Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 30, No.2, 2006
Pages 445-447

AN OBSERVATION ON STRONGLY HEWITT SPACE

Z. ERCAN

Abstract. We give a characterization of strongly Hewitt spaces. In particular we remark that Lindelöf space is not necessarily strongly Hewitt space.

Throughout this paper the word *space* will mean *completey regular Hausdorff topological space*. Stone-Cech compactification of a space X is denoted by βX. Recall that a space X is called a *Hewitt space* (or *a realcompact space*) if it is a closed subspace of a product space of $\mathbf{R}$. For a space X let $C(X)$ (respectively $C_b(X)$) be the ring of continuous (respectively continuous and bounded) real-valued functions on X. A space X is a Hewitt space if and only if for each $a \in \beta X \setminus X$ there exists $f \in C(\beta X)$ such that $0 < f(x)$ for each $x \in X$ and $f(a) = 0$ (see [**1**], p.215). This motivates the following definition, which is recently introduced in [**2**].

Definition 1. A space X is called a strongly Hewitt space if for each sequence (x_n) in $\beta X \setminus X$ there exist a subsequence (x_{k_n}) of (x_n) and $f \in C(\beta X)$ such that $0 < f(x)$ for each $x \in X$ and $f(x_{k_n}) = 0$ for each n.

Strongly Hewitt spaces are studied in [**2**] and [**3**]. It is natural to ask what happens in the above definition if the term $f(x_{k_n}) = 0$ is replaced by $f(x_{k_n}) \longrightarrow 0$. The answer to this is given in the next theorem. There are many ways construct of the Stone-Cech compactification βX of a space X. One of them is as follows:

2000 *Mathematics Subject Classification.* 54C35.
Key words and phrases. Hewitt space, strongly Hewitt space.

βX can be identified with the space of the ring homomorphisms π from $C_b(X)$ into $\mathbf{R}$ with $\pi(\mathbf{1}) = 1$, under the convergence

$$\pi_\alpha \longrightarrow \pi \Longleftrightarrow \pi_\alpha(f) \longrightarrow \pi(f)$$

for each $f \in C_b(X)$ and each $x \in X$ is identified by π_x, $\pi_x(f) = f(x)$. Under this approach a space X is strongly Hewitt if and only if for each sequence (π_n) in $\beta X \setminus X$ there exist $f \in C_b(X)$ with $0 < f(x)$ for each $x \in X$ and a subsequence (π_{k_n}) such that $\pi_{k_n}(f) = 0$ for each n. Now we can give a characterization of strongly Hewitt spaces as follows.

Theorem 2. *Let X be a space. The followings are equivalent.*
a. *X is a strongly Hewitt space.*
b. *X is a Hewitt space and $\beta X \setminus X$ is countably compact.*
c. *For each sequence (π_n) in $\beta X \setminus X$ there exist a subsequence (π_{k_n}) of (π_n) and $f \in C_b(X)$ with $0 < f(x)$ for each $x \in X$ such that $\pi_{k_n}(f) \longrightarrow 0$.*

Proof. The equivalence of **a** and **b** is proved in **[2]**. By the definition it is obvious that **a** implies **c**. It remains to prove **c** implies **b**. Suppose that **c** holds. By choosing (π_n) to be a constant sequence immediately we can see that X is a Hewitt space. To show $\beta X \setminus X$ is countably compact it is enough to show that each countable set in $\beta X \setminus X$ has an accumulation point. Let (π_n) be a sequence in $\beta X \setminus X$. Choose $f \in C_b(X)$ with $0 < f(x)$ for each $x \in X$ such that $\pi_{k_n}(f) \longrightarrow 0$ for some subsequence (π_{k_n}). As βX is compact there exists a subnet (π_α) of (π_{k_n}) such that $\pi_\alpha \longrightarrow \pi$ in βX. As $0 < f(x)$ for each $x \in X$ and $\pi_\alpha(f) \longrightarrow \pi(f)$, $\pi \notin X$. So $\pi_\alpha \longrightarrow \pi$ in $\beta X \setminus X$. This shows that π is an accumulation point of (π_n). This completes the proof. □

Remarks

1. Recall that a space X is *Lindelöf* if each open cover of X has a countable subcover. It is well known that each Lindelöf space is a Hewitt space (see **[1]**, p.216). It is natural to ask whether each Lindelöf sapce is strongly Hewitt space. We can use the above theorem to show that Lindelöf spaces are not necessarily strongly Hewitt. Namely, let I be the space of all irrational numbers with the topology induced from the real line. As I has a countable base it is the Lindelöf space. But I is not strongly Hewitt.

To see this suppose that I is a strongly Hewitt space. Then from the above Theorem $\beta I \setminus I$ is countably compact. But $\beta I \setminus I$ is σ-compact, and hence is the Lindelöf. Thus $\beta I \setminus I$ turned out to be a countably compact and Lindelöf one. Hence $\beta I \setminus I$ should be compact. But this is impossible, because I is not locally compact at none of the points of I.

2. The space $X = \mathbb{R}^w$ is Lindelöf. In **[2]** it is remarked that X is not Strongly Hewitt.

References

[1] R. Engelking, *General Topology*, Heldermann Verlag Berlin, 1989.

[2] J. Kakol and W. Sliwa, *Strongly Hewitt spaces*, Topology Appl. **119** (2002), 219-227.

[3] A. R. Todd and H. Render, *Continuous function spaces, (db)-spaces and strongly Hewitt spaces*, Topology Appl. **141** (2004), 171–186.

Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey

E-mail address: `zercan@metu.edu.tr`