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AN INSERTION THEOREM CHARACTERIZING PARACOMPACTNESS

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Abstract. For a space Y, $C_0(Y)$ denotes the Banach space of all real-valued continuous functions on Y vanishing at infinity. We prove that a Hausdorff space X is paracompact if and only if for every space Y and every two maps $g, h : X \to C_0(Y)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C_0(Y)$ with $g \leq f \leq h$. This result fails if 'for every space Y' is replaced by 'for every space Y which is the initial segment of an infinite cardinal'.

1. Introduction

By a space we mean a Hausdorff space. For a space Y, let $C_0(Y)$ be the Banach space of all real-valued continuous functions s on Y such that for each $\varepsilon > 0$ the set $\{y \in Y : |s(y)| \geq \varepsilon\}$ is compact, where $\|s\| \equiv \sup_{y \in Y} |s(y)|$ for $s \in C_0(Y)$. Gutev-Ohta-Yamazaki [2] defined upper and lower semi-continuity of a $C_0(Y)$-valued map (see Section 2 below) and proved the following theorem [2, Corollary 5.8]:

Theorem 1 (Gutev-Ohta-Yamazaki)**.** *For a space X, the following are equivalent*:

(1) *X is paracompact.*

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(2) *For every space Y and every upper semi-continuous map $g : X \to C_0(Y)$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f$.*
(3) *For every infinite cardinal κ and every upper semi-continuous map $g : X \to C_0(\kappa)$, there exists a continuous map $f : X \to C_0(\kappa)$ such that $g \leq f$.*

In view of usual insertion theorems such as Katětov-Tong's theorem [3, 6] characterizing normality (see also [1, 1.7.15 (b)]), it is natural to ask if each of the conditions (2) and (3) above can be replaced by the following conditions (4) and (5), respectively.

(4) *For every space Y and every two maps $g, h : X \to C_0(Y)$ such that g is upper semi- continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C_0(Y)$ such that $g \leq f \leq h$.*
(5) *For every infinite cardinal κ and every two maps g, $h : X \to C_0(\kappa)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map*
$f : X \to C_0(\kappa)$ such that $g \leq f \leq h$.

As we shall see in Section 2, if X is paracompact, then (4) holds by Michael's selection theorem, and clearly, (4) implies (5), i.e., we have (1) $\Rightarrow$ (4) $\Rightarrow$ (5). The purpose of this note is to show that (4) implies (1) but (5) is strictly weaker than (1). The former answers [2, Problem 5.9] positively.

Throughout this note, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. We always consider an ordinal a space with the usual order topology.

2. Definitions and results from the literature

Let X and Y be spaces. Recall from [2] that a map $f : X \to C_0(Y)$ is *lower* (resp. *upper*) *semi-continuous* if for every $x \in X$ and every $\varepsilon > 0$, there exists a neighborhood G of x in X such that if $x' \in G$, then $f(x')(y) > f(x)(y) - \varepsilon$ (resp. $f(x')(y) < f(x)(y) + \varepsilon$) for each $y \in Y$. For $s, t \in C_0(Y)$, we write $s \leq t$ if $s(y) \leq t(y)$ for each $y \in Y$, and for $f, g : X \to C_0(Y)$, we write $f \leq g$ if $f(x) \leq g(x)$ for each $x \in X$. For each $s, t \in C_0(Y)$ with $s \leq t$, the set $[s, t] = \{u \in C_0(Y) : s \leq u \leq t\}$ is a closed covex set in $C_0(Y)$. Let $\mathcal{F}_c(C_0(Y))$ denote the family of all nonempty closed convex sets in $C_0(Y)$. For two maps $g, h : X \to C_0(Y)$ with $g \leq h$, we define a map $[g, h] : X \to \mathcal{F}_c(C_0(Y))$ by $[g, h](x) = [g(x), h(x)]$ for $x \in X$. The following lemma is [2, Lemma 2.6]; we include a direct proof here, since the original proof in [2] is incorrect.

