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EXISTENCE OF PERIODIC ORBITS FOR A PERTURBED VECTOR FIELD

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ABSTRACT. Let $\gamma(t)$ be a periodic isolated orbit of period T for a vector field v on a manifold M of dimension 3 or 4. We prove that under some conditions on the Poincaré map, expressed in terms of fixed point indices of iterations, every small perturbation of v has a periodic orbit of period close to either T or $2T$.

1. INTRODUCTION

We consider a vector field $v(x)$ on a manifold M^m, where $m = 3, 4$, and its periodic isolated trajectory $\gamma(t)$ of period T, $\dot{\gamma}(t) = v(\gamma(t))$. In case of a continuous v, we consider fields (and their perturbations) that generate a flow near $\gamma(t)$, thus we assume uniqueness of the solutions of $\dot{x}(t) = v(x(t))$.

One of the typical methods of studying periodic orbits is to reduce the problem to a discrete case via the Poincaré map. We take a small enough transversal slice Σ to the orbit γ and $x_0 = \Sigma \cap \gamma$. In such a situation, the Poincaré map P (first return map) is defined in a neighborhood of x_0 and is a local homeomorphism. The one-to-one correspondence between periodic orbits and fixed points of the associated Poincaré maps enables us to use recent results on the forms of fixed point indices of iterations of 2- and 3-dimensional

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maps to determine the behavior of perturbations of P near x_0 and thus perturbations of v near the periodic orbit γ.

First, let us recall the notion of fixed point indices of iterations. We consider $f : U \to \mathbb{R}^m$ (U is an open neighborhood of x_0) such that for each n, x_0 is an isolated fixed point for f^n, though the neighborhood of isolation may depend on n. Then the fixed point index $\mathrm{ind}(f^n, x_0)$ is defined for f^n restricted to a small enough neighborhood of x_0.

In [5], Shui-Nee Chow, John Mallet-Paret, and James A. Yorke (see also, [1] and [10]) introduced a so-called bifurcation invariant, $\Phi(f, x_0)$, which carries the information on the whole sequence of indices of iterations in one integer number:

$$\Phi(f, x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{ind}(f^n, x_0).$$

For a periodic orbit γ of a vector field v, we may define the counterpart invariant by the Poincaré map:

$$\Phi(v, \gamma) = \Phi(P, x_0).$$

The bifurcation invariant $\Phi(v, \gamma)$ provides, in many cases, important information on the behavior of perturbed vector fields in the neighborhood of γ. It can be used to state the existence of periodic orbits of the period close to the period of γ or to the double period of γ.

Definition 1.1. We say that an isolated T-periodic orbit $\gamma(t)$ of a vector field v has the 2-period property if for each $\delta > 0$, there is $\varepsilon > 0$ such that each ε perturbation $\tilde{v}$ of v has in a small neighborhood of $\gamma(t)$ a periodic orbit of period T' with either $|T - T'| < \delta$ or $|2T - T'| < \delta$.

In [2, Theorem 4.4, p. 25], I. K. Babenko and S. A. Bogatyĭ proved the following theorem.

Theorem 1.2. *Let v be a smooth vector field on a 3-dimensional manifold M^3 and $\gamma(t)$ be an isolated periodic orbit. If $\Phi(v, \gamma) \neq 0$, then $\gamma(t)$ has the 2-period property.*

This note will generalize Theorem 1.2 for 4-dimensional manifolds and, by weakening the assumption of smoothness, for 3-dimensional ones. We will also replace the bifurcation invariant by the bifurcation function, which makes the statements more general.

2. Bifurcation function

In this section, we define the bifurcation function and establish its relation to the bifurcation invariant Φ.

Definition 2.1. For a given k, we define the sequence

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if} \quad k|n, \\ 0 & \text{if} \quad k \not| n. \end{cases}$$

In other words, each reg_k (where k is fixed) is a periodic sequence which has the form

$$(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots),$$

where the non-zero entries appear for indices of the sequence divisible by k.

