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ON THE UNIQUENESS OF THE HYPERSPACES 2^X AND $\mathcal{C}_n(X)$ OF RIM-METRIZABLE CONTINUA

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ABSTRACT. A continuum X has unique hyperspace $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$ provided that if Y is a continuum and Γ_X is homeomorphic to Γ_Y, then X is homeomorphic to Y. In [4], I. Lončar proved that, in the realm of rim-metrizable continua, the following classes of spaces have unique hyperspace $\mathcal{C}_1(X)$: hereditarily indecomposable continua, smooth fans and indecomposable continua whose proper and non-degenerate subcontinua are arcs. In this paper, we prove that every rim-metrizable hereditarily indecomposable continuum has unique hyperspace $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$.

1. Introduction

In [7], Professor S. Nadler Jr. proved that hereditarily indecomposable metric continua have unique hyperspace $\mathcal{C}(X)$, then Professor S. Macías proved that those continua have unique hyperspaces 2^X and $\mathcal{C}_n(X)$ (see [5, p. 416] and [6, 6.1], respectively). Later, Professor I. Lončar proved that rim-metrizable hereditarily indecomposable continua have unique hyperspace $\mathcal{C}(X)$ (see [4, Theorem 2.4]). In this paper, we prove that rim-metrizable hereditarily indecomposable continua have unique hyperspaces 2^X and $\mathcal{C}_n(X)$. The paper is divided into 2 sections. In section 2, we give the definitions and notation for understanding the paper. In section 3, we present the main result of the paper.

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2. Definitions and notation

By a *space* we mean a topological space. The closed interval $[0, 1]$ is denoted by I. A *Hilbert cube* is a space homeomorphic to $\prod\{I_n : n \in \mathbb{N}\}$, where each $I_n = I$. By a *map* we mean a continuous function. A map $f : X \to Y$ between spaces is *monotone* provided that all fibers $f^{-1}(y)$ are connected. The weight of a space X is denoted by $\omega(X)$. A *continuum* is a non-empty Hausdorff compact connected space. A *subcontinuum* is a continuum contained in a space. A continuum X is *decomposable* provided that $X = A \cup B$, where A and B are proper subcontinua of X. A continuum is *indecomposable* if it is not decomposable. A continuum is *hereditarily indecomposable* provided that each subcontinuum of it is indecomposable. The symbol $\mathbb{N}$ denotes the set of the positive integers.

Given a Hausdorff compact space X, we denote by 2^X the family of all non-empty closed subsets of X. Given $n \in \mathbb{N}$, we denote by $\mathcal{C}_n(X)$ the family of all non-empty closed subsets of X having at most n components and by $\mathcal{F}_n(X)$ the family of all non-empty closed subsets of X having at most n points. The topology on 2^X is the Vietoris Topology (see [2, 2.7.20. (a)]) and the spaces $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are considered as subspaces of 2^X. The spaces 2^X, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are called *hyperspaces* of X. Note that the hyperspace $\mathcal{F}_1(X)$ is homeomorphic to X.

Given a map $f : X \to Y$ between Hausdorff compact spaces, we define the function $2^f : 2^X \to 2^Y$ by $2^f(E) = f[E]$ for $E \in 2^X$. By [2, 3.12.27. (e)], the function 2^f is continuous. Note that $2^f[\mathcal{C}_n(X)] \subseteq \mathcal{C}_n(Y)$ and $2^f[\mathcal{F}_n(X)] \subseteq \mathcal{F}_n(Y)$. The restriction $2^f\mid_{\mathcal{C}_n(X)}$ is denoted by $\mathcal{C}_n(f)$ and the restriction $2^f\mid_{\mathcal{F}_n(X)}$ is denoted by $\mathcal{F}_n(f)$.

From [2, 3.12.27. (a) and 3.12.27. (b)] we have the following result:

Theorem 2.1. *If X is a Hausdorff compact space, then the hyperspace 2^X is a Hausdorff compact space and $\omega(2^X) = \omega(X)$.*

By [3, 14.9 and 15.12], we obtain:

Theorem 2.2. *If X is a metrizable continuum, then the hyperspace 2^X and each of the hyperspaces $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are metrizable continua.*

An *inverse system* is a family $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$, where $(\Lambda, \leq)$ is a directed set, X_α is a space for every $\alpha \in \Lambda$, and for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$, $f_\alpha^\beta : X_\beta \to X_\alpha$ is a map such that:

i) f_α^α is the identity map on X_α for every $\alpha \in \Lambda$ and
ii) $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$ for any $\alpha, \beta, \gamma \in \Lambda$ satisfying $\alpha \leq \beta \leq \gamma$.