Lemma 1. *For every two maps $g, h : X \to C_0(Y)$ with $g \leq h$, if g is upper semi-continuous and h is lower semi-continuous, then the map $[g, h] : X \to \mathcal{F}_c(C_0(Y))$ is lower semi-continuous, i.e., the set $[g, h]^{-1}(U) = \{x \in X : [g, h](x) \cap U \neq \emptyset\}$ is open in X for every open set U in $C_0(Y)$.*

Proof. Let U be an open set in $C_0(Y)$ and $x \in [g, h]^{-1}[U]$. Since $[g, h](x) \cap U \neq \emptyset$, there exists $s \in U$ such that $g(x) \leq s \leq h(x)$. Choose $\varepsilon > 0$ such that $\{u \in C_0(Y) : \|s - u\| < \varepsilon\} \subseteq U$. Since g is upper semi-continuous and h is lower semi-continuous, there exists a neighbourhood G of x such that if $x' \in G$, then

$$g(x')(y) < g(x)(y) + \varepsilon/2 \leq s(y) + \varepsilon/2 \tag{2.1}$$

and

$$h(x')(y) > h(x)(y) - \varepsilon/2 \geq s(y) - \varepsilon/2 \tag{2.2}$$

for each $y \in Y$. It suffices to show show that $G \subseteq [g, h]^{-1}[U]$. To see this, take a point $x' \in G$, and define two elements $t, u \in C_0(Y)$ by $t(y) = \max\{g(x')(y), s(y)\}$ and $u(y) = \min\{h(x')(y), t(y)\}$ for $y \in Y$. Then $t \geq g(x')$ and $\|s - t\| < \varepsilon/2$ by (2.1), and hence, $g(x') \leq u \leq h(x')$ and $\|s - u\| < \varepsilon$ by (2.2). Thus, $u \in [g, h](x') \cap U$, which implies that $x' \in [g, h]^{-1}[U]$. Hence, $G \subseteq [g, h]^{-1}[U]$. □

Now, we show that every paracompact space X satisfies the condition (4). Assume that X is paracompact, and let $g, h : X \to C_0(Y)$ be maps such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$. Then $[g, h] : X \to \mathcal{F}_c(C_0(Y))$ is lower semi-continuous by Lemma 1. Hence, it follows from Michael's selection theorem [5, Theorem 3.2″] that there exists a continuous map $f : X \to C_0(Y)$ such that $f(x) \in [g, h](x)$ for each $x \in X$, which implies that $g \leq f \leq h$.

3. The condition (4)

We prove that every space X satisfying (4) is paracompact. As usual, we use the symbol $C(Y)$ instead of $C_0(Y)$ for a compact space Y. Let us consider the following condition on a space X:

(4′) For every compact space Y and every two maps $g, h : X \to C(Y)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C(Y)$ such that $g \leq f \leq h$.

Clearly (4) implies (4′), and the converse is also true in the realm of Tychonoff spaces since for a Tychonoff space Y, $C_0(Y)$ is isometrically embedded in $C(\beta Y)$, where βY is the Čech-Stone compactification of Y. Thus, it is enough to prove the following theorem:

Theorem 2. *If a space X satisfies* (4′), *then X is paracompact.*

Proof. Assume that X satisfies (4′). By [4, Theorem 5], it suffices to show that for every infinite cardinal κ, every monotone increasing open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X has a locally finite open refinement. Let Y be the quotient space obtained from the product space $(\kappa + 1) \times 2$ by identifying the points $\langle \kappa, 0\rangle$ and $\langle \kappa, 1\rangle$. We write

$$Y = \{\langle \alpha, i\rangle : \alpha < \kappa, i = 0, 1\} \cup \{\langle \kappa, 0\rangle\},$$

where $\langle \kappa, 0\rangle = \langle \kappa, 1\rangle$. Let $\lambda(x) = \min\{\alpha < \kappa : x \in U_\alpha\}$ for each $x \in X$. Define two maps $g, h : X \to C(Y)$ by

$$g(x)(\langle \alpha, i\rangle) = \begin{cases} 1 & \text{if } \alpha \leq \lambda(x) \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$h(x)(\langle \alpha, i\rangle) = \begin{cases} 0 & \text{if } \alpha \leq \lambda(x) \text{ and } i = 1, \\ 1 & \text{otherwise,} \end{cases}$$

