $\mathcal{K}$, and a set $F \in \mathcal{F}$, such that $F^c \in \mathcal{K}$. The supercontour $\mathcal{G} = \mathcal{F} \wedge \mathcal{K}$ is not fractal because $\mathcal{G} |_F \approx \mathcal{F}$ and $\mathcal{G} |_{F^c} \approx \mathcal{K}$. $\square$

References

[1] B. Boldjiev and V. Malyhin, *The sequentiality is equivalent to the $\mathcal{F}$-Fréchet-Urysohn property*, Comment. Math. Univ. Carolin. **31** (1990), no. 1, 23–25.

[2] W. W. Comfort and S. Negropontis, *The Theory of Ultrafilters.* Die Grundlehren der mathematischen Wissenschaften, Band 211. New York-Heidelberg: Springer-Verlag, 1974.

[3] Szymon Dolecki, private communication.

[4] Szymon Dolecki and Frédéric Mynard, *Cascades and multifilters*, Topology Appl. **104** (2000), no. 1-3, 53–65.

[5] Szymon Dolecki, Andrzej Starosolski, and Stephen Watson, *Extension of multisequences and countably uniradial classes of topologies,* Comment. Math. Univ. Carolin. **44** (2003), no. 1, 165–181.

[6] S. P. Franklin and M. Rajagopalan, *On subsequential spaces*, Topology Appl. **35** (1990), no. 1, 1–19.

[7] Zdeněk Frolík, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91.

[8] ———, *Fixed points of maps of βN,* Bull. Amer. Math. Soc. 74 (1968), 187–191.

[9] Gerhard Grimeisen, *Gefilterte Summation von Filtern und iterierte Grenzprozesse. I,* Math. Annalen **141** (1960), 318–342.

[10] ———, *Gefilterte Summation von Filtern und iterierte Grenzprozesse. II,* Math. Annalen **144** (1961), 386–417.

[11] Kenneth Kunen, *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. **172** (1972), 299–306.

[12] Peter J. Nyikos, *Convergence in Topology,* in Recent Progress in General Topology. Ed. Michael Hrušek and Jan van Mill. Amsterdam: North-Holland, 1992. 537–570.

[13] Mary Ellen Rudin, *Partial orders on the types of βN,* Trans. Amer. Math. Soc. **155** (1971), 353–362.

[14] Andrzej Starosolski, *Fractal filters and supercontours.* Doctoral Dissertation, Universite de Bourgogne, 2003.

[15] Jan van Mill, *An Introduction to $\beta\omega$*, in Handbook of Set-Theoretic Topology. 1984. Ed. Kenneth Kunen and Jerry E. Vaughan. New York: Elsevier Science Publishing Co., 1988. 503–567.

Wydział Matemetyczno-Fizyczny; Politechnika Śląska; Gliwice, Poland

E-mail address: `andrzej.starosolski@polsl.pl`