The maps f_α^β are called *bonding maps* and the spaces X_α are called *coordinate spaces*. A subset Σ of a directed set Λ is *cofinal* provided that for every $\alpha \in \Lambda$ there exists $\beta \in \Sigma$ such that $\alpha \leq \beta$.

Given a point $\hat{x}$ in a product $\prod \{X_\alpha : \alpha \in \Lambda\}$, we write $\hat{x} = (x_\alpha)_{\alpha \in \Lambda}$.

Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse system. The subspace of the product $\prod \{X_\alpha : \alpha \in \Lambda\}$ consisting of all points $\hat{x}$ such that $x_\alpha = f_\alpha^\beta(x_\beta)$ for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$ is called the *inverse limit* of the inverse system S, which is denoted by $\underleftarrow{\text{Lim}} S$ or by X_Λ. We define the projection map $f_\alpha^\Lambda : X_\Lambda \to X_\alpha$ by $f_\alpha^\Lambda(\hat{x}) = x_\alpha$.

The following result is well known but we include it for the convenience of the reader since the proof is short.

Theorem 2.3. *Let $f : X \to Y$ be an onto monotone map between continua. If X is an indecomposable continuum (hereditarily indecomposable continuum), then Y is an indecomposable continuum (hereditarily indecomposable continuum).*

Proof. Suppose Y is decomposable. Let E and F be two proper subcontinua of Y such that $Y = E \cup F$. By [2, 6.1.29], the sets $f^{-1}[E]$ and $f^{-1}[F]$ are connected, then they are continua. Since $X = f^{-1}[E] \cup f^{-1}[F]$ and $f^{-1}[E]$ and $f^{-1}[F]$ are proper subcontinua of X we conclude that X is decomposable.

If X is hereditarily indecomposable and Z is a subcontinuum of Y, then, by [2, 6.1.29], we deduce that the set $f^{-1}[Z]$ is a continuum. Since the map $f\,|_{f^{-1}[Z]}$: $f^{-1}[Z] \to Z$ is monotone, by the first part of this Theorem, we conclude that Z is indecomposable. Therefore, Y is hereditarily indecomposable. $\square$

Notation 2.4. Given an inverse system $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ of Hausdorff compact spaces, let $2^S = \{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Lambda\}$, let $\mathcal{C}_n(S) = \{\mathcal{C}_n(X_\alpha), \mathcal{C}_n(f_\alpha^\beta), \Lambda\}$ and let $\mathcal{F}_n(S) = \{\mathcal{F}_n(X_\alpha), \mathcal{F}_n(f_\alpha^\beta), \Lambda\}$.

Theorem 2.5. *Let* $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ *be an inverse system of Hausdorff compact spaces. Then the families* 2^S, $\mathcal{C}_n(S)$, *and* $\mathcal{F}_n(S)$ *are inverse systems and the map* $h : 2^{\underleftarrow{\lim} S} \to \underleftarrow{\lim} 2^S$ *defined by* $h(E) = \left(f_\alpha^\Lambda[E]\right)_{\alpha\in\Lambda}$ *is a homeomorphism. Moreover,* $h\left[\mathcal{C}_n\left(\underleftarrow{\lim} S\right)\right] = \underleftarrow{\lim}\mathcal{C}_n(S)$ *and* $h\left[\mathcal{F}_n\left(\underleftarrow{\lim} S\right)\right] = \underleftarrow{\lim}\mathcal{F}_n(S)$.

Proof. It is not difficult to see that the families 2^S, $\mathcal{C}_n(S)$, and $\mathcal{F}_n(S)$ are inverse systems. Given $\alpha \in \Lambda$, let q_α be the projection map from $\underleftarrow{\lim} 2^S$ into 2^{X_α}. Note that, by [2, 3.2.13], $\underleftarrow{\lim} S$ is a Hausdorff compact space. Then, by 2.1, the hyperspace 2^{X_α} is a Hausdorff compact space for every $\alpha \in \Lambda \cup \{\Lambda\}$.

Since $2^{f_\alpha^\Lambda} = 2^{f_\alpha^\beta} \circ 2^{f_\beta^\Lambda}$ for all $\alpha, \beta \in \Lambda$ such that $\alpha \leq \beta$, by [2, 2.5.F], the family $\{2^{f_\alpha^\Lambda} : \alpha \in \Lambda\}$ induces a map $f : 2^{\underleftarrow{\lim} S} \to \underleftarrow{\lim} 2^S$ such that $2^{f_\alpha^\Lambda} = q_\alpha \circ f$ for each $\alpha \in \Lambda$. Then $h = f$.