for $x \in X$, respectively. To show that g is upper semi-continuous, let $x \in X$ and $\varepsilon > 0$ be fixed. For every $x' \in U_{\lambda(x)}$, if $\alpha \leq \lambda(x)$ and $i = 0$, then $g(x')(\langle \alpha, i\rangle) \leq 1 < g(x)(\langle \alpha, i\rangle) + \varepsilon$, and if $\alpha > \lambda(x)$ or $i = 1$, then $g(x')(\langle \alpha, i\rangle) = 0 < g(x)(\langle \alpha, i\rangle) + \varepsilon$, because $\lambda(x') \leq \lambda(x)$. Hence, g is upper semi-continuous. Similarly, we can prove that h is lower semi-continuous. Since $g \leq h$, it follows from (4′) that there exists a continuous map $f : X \to C(Y)$ such that $g \leq f \leq h$. Take a locally finite open cover $\mathcal{V}$ of X such that diameter $f[V] \leq 1/3$ for each $V \in \mathcal{V}$. To show that $\mathcal{V}$ is a refinement of $\mathcal{U}$, let $V \in \mathcal{V}$ and fix a point $x \in V$. We distinguish two cases: If $f(x)(\langle \kappa, 0\rangle) \leq 1/2$, then by the continuity of f, $f(x)(\langle \alpha, 0\rangle) < 2/3$ for some $\alpha < \kappa$. If there exists a point $y \in V \setminus U_\alpha$, then $\lambda(y) > \alpha$, and hence, $f(y)(\langle \alpha, 0\rangle) \geq g(y)(\langle \alpha, 0\rangle) = 1$ by the definition of g.

Thus, $\|f(x)-f(y)\| > 1/3$, which contradicts the fact that diameter $f[V] \leq 1/3$. Hence, $V \subseteq U_\alpha$. If $f(x)(\langle\kappa, 0\rangle) \geq 1/2$, then by the continuity of f, $f(x)(\langle\beta, 1\rangle) > 1/3$ for some $\beta < \kappa$. If there exists a point $y \in V \setminus U_\beta$, then $\lambda(y) > \beta$, and hence, $f(y)(\langle\beta, 1\rangle) \leq h(y)(\langle\beta, 1\rangle) = 0$ by the definition of h. Thus, $\|f(x) - f(y)\| > 1/3$, which also contradicts the fact that diameter $f[V] \leq 1/3$. Hence, $V \subseteq U_\beta$. Consequently, $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. □

4. The condition (5)

We show that a non-paracompact space can satisfy (5) by proving that every infinite cardinal with uncountable cofinality (in particular, the first uncountable cardinal ω_1) satisfies (5). First, we consider the following condition on a space X:

(5′) For every infinite cardinal κ and every two maps $g, h : X \to C(\kappa + 1)$ such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$, there exists a continuous map $f : X \to C(\kappa + 1)$ such that $g \leq f \leq h$.

The condition (5′) implies (5) since $C_0(\kappa)$ can be isometrically embedded in $C(\kappa+1)$. Thus, it suffices to prove the following theorem:

Theorem 3. *If τ is an infinite cardinal with uncountable cofinality, then the ordinal space τ satisfies* (5′).

Before proving this, we give some lemmas. Let τ be an infinite cardinal with uncountable cofinality, and fix an infinite cardinal κ. For a map $f : \tau \to C(\kappa + 1)$, the capital letter F denotes the real-valued function on $\tau \times (\kappa + 1)$ defined by $F(\langle\alpha, \lambda\rangle) = f(\alpha)(\lambda)$ for $\langle\alpha, \lambda\rangle \in \tau \times (\kappa+1)$. The proof of the following lemma is left to the reader.

Lemma 2. *Let $f : \tau \to C(\kappa + 1)$ be a map. If f is upper semi-continuous, then so is F. If f is lower semi-continuous, then so is F.*

Next, we associate two real-valued functions F^* and F_* on $\kappa + 1$ with a map $f : \tau \to C(\kappa + 1)$. For each $\lambda \leq \kappa$, let

$$\mathcal{N}(\lambda) = \{(\tau \setminus \alpha) \times ((\lambda + 1) \setminus \mu) : \alpha < \tau,\, \mu < \lambda\}.$$

The functions F^* and F_* are defined by

$$F^*(\lambda) = \inf_{U \in \mathcal{N}(\lambda)} \sup_{p \in U} F(p) \quad \text{and} \quad F_*(\lambda) = \sup_{U \in \mathcal{N}(\lambda)} \inf_{p \in U} F(p),$$

for $\lambda \le \kappa$, respectively. Note that every member of $\mathcal{N}(\lambda)$ is countably compact and every real-valued, upper (resp. lower) semi-continuous function on a countably compact space is bounded above (resp. below). Hence, by Lemma 2, if f is upper (resp. lower) semi-continuous, then F^* (resp. F_*) is well-defined.

