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THE WHITEHEAD MANIFOLD HAS NO ORIENTATION REVERSING HOMEOMORPHISM

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Abstract. We show that every homeomorphism of the Whitehead manifold to itself is orientation preserving.

1. Introduction

Throughout this paper we will be working in the piecewise-linear category. The Whitehead contractible 3-manifold W is the monotone union of 3-dimensional solid tori $T_i, i \in \{0, 1, 2, \cdots\}$. A fixed orientation on T_0 orients all of the other T_i. Each T_i is embedded in T_{i+1} in the same manner as T_0 is embedded in T_1 as shown in the Figure (see page 2) with the same orientation. The results of this paper can be interpreted to show that the Whitehead manifold is unique; i.e., any two manifolds defined in this manner are homeomorphic. Recall that the Whitehead manifold is contractible, yet it is not homeomorphic to the Euclidean 3-space.

A simple way to construct the Whitehead manifold is to embed T_1 in Euclidean 3-space, let h be an orientation preserving homeomorphism of 3-space to itself that takes T_0 onto T_1, set $T_i = h^i(T_0)$, and then $W = \bigcup T_i$ is the Whitehead manifold. The homeomorphism h restricts to an orientation preserving homeomorphism of W onto itself which we also denote by h and call a *ratchet homeomorphism*. Notice that for each integer k, $h^k(T_n) = T_{n+k}$ for all n.

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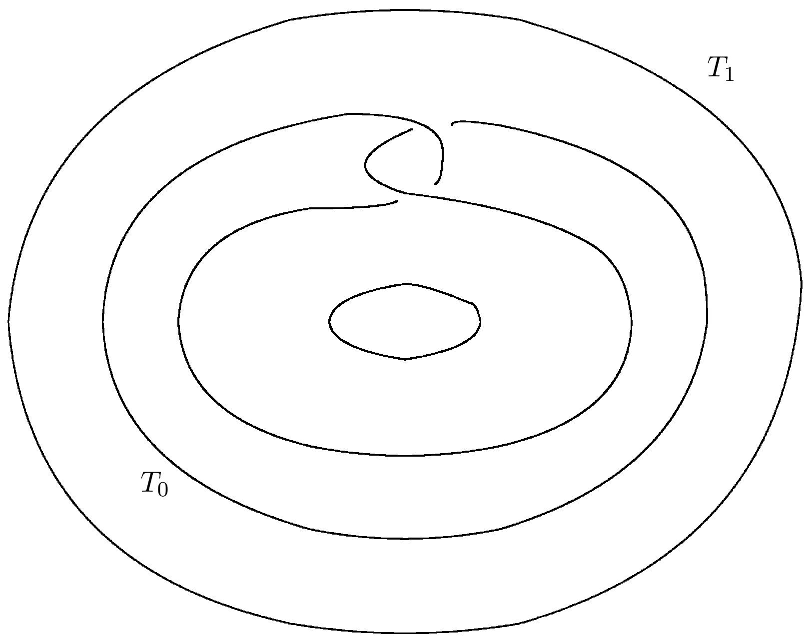

Figure

2. Definitions

We begin by reviewing some basic definitions. For a manifold M, let *int* M and ∂M denote the interior and the boundary of M, respectively. For a set A, let $|A|$ denote the cardinality of A. A solid torus is a space homeomorphic to the product of a 2-dimensional disk and a circle. Recall that a *meridional disk* of a solid torus T is a disk D in T such that $D \cap \partial T = \partial D$ and ∂D does not separate ∂T.

3. Strategy of the proof

In [4], Robert Myers showed that any homeomorphism of the Whitehead manifold to itself has the property that it is isotopic to a homeomorphism that sends T_n to T_{n+k} for all large n and for some fixed integer k. Hence, by using the ratchet homeomorphism described in section 1, we may conclude that given any homeomorphism of W, there is a homeomorphism of the same orientation type that sends T_0 onto T_0 and T_1 onto T_1. To show there are no orientation reversing homeomorphisms, it will be sufficient to show that any homeomorphism of W to itself that is fixed setwise on T_0 and T_1 must be orientation preserving.

We mention only that Myers' proof uses the fact that the manifold $R = T_1 - int\ T_0$ is an irreducible 3-manifold with incompressible boundary that is both torus free and annulus free. We refer the interested reader to [4] for definitions and details.

4. Knot polynomials

In this section we will use the Homfly polynomial [2] and a result from [3] to show that, up to isotopy, there are only two homeomorphisms of T_1 onto itself fixing T_0 setwise. Both of these homeomorphisms preserve orientation.

Lemma 4.1. *Let $f, g : T \to T$ be homeomorphisms from a solid torus to itself that induce the same homomorphism*

$$f_*, g_* : H_1(\partial T) \to H_1(\partial T)$$

on the first homology, then f and g are isotopic. Furthermore, if f and g agree on ∂T, then the isotopy between f and g can be chosen to remain constant on ∂T.

Proof: It is well known that there is an isotopy of ∂T between $f|\partial T$ and $g|\partial T$ [1], [5]. This isotopy can be extended to all of T by feathering the isotopy over a collar of ∂T. To complete the proof, it will be sufficient to consider the case where f is the identity and $g|\partial T$ is the identity. By general position techniques, we may isotope g so that g takes a meridional disk D to itself. By the Alexander trick, we can further isotope g so that it is the identity on D as well as ∂T. Cutting T open along D yields a 3-ball, so the Alexander trick can be employed once more to isotope g to the identity. □

Theorem 4.2. *Let $h : T_1 \to T_1$ be a homeomorphism of the solid torus T_1 to itself that fixes $T_0 \subset T_1$. Then h is isotopic to the identity or to the (orientation preserving) homeomorphism that flips the orientation of both the longitudinal and the meridional curves.*

Proof: Let (m, n) denote elements of the homology group $H_1(\partial T_1)$ where $(1, 0)$ represents an oriented meridian and $(0, 1)$ represents an oriented longitude. Since a meridional curve must go to a meridional curve, $h_*(1, 0) = (\pm 1, 0)$ and $h_*(0, 1) = (m, \pm 1)$. Suppose $h_*(1, 0) = (1, 0)$ and $h_*(0, 1) = (m, 1)$. We may suppose that $h|\partial T_1$ is the result of a twist homeomorphism. By the Lemma, there is an isotopy between h and a twist homeomorphism of T_1 that is fixed on the boundary. This induces an isotopy of the 3-space that takes the unknotted T_0 to the image of T_0 under a twist homeomorphism. But Masaharu Kouno, Kimihiko Motegi, and Tetsuo Shibuya showed that the image of T_0 under a nontrivial twist homeomorphism is knotted [3]. Therefore, $m = 0$. A similar argument shows that in all cases $m = 0$.

We now need to consider only the cases when $h_*(1, 0) = (\pm 1, 0)$ and $h_*(0, 1) = (0, \pm 1)$. If $h_*(1, 0) = (1, 0)$ and $h_*(0, 1) = (0, -1)$ or if $h_*(1, 0) = (-1, 0)$ and $h_*(0, 1) = (0, 1)$, then there is an isotopy of the 3-space, fixed outside T_1 that takes T_0 to its mirror image. But the Homfly polynomials of T_0 and its mirror image are distinct, so there is no such isotopy. Thus, if $h_*(1, 0) = (1, 0)$, then $h_*(0, 1) = (0, 1)$ and if $h_*(1, 0) = (-1, 0)$, then $h_*(0, 1) = (0, -1)$, giving the desired result. □

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