Now, we define the inverse function of h. Given $(E_\alpha)_{\alpha\in\Lambda} \in \underleftarrow{\lim} 2^S$, we have that $E_\alpha = 2^{f_\alpha^\beta}(E_\beta) = f_\alpha^\beta[E_\beta]$. Then, by [2, 3.2.13], the space $E_\Lambda = \underleftarrow{\lim}\{E_\alpha, f_\alpha^\beta|_{E_\beta}, \Lambda\}$ is a non-empty compact space and, by [2, 3.2.15], $E_\alpha = f_\alpha^\Lambda[E_\Lambda]$ for each $\alpha \in \Lambda$. Since E_Λ is contained in X_Λ, we can define $h'\left((E_\alpha)_{\alpha\in\Lambda}\right) = E_\Lambda$. Moreover, every $E \in 2^{\underleftarrow{\lim} S}$ can be written as $E = \underleftarrow{\lim}\{f_\alpha^\Lambda[E], f_\alpha^\beta|_{f_\beta^\Lambda[E]}, \Lambda\}$, by [2, 2.5.6]. Then h' is the inverse function of h. Thus, the map h is a homeomorphism since the space $2^{\underleftarrow{\lim} S}$ is compact.

In order to see that $h\left[\mathcal{C}_n\left(\underleftarrow{\lim} S\right)\right] = \underleftarrow{\lim}\mathcal{C}_n(S)$, let $(E_\alpha)_{\alpha\in\Lambda} \in \underleftarrow{\lim}\mathcal{C}_n(S)$. By [8, Lemma 1], we have that $h'\left((E_\alpha)_{\alpha\in\Lambda}\right) = E_\Lambda \in \mathcal{C}_n\left(\underleftarrow{\lim} S\right)$. The other inclusion is clear. In a similar way we get $h\left[\mathcal{F}_n\left(\underleftarrow{\lim} S\right)\right] = \underleftarrow{\lim}\mathcal{F}_n(S)$. □

3. Uniqueness of hyperspaces 2^X and $\mathcal{C}_n(X)$

Our main result is Theorem 3.20, in which we prove that hereditarily indecomposable rim-metrizable continua X have unique hyperspaces 2^X and $\mathcal{C}_n(X)$. We begin with a couple of definitions and continue with all the required results to obtain our main Theorem.

Definition 3.1. A subset Σ of a directed set Λ is a *chain* provided that for any $\alpha, \beta \in \Sigma$ we have that $\alpha \leq \beta$ or $\beta \leq \alpha$. A directed set Λ is called *σ-complete* provided that for every sequence $\{\alpha_n : n \in \mathbb{N}\}$ in Λ there exists $\sup\{\alpha_n : n \in \mathbb{N}\} \in \Lambda$.

Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse system and let $\Sigma \subseteq \Lambda$ be a chain with $\gamma = \sup\Sigma \in \Lambda$. By [2, 2.5.F], the family $\{f_\alpha^\gamma : \alpha \in \Sigma\}$ induces a map $h_\gamma : X_\gamma \to \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$ such that $f_\alpha^\gamma = f_\alpha^\Sigma \circ h_\gamma$ for every $\alpha \in \Sigma$. Note that h_γ is defined by $h_\gamma(x_\gamma) = (f_\alpha^\gamma(x_\gamma))_{\alpha\in\Sigma}$.

Definition 3.2. An inverse system $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ is *continuous* provided that for each chain $\Sigma \subseteq \Lambda$, with $\gamma = \sup\Sigma \in \Lambda$, the induced map, $h_\gamma : X_\gamma \to \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$, by the family $\{f_\alpha^\gamma : \alpha \in \Sigma\}$ is a homeomorphism.

Theorem 3.3. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be a continuous inverse system of Hausdorff compact spaces. If A is a closed subset of X_Λ, then the inverse system $\{f_\alpha^\Lambda[A], f_\alpha^\beta|_{f_\beta^\Lambda[A]}, \Lambda\}$ is continuous.*

Proof. Let Σ be a chain contained in Λ, with $\gamma = \sup\Sigma \in \Lambda$, and let $h_\gamma : X_\gamma \to \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$ be the induced map by the family $\{f_\alpha^\gamma : \alpha \in \Sigma\}$ $(f_\alpha^\gamma = f_\alpha^\Sigma \circ h_\gamma)$. Since S is continuous, the map h_γ is a homeomorphism.