Lemma 3. *Let $f : \tau \to C(\kappa + 1)$ be a map. If f is upper semi-continuous, then F^* is continuous. If f is lower semi-continuous, then F_* is continuous.*

Proof. We prove only the first statement since the second can be proved similarly. Assume that f is upper semi-continuous. It is easy to prove that F^* is upper semi-continuous whether or not f is too. Thus, it remains to show that F^* is lower semi-continuous. Suppose on the contrary that there exists $r \in \mathbb{R}$ such that the set $G = \{\lambda \le \kappa : F^*(\lambda) > r\}$ is not open. Fix $\lambda \in G \setminus \mathrm{int}_{\kappa+1} G$ and choose $s, t \in \mathbb{R}$ with $F^*(\lambda) > t > s > r$. Note that λ is a limit ordinal. First, put $U_0 = \tau \times (\lambda + 1)$. Then $\sup_{p \in U_0} F(p) > t$ because $F^*(\lambda) > t$ and $U_0 \in \mathcal{N}(\lambda)$, and hence, there exists a point $p_0 = \langle \alpha(0), \mu(0) \rangle \in U_0$ such that $F(p_0) > t$. Since λ is a limit and $f(\alpha(0)) = F(\langle \alpha(0), \cdot \rangle)$ is continuous, we may assume that $\mu(0) < \lambda$. Since $\lambda \in G \setminus \mathrm{int}_{\kappa+1} G$, there exists $\nu(0) \in \lambda \setminus \mu(0)$ such that $\nu(0) \notin G$, i.e., $F^*(\nu(0)) \le r$, which implies that $\sup_{p \in V} F(p) < s$ for some $V \in \mathcal{N}(\nu(0))$. Thus, we can find $\beta(0) \in \tau \setminus \alpha(0)$ such that $F(\langle \gamma, \nu(0) \rangle) < s$ for each $\gamma \in \tau \setminus \beta(0)$. Next, if we put $U_1 = (\tau \setminus \beta(0)) \times ((\lambda+1) \setminus \nu(0))$, then $\sup_{p \in U_1} F(p) > t$, because $F^*(\lambda) > t$ and $U_1 \in \mathcal{N}(\lambda)$. Hence, we can find a point $p_1 = \langle \alpha(1), \mu(1) \rangle \in U_1$ such that $F(p_1) > t$ and $\mu(1) < \lambda$ as above. Repeating this process, we can define sequences $\alpha, \beta : \omega \to \tau$ and $\mu, \nu : \omega \to \lambda$ such that

$$\forall n \, (\alpha(n) \le \beta(n) \le \alpha(n+1) \text{ and } \mu(n) \le \nu(n) \le \mu(n+1)), \tag{4.1}$$

$$\forall n \, (F(\langle \alpha(n), \mu(n) \rangle) > t), \text{ and} \tag{4.2}$$

$$\forall n, \gamma \, (\text{if } \beta(n) \le \gamma < \tau, \text{then } F(\langle \gamma, \nu(n) \rangle) < s). \tag{4.3}$$

Put $\alpha_0 = \sup_{n<\omega} \alpha(n)$ and $\mu_0 = \sup_{n<\omega} \mu(n)$. Then, $\alpha_0 < \tau$ since the cofinality of τ is uncountable, and $\mu_0 \le \lambda$. Since f is upper semi-continuous, so is F by Lemma 2. Thus, the set

$\{p \in \tau \times (\kappa+1) : F(p) \geq t\}$ is closed in $\tau \times (\kappa+1)$, which implies that $F(\langle \alpha_0, \mu_0 \rangle) \geq t$ by (4.2). On the other hand, $F(\langle \alpha_0, \nu(n) \rangle) < s$ for all $n < \omega$ by (4.1) and (4.3). Since the sequence ν converges to μ_0 by (4.1), this contradicts the continuity of $f(\alpha_0) = F(\langle \alpha_0, \cdot \rangle)$. Hence, F^* is continuous. □

We now prove Theorem 3.