Note that, by [2, 3.2.13], X_Λ is a Hausdorff compact space. Then, by [2, 2.5.6], we have that:

$$\begin{aligned} h_\gamma\left[f_\gamma^\Lambda[A]\right] &= \varprojlim\left\{f_\alpha^\Sigma\left[h_\gamma\left[f_\gamma^\Lambda[A]\right]\right], f_\alpha^\beta|_{f_\beta^\Sigma\left[h_\gamma\left[f_\gamma^\Lambda[A]\right]\right]}, \Sigma\right\} \\ &= \varprojlim\left\{f_\alpha^\gamma\left[f_\gamma^\Lambda[A]\right], f_\alpha^\beta|_{f_\beta^\gamma\left[f_\gamma^\Lambda[A]\right]}, \Sigma\right\} \\ &= \varprojlim\left\{f_\alpha^\Lambda[A], f_\alpha^\beta|_{f_\beta^\Lambda[A]}, \Sigma\right\}. \end{aligned}$$

Note that the homeomorphism

$$h_\gamma|_{f_\gamma^\Lambda[A]} : f_\gamma^\Lambda[A] \to \varprojlim\left\{f_\alpha^\Lambda[A], f_\alpha^\beta|_{f_\beta^\Lambda[A]}, \Sigma\right\}$$

satisfies $f_\alpha^\gamma|_{f_\gamma^\Lambda[A]} = f_\alpha^\Sigma \circ h_\gamma|_{f_\gamma^\Lambda[A]}$ for each $\alpha \in \Sigma$. Then, by [2, 2.5.F], the map $h_\gamma|_{f_\gamma^\Lambda[A]}$ is the induced map by $\{f_\alpha^\gamma|_{f_\gamma^\Lambda[A]} : \alpha \in \Sigma\}$. Hence, the inverse system $\{f_\alpha^\Lambda[A], f_\alpha^\beta|_{f_\beta^\Lambda[A]}, \Lambda\}$ is continuous. $\square$

The next theorem tells us that the induced hyperspace inverse system of a continuous inverse system is continuous.

Theorem 3.4. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be a continuous inverse system of Hausdorff compact spaces. Then the inverse systems $2^S = \{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Lambda\}$ and $\mathcal{C}_n(S) = \{\mathcal{C}_n(X_\alpha), \mathcal{C}_n(f_\alpha^\beta), \Lambda\}$ are continuous.*

Proof. Let Σ be a chain contained in Λ, with $\gamma = \sup\Sigma \in \Lambda$, and let $h_\gamma : X_\gamma \to \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$ be the induced map by $\{f_\alpha^\gamma : \alpha \in \Sigma\}$ $(f_\alpha^\gamma = f_\alpha^\Sigma \circ h_\gamma)$.

Since S is continuous, the map h_γ is a homeomorphism. Moreover, $2^{f_\alpha^\gamma} = 2^{f_\alpha^\Sigma} \circ 2^{h_\gamma}$ since $f_\alpha^\gamma = f_\alpha^\Sigma \circ h_\gamma$.

Let q_α^Σ denote the projection map from $\varprojlim\{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Sigma\}$ into 2^{X_α}.

By 2.5, the map $h : 2^{X_\Sigma} \to \varprojlim\{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Sigma\}$ defined by $h(E) = \big(f_\alpha^\Lambda[E]\big)_{\alpha\in\Sigma}$ is a homeomorphism. Since $2^{f_\alpha^\Sigma} = q_\alpha^\Sigma \circ h$ for every $\alpha \in \Sigma$, we have that $2^{f_\alpha^\gamma} = 2^{f_\alpha^\Sigma} \circ 2^{h_\gamma} = q_\alpha^\Sigma \circ h \circ 2^{h_\gamma}$ for each $\alpha \in \Sigma$. Then, by [2, 2.5.F], the homeomorphism $h \circ 2^{h_\gamma} : 2^{X_\gamma} \to \varprojlim\{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Sigma\}$ is the induced map by the family $\{2^{f_\alpha^\gamma} : \alpha \in \Sigma\}$. Hence the inverse system 2^S is continuous.

In a similar way, we can prove that the inverse system $\mathcal{C}_n(S)$ is continuous. □

Definition 3.5. An inverse system $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is *σ-complete* if S is continuous and Λ is σ-complete.

From 3.4 and the definition of a σ-complete inverse system, we obtain the following result:

Theorem 3.6. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse system of Hausdorff compact spaces. If S is σ-complete, then the inverse systems 2^S and $\mathcal{C}_n(S)$ are σ-complete.*

Definition 3.7. An inverse system $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is an *inverse σ-system* if S is σ-complete and $\omega(X_\alpha) \leq \aleph_0$ for each $\alpha \in \Lambda$.

Theorem 3.8. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse σ-system of Hausdorff compact spaces. Then the inverse systems $2^S = \{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Lambda\}$ and $\mathcal{C}_n(S) = \{\mathcal{C}_n(X_\alpha), \mathcal{C}_n(f_\alpha^\beta), \Lambda\}$ are inverse σ-sytems.*

Proof. By 2.1, we have that $\omega(2^{X_\alpha}) = \omega(X_\alpha) \leq \aleph_0$. Then, by 3.6, we deduce that 2^S and $\mathcal{C}_n(S)$ are inverse σ-systems. □

From [9, Theorem 15], we have the following result:

Theorem 3.9. *Let $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ and $\{Y_\alpha, g_\alpha^\beta, \Lambda\}$ be two inverse σ-systems of Hausdorff compact spaces with onto bonding maps. If $l : X_\Lambda \to Y_\Lambda$ is a map, then there exist a cofinal subset Σ of Λ and maps $l_\alpha : X_\alpha \to Y_\alpha$ for every $\alpha \in \Sigma$, such that $l_\alpha \circ f_\alpha^\Lambda = g_\alpha^\Lambda \circ l$ and $l_\alpha \circ f_\alpha^\beta = g_\alpha^\beta \circ l_\beta$ for any $\alpha, \beta \in \Sigma$ satisfying $\alpha \leq \beta$. Moreover, if $l : X_\Lambda \to Y_\Lambda$ is a homeomorphism, then each l_α is a homeomorphism.*

Remark 3.10. Let $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ and $\{Y_\alpha, g_\alpha^\beta, \Lambda\}$ be a pair of inverse σ-systems of Hausdorff compact spaces with onto bonding maps and let $l : X_\Lambda \to Y_\Lambda$ be a map. Let Σ be the cofinal subset of Λ and let $l_\alpha : X_\alpha \to Y_\alpha$ be the maps satisfying 3.9.

Let $X_\Sigma = \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$ and let $Y_\Sigma = \varprojlim\{Y_\alpha, g_\alpha^\beta, \Sigma\}$. Define the maps $g : X_\Lambda \to X_\Sigma$ and $g' : Y_\Lambda \to Y_\Sigma$ by $g((x_\alpha)_{\alpha\in\Lambda}) = (x_\alpha)_{\alpha\in\Sigma}$ and $g'((y_\alpha)_{\alpha\in\Lambda}) = (y_\alpha)_{\alpha\in\Sigma}$. By [2, 2.5.11], the maps g and g' are homeomorphisms. Note that $f_\alpha^\Lambda = f_\alpha^\Sigma \circ g$ and $g_\alpha^\Lambda = g_\alpha^\Sigma \circ g'$ for each $\alpha \in \Sigma$.

The family $\{l_\alpha : \alpha \in \Sigma\}$ induces a map $l_\Sigma : X_\Sigma \to Y_\Sigma$ such that $l_\alpha \circ f_\alpha^\Sigma = g_\alpha^\Sigma \circ l_\Sigma$.

Since $g_\alpha^\Sigma \circ g' \circ l \circ g^{-1} = g_\alpha^\Lambda \circ l \circ g^{-1} = l_\alpha \circ f_\alpha^\Lambda \circ g^{-1} = l_\alpha \circ f_\alpha^\Sigma$ we deduce that $l_\Sigma = g' \circ l \circ g^{-1}$.

Theorem 3.11. *Let $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ and $\{Y_\alpha, g_\alpha^\beta, \Lambda\}$ be a pair of inverse σ-systems of metrizable continua with onto bonding maps. If each f_α^β is monotone and $\mathcal{C}_n(X_\Lambda)$ is homeomorphic to $\mathcal{C}_n(Y_\Lambda)$, then there exists a cofinal subset Σ of Λ such that the maps g_α^β are monotone for any $\alpha, \beta \in \Sigma$ satisfying $\alpha \leq \beta$.*

Proof. By 2.5, the hyperspace $\mathcal{C}_n(X_\Lambda)$ is homeomorphic to the space $X = \varprojlim\left\{\mathcal{C}_n(X_\alpha), \mathcal{C}_n(f_\alpha^\beta), \Lambda\right\}$ and the hyperspace $\mathcal{C}_n(Y_\Lambda)$ is homeomorphic to the space $Y = \varprojlim\left\{\mathcal{C}_n(Y_\alpha), \mathcal{C}_n(g_\alpha^\beta), \Lambda\right\}$. Let $l : X \to Y$ be a homeomorphism.

By 2.2, the hyperspaces $\mathcal{C}_n(X_\alpha)$ and $\mathcal{C}_n(Y_\alpha)$ are metrizable continua. Then, by [2, 6.1.20], X and Y are continua. By 3.8, the inverse systems $\{\mathcal{C}_n(X_\alpha), \mathcal{C}_n(f_\alpha^\beta), \Lambda\}$ and $\{\mathcal{C}_n(Y_\alpha), \mathcal{C}_n(g_\alpha^\beta), \Lambda\}$ are inverse σ-systems.