Proof of Theorem 3. Let τ be an infinite cardinal with uncountable cofinality, and fix an infinite cardinal κ. Let $g, h : \tau \to C(\kappa+1)$ be maps such that g is upper semi-continuous, h is lower semi-continuous and $g \leq h$. Then, we can define the functions $G, H : \tau \times (\kappa+1) \to \mathbb{R}$ and $G^*, H_* : \kappa+1 \to \mathbb{R}$ as above. By Lemma 3, both G^* and H_* are continuous. Hence, we can extend g to $g^* : \tau+1 \to C(\kappa+1)$ by letting $g^*(\tau) = G^*$ and $g^*|_\tau = g$, and h to $h_* : \tau+1 \to C(\kappa+1)$ by letting $h_*(\tau) = H_*$ and $h_*|_\tau = h$.

Fact 1. *$g^* \leq h_*$.*

Proof. Since $g \leq h$, it is enough to show that $G^*(\lambda) \leq H_*(\lambda)$ for each $\lambda \leq \kappa$. Since G^* and H_* are continuous, it is enough to show that this holds for every isolated ordinal $\lambda \leq \kappa$. Fix an isolated ordinal $\lambda \leq \kappa$, and define functions G_λ and H_λ on τ by $G_\lambda(\alpha) = G(\langle \alpha, \lambda \rangle)$ and $H_\lambda(\alpha) = H(\langle \alpha, \lambda \rangle)$ for $\alpha < \tau$, respectively. Then, by Lemma 2, G_λ is upper semi-continuous and H_λ is lower semi-continuous. Since $G_\lambda \leq H_\lambda$ and τ is normal, it follows from Katětov-Tong's insertion theorem (see [1, 1.7.15 (b)]) that there exists a continuous function $k : \tau \to \mathbb{R}$ such that $G_\lambda \leq k \leq H_\lambda$. Then, since the cofinality of τ is uncountable, we can find $\beta < \tau$ such that k takes a constant value r on $\tau \setminus \beta$. Hence, $G(\langle \gamma, \lambda \rangle) \leq r \leq H(\langle \gamma, \lambda \rangle)$ for each $\gamma \in \tau \setminus \beta$. Since λ is an isolated ordinal, this implies that $G^*(\lambda) \leq H_*(\lambda)$. □

Fact 2. *The map g^* is upper semi-continuous and h_* is lower semi-continuous.*

Proof. We only prove the upper semi-continuity of g^* since the proof for h_* is similar. It suffices to show that for every $\varepsilon > 0$, there exists $\beta < \tau$ such that

$$g(\gamma)(\lambda) \leq g^*(\tau)(\lambda) + \varepsilon \tag{4.4}$$

for each $\gamma \in \tau \setminus \beta$ and each $\lambda \leq \kappa$. Let $\varepsilon > 0$ be fixed. Since $g : \tau \to C(\kappa+1)$ is upper semi-continuous, for every $\alpha < \tau$, there is $\tau(\alpha) < \alpha$ such that $g(\gamma)(\lambda) < g(\alpha)(\lambda) + \varepsilon/2$ for each $\gamma \in (\alpha+1) \setminus \tau(\alpha)$ and each $\lambda \leq \kappa$. By the pressing down lemma, we can find $\beta < \tau$ such that the set $A = \{\alpha < \tau : \tau(\alpha) = \beta\}$ is cofinal in τ. To show that this β is a required one, let $\gamma \in \tau \setminus \beta$ and $\lambda \leq \kappa$. Then, by the definition of $g^*(\tau)(\lambda)$ $(= G^*(\lambda))$, there exists $\alpha_1 < \tau$ such that $g(\alpha)(\lambda) \leq g^*(\tau)(\lambda) + \varepsilon/2$ for each $\alpha \in \tau \setminus \alpha_1$. Pick $\alpha_2 \in A \cap (\tau \setminus \max\{\gamma, \alpha_1\})$. Then, since

$$g(\gamma)(\lambda) < g(\alpha_2)(\lambda) + \varepsilon/2 \quad \text{and} \quad g(\alpha_2)(\lambda) < g^*(\tau)(\lambda) + \varepsilon/2,$$

we have (4.4). Hence, g^* is upper semi-continuous. □

By Facts 1, 2 and Lemma 1, the map $[g^*, h_*] : \tau+1 \to \mathcal{F}_c(C(\kappa+1))$ is lower semi-continuous. Hence, it follows from Michael's selection theorem [5, Theorem 3.2″] that there exists a continuous map $f : \tau + 1 \to C(\kappa + 1)$ such that $f(x) \in [g^*, h_*](x)$ for each $x \in X$, which implies that $g^* \leq f \leq h_*$. Then, the restriction $f|_\tau$ is also continuous and $g \leq f|_\tau \leq h$. □

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