Given $\alpha \in \Lambda$, let p_α^Λ be the projection map from X into $\mathcal{C}_n(X_\alpha)$, let q_α^Λ be the projection map from Y into $\mathcal{C}_n(Y_\alpha)$ and let $Z_\alpha = q_\alpha^\Lambda[Y]$. Then, by [2, 2.5.6], $Y = \varprojlim \left\{ Z_\alpha, \mathcal{C}_n(g_\alpha^\beta)\mid_{Z_\beta}, \Lambda \right\}$.

By 3.3, the inverse system $\left\{ Z_\alpha, \mathcal{C}_n(g_\alpha^\beta)\mid_{Z_\beta}, \Lambda \right\}$ is continuous, then it is an inverse σ-system with onto bonding maps. Moreover, each $\mathcal{C}_n(f_\alpha^\beta)$ is onto since every f_α^β is monotone (see [1, Proposition 1]).

By 3.9, there exist a cofinal subset Σ of Λ and homeomorphisms $l_\alpha : \mathcal{C}_n(X_\alpha) \to Z_\alpha$ for every $\alpha \in \Sigma$, such that $l_\alpha \circ p_\alpha^\Lambda = q_\alpha^\Lambda \circ l$ and $l_\alpha \circ \mathcal{C}_n(f_\alpha^\beta) = \mathcal{C}_n(g_\alpha^\beta)\mid_{Z_\beta} \circ l_\beta$ for any $\alpha, \beta \in \Sigma$ satisfying $\alpha \leq \beta$.

By [1, Theorem 4], the maps $\mathcal{C}_n(f_\alpha^\beta)$ are monotone, then each map $\mathcal{C}_n(g_\alpha^\beta)\mid_{Z_\beta} = l_\alpha \circ \mathcal{C}_n(f_\alpha^\beta) \circ l_\beta^{-1}$ is monotone.

By [2, 3.2.15], the projections g_α^Λ are onto, then $\mathcal{F}_1(Y_\alpha) \subseteq Z_\alpha$. Let $\alpha, \beta \in \Sigma$ with $\alpha \leq \beta$ and let $y_\alpha \in Y_\alpha$. By [3, 15.9 (2)], the set $\bigcup (\mathcal{C}_n(g_\alpha^\beta)\mid_{Z_\beta})^{-1}(\{y_\alpha\})$ is connected. It is not difficult to see that $(g_\alpha^\beta)^{-1}(y_\alpha) = \bigcup (\mathcal{C}_n(g_\alpha^\beta)\mid_{Z_\beta})^{-1}(\{y_\alpha\})$. Then g_α^β is monotone. □

Let us recall the following result due to I. Lončar

Theorem 3.12. [4, Theorem 3.4] *Let X be a Hausdorff compact space with $\omega(X) \geq \aleph_1$. Then there exists an inverse σ-system $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ such that X is homeomorphic to X_Λ.*

Remark 3.13. In the previous Theorem, the directed set Λ only depends on $\omega(X)$ and each space X_α is contained in a Hilbert cube (see [4, Theorem 3.3]). So, we can assume that the spaces X_α are compact and metrizable. Moreover, if two Hausdorff compact spaces have the same weight, then the inverse σ-systems satisfying 3.12, for those two spaces, can be chosen with the same directed set.

Definition 3.14. A space X is *rim-metrizable* if it has a basis $\mathcal{B}$ such that every $U \in \mathcal{B}$ has metrizable boundary.

The following result is used in the proof of Theorem 3.20.

Theorem 3.15. [4, Theorem 3.7] *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse system of Hausdorff compact spaces with onto bonding maps. Then:*

1. *There exists an inverse system $M(S) = \{M_\alpha, m_\alpha^\beta, \Lambda\}$ of Hausdorff compact spaces such that the bonding maps m_α^β are monotone surjections and the space $\varprojlim S$ is homeomorphic to the space $\varprojlim M(S)$,*
2. *If S is σ-directed, then the inverse system $M(S)$ is σ-directed,*
3. *If S is σ-complete, then the inverse system $M(S)$ is σ-complete,*
4. *If every X_α is a metric space and $\varprojlim S$ is locally connected (a rim-metrizable continuum) then every M_α is metrizable.*

Definition 3.16. Let X be a continuum and let $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$, where $n \in \mathbb{N}$. The continuum X *has unique hyperspace* Γ_X provided that:

- if Y is a continuum and Γ_Y is homeomorphic to Γ_X, then Y is homeomorphic to X.

In the previous definition, the hyperspace $\Gamma_Y = 2^Y$ when $\Gamma_X = 2^X$ and $\Gamma_Y = \mathcal{C}_n(Y)$ when $\Gamma_X = \mathcal{C}_n(X)$.

Given a map $f : X \to Y$ between Hausdorff compact spaces, let Γ_f denote the induced map between the hyperspaces Γ_X and Γ_Y.

From [7, (0.60) and (1.61)] we have:

Theorem 3.17. *Hereditarily indecomposable metrizable continua have unique hyperspace $\mathcal{C}_1(X)$. In fact, if $f : \mathcal{C}_1(X) \to \mathcal{C}_1(Y)$ is a homeomorphism, where X is a hereditarily indecomposable metrizable continuum and Y is a metrizable continuum, then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.*

From [5, p. 416] and [6, 6.1] we obtain:

Theorem 3.18. *Hereditarily indecomposable metrizable continua have unique hyperspace Γ_X, where $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$ and $n \in \mathbb{N}$. In fact, if $f : \Gamma_X \to \Gamma_Y$ is a homeomorphism, where X is a hereditarily indecomposable metrizable continuum and Y is a metrizable continuum, then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.*

Theorem 3.19. [4, Theorem 2.4] *Hereditarily indecomposable rim-metrizable continua have unique hyperspace $\mathcal{C}_1(X)$, i.e., if X is a hereditarily indecomposable non-metric rim-metrizable continuum and Y is a continuum such that $\mathcal{C}_1(X)$ is homeomorphic to $\mathcal{C}_1(Y)$, then X is homeomorphic to Y. In fact, if $f : \mathcal{C}_1(X) \to \mathcal{C}_1(Y)$ is a homeomorphism, then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.*

The following Theorem is the main result of this paper.

Theorem 3.20. *Hereditarily indecomposable rim-metrizable continua have unique hyperspace Γ_X, where $\Gamma_X \in \{2^X, \mathcal{C}_n(X)\}$ and $n \in \mathbb{N}$. In fact, if $f : \Gamma_X \to \Gamma_Y$ is a homeomorphism, where X is a hereditarily indecomposable rim-metrizable continuum and Y is a continuum then $f[\mathcal{F}_1(X)] = \mathcal{F}_1(Y)$.*

Proof. Suppose Γ_X is homeomorphic to Γ_Y and let $f : \Gamma_X \to \Gamma_Y$ be a homeomorphism. Since $\mathcal{F}_1(X) \subseteq \Gamma_X \subseteq 2^X$, by 2.1, we have that $\omega(\Gamma_X) = \omega(X)$. In a similar way, we obtain that $\omega(\Gamma_Y) = \omega(Y)$. Then $\omega(Y) = \omega(X)$. If $\omega(X) \leq \aleph_0$, then X and Y are metrizable. Hence, it follows by 3.18.

Suppose $\omega(X) \geq \aleph_1$. By 3.12 and 3.13, there exist two inverse σ-systems $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ and $\{Y_\alpha, g_\alpha^\beta, \Lambda\}$ of metrizable continua such that X is homeomorphic to X_Λ and Y is homeomorphic to Y_Λ.

Note that, if $g : X \to X_\Lambda$ is a homeomorphism, then the homeomorphism $\Gamma_g : \Gamma_X \to \Gamma_{X_\Lambda}$ satisfies $\Gamma_g[\mathcal{F}_1(X)] = \mathcal{F}_1(X_\Lambda)$. Hence, we can assume that $X = X_\Lambda$ and $Y = Y_\Lambda$.

By [2, 2.5.6], we also assume that each of the bonding maps f_α^β and g_α^β are onto. Since every X_α is metrizable and X_Λ is rim-metrizable, by 3.15, we assume that the bonding maps f_α^β are monotone. By [2, 3.2.15], every projection map f_α^Λ is onto and, by [2, 6.3.16.(a)], they are monotone. Then, by 2.3, every metrizable continuum X_α is hereditarily indecomposable.

If $\Gamma_{X_\Lambda} = 2^{X_\Lambda}$, each of the maps $\Gamma_{f_\alpha^\beta}$ and $\Gamma_{g_\alpha^\beta}$ are onto.

If $\Gamma_{X_\Lambda} = \mathcal{C}_n(X_\Lambda)$, by 3.11, we may assume that the maps g_α^β are monotone. Then, by [2, 6.1.29], we deduce that each of the maps $\Gamma_{f_\alpha^\beta}$ are $\Gamma_{g_\alpha^\beta}$ are onto.

In both cases, the inverse systems $\{\Gamma_{X_\alpha}, \Gamma_{f_\alpha^\beta}, \Lambda\}$ and $\{\Gamma_{Y_\alpha}, \Gamma_{g_\alpha^\beta}, \Lambda\}$ have onto bonding maps.

By 2.2 and 3.8, the inverse systems $\{\Gamma_{X_\alpha}, \Gamma_{f_\alpha^\beta}, \Lambda\}$ and $\{\Gamma_{Y_\alpha}, \Gamma_{g_\alpha^\beta}, \Lambda\}$ are inverse σ-systems of metrizable continua.

By 2.5, the map $h : \Gamma_{X_\Lambda} \to \varprojlim\{\Gamma_{X_\alpha}, \Gamma_{f_\alpha^\beta}, \Lambda\}$ defined by $h(C) = (f_\alpha^\Lambda[C])_{\alpha\in\Lambda}$ and the map $h' : \Gamma_{Y_\Lambda} \to \varprojlim\{\Gamma_{Y_\alpha}, \Gamma_{g_\alpha^\beta}, \Lambda\}$ defined by $h'(C) = (g_\alpha^\Lambda[C])_{\alpha\in\Lambda}$ are homeomorphisms. Moreover, $h\left[\mathcal{F}_1\left(X_\Lambda\right)\right] = \varprojlim\{\mathcal{F}_1(X_\alpha), \mathcal{F}_1(f_\alpha^\beta), \Lambda\}$ and $h'\left[\mathcal{F}_1\left(Y_\Lambda\right)\right] = \varprojlim\{\mathcal{F}_1(Y_\alpha), \mathcal{F}_1(g_\alpha^\beta), \Lambda\}$.

Let $l = h' \circ f \circ h^{-1} : \varprojlim\{\Gamma_{X_\alpha}, \Gamma_{f_\alpha^\beta}, \Lambda\} \to \varprojlim\{\Gamma_{Y_\alpha}, \Gamma_{g_\alpha^\beta}, \Lambda\}$. Then, by 3.9, there exist a cofinal subset Σ of Λ and homeomorphisms $l_\alpha : \Gamma_{X_\alpha} \to \Gamma_{Y_\alpha}$ for every $\alpha \in \Sigma$, such that $l_\alpha \circ p_\alpha^\Lambda = q_\alpha^\Lambda \circ l$ and $l_\alpha \circ \Gamma_{f_\alpha^\beta} = \Gamma_{g_\alpha^\beta} \circ l_\beta$ for any $\alpha, \beta \in \Sigma$ satisfying $\alpha \leq \beta$, where $p_\alpha^\Lambda : \varprojlim\{\Gamma_{X_\alpha}, \Gamma_{f_\alpha^\beta}, \Lambda\} \to \Gamma_{X_\alpha}$ and $q_\alpha^\Lambda : \varprojlim\{\Gamma_{Y_\alpha}, \Gamma_{g_\alpha^\beta}, \Lambda\} \to \Gamma_{Y_\alpha}$ are the projection maps. By 3.10, we may assume that $\Sigma = \Lambda$.

By 3.18, the spaces X_α and Y_α are homeomorphic and $l_\alpha[\mathcal{F}_1(X_\alpha)] = \mathcal{F}_1(Y_\alpha)$. Thus, the homeomorphism

$$l_\alpha |_{\mathcal{F}_1(X_\alpha)}: \mathcal{F}_1(X_\alpha) \to \mathcal{F}_1(Y_\alpha)$$

induces a homeomorphism:

$$l' : \varprojlim\{\mathcal{F}_1(X_\alpha), \mathcal{F}_1(f_\alpha^\beta), \Lambda\} \to \varprojlim\{\mathcal{F}_1(Y_\alpha), \mathcal{F}_1(g_\alpha^\beta), \Lambda\}$$

such that $l |_{\varprojlim\{\mathcal{F}_1(X_\alpha), \mathcal{F}_1(f_\alpha^\beta), \Lambda\}} = l'$. Hence:

$$\begin{aligned} f[\mathcal{F}_1(X_\Lambda)] &= (h')^{-1} \circ h' \circ f \circ h^{-1}[\varprojlim\{\mathcal{F}_1(X_\alpha), \mathcal{F}_1(f_\alpha^\beta), \Lambda\}] \\ &= (h')^{-1} \circ l[\varprojlim\{\mathcal{F}_1(X_\alpha), \mathcal{F}_1(f_\alpha^\beta), \Lambda\}] \\ &= (h')^{-1}[\varprojlim\{\mathcal{F}_1(Y_\alpha), \mathcal{F}_1(g_\alpha^\beta), \Lambda\}] \\ &= \mathcal{F}_1(Y_\Lambda). \end{aligned}$$

Since $\mathcal{F}_1(X_\Lambda)$ and X_Λ are homeomorphic, and $\mathcal{F}_1(Y_\Lambda)$ is homeomorphic to Y_Λ, we conclude that X_Λ and Y_Λ are homeomorphic. □

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