A sequence of indices of iterations (as well as any integer sequence) may be written in the form of so-called *periodic expansion* (cf. [9] and [11])

$$\mathrm{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \mathrm{reg}_k(n), \tag{2.1}$$

where $a_n = \frac{1}{n} \sum_{k|n} \mu(k) \mathrm{ind}(f^{(n/k)}, x_0)$ and μ is the classical Möbius function; i.e., $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^s$ if k is a product of s different primes, and $\mu(k) = 0$, otherwise.

Theorem 2.2 (Dold relations, [6])**.** *For each natural n, we have*

$$\sum_{k|n} \mu(k) \mathrm{ind}(f^{(n/k)}, x_0) \equiv 0 \pmod{n}.$$

Due to Dold relations, each sequence of indices of iterations has the periodic expansion with integer coefficients, $a_k \in \mathbb{Z}$.

Definition 2.3 ([2])**.** For a fixed point x_0 of a map f, which has the sequence of indices of the form (2.1), we define a function of the variable z, called the bifurcation function,

$$A(f, x_0; z) = \sum_{k=1}^{\infty} a_k z^k.$$

For a periodic orbit γ of a vector field v, we define the bifurcation function by the Poincaré map

$$A(v, \gamma; z) = A(P, x_0; z).$$

The following lemma shows the relation between the bifurcation invariant and the bifurcation function.

Lemma 2.4 ([2])**.** *Assume that* $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ *is bounded and has the periodic expansion* $\mathrm{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \mathrm{reg}_k(n)$*, then the set* $O = \{k : a_k \neq 0\}$ *is finite and* $\Phi(f, x_0) = \sum_{k \in O} a_k$*. As a result,* $\Phi(f, x_0) = A(f, x_0; 1)$*.*

Remark 2.5. Let us note an obvious fact that $\Phi(f, x_0) \neq 0$ implies $A(f, x_0; z) \not\equiv 0$, but the converse implication is not true.

3. Main results

The reader is reminded that in the case of continuous vector fields, we consider only fields which generate a flow near the periodic orbit.

Theorem 3.1. *Let v be a continuous vector field on a 3-dimensional manifold M^3 and $\gamma(t)$ be an isolated periodic orbit. If $A(v, \gamma; z) \not\equiv 0$, then $\gamma(t)$ has the 2-period property.*

Proof: We will show that the fact that the bifurcation function $A(v, \gamma; z)$ is non zero implies that either $\mathrm{ind}(P, x_0) \neq 0$ or $\mathrm{ind}(P^2, x_0) \neq 0$. This gives the theorem, as every small enough perturbation $\tilde{P}$ of P has in a neighborhood of x_0, either a fixed point or a point of minimal period 2.

We consider two cases.

Case 1. P is an orientation-preserving local planar homeomorphism. By the result of Morton Brown [3], there is an integer $g \neq 1$ such that for each $n \neq 0$,

$$\mathrm{ind}(P^n, x_0) = \begin{cases} g & \text{if } \mathrm{ind}(P, x_0) = g, \\ 1 \text{ or } g & \text{if } \mathrm{ind}(P, x_0) = 1. \end{cases} \tag{3.1}$$

Thus, in the case $\mathrm{ind}(P, x_0) = 1$ or $g \neq 0$, the theorem is proved. However, if $g = 0$, then by (3.1), $\mathrm{ind}(P^n, x_0) = g\mathrm{reg}_1(n)$, and so $A(v, \gamma; z) = gz \equiv 0$, which is excluded by the assumption.

Case 2. P is an orientation-reversing local planar homeomorphism. By the result of Marc Bonino [4], $\mathrm{ind}(P, x_0) \in \{-1, 0, 1\}$. If $\mathrm{ind}(P, x_0) = \pm 1$, the theorem is proved.

Let us consider the case $\mathrm{ind}(P, x_0) = 0$.

Assume $n > 0$ is odd. For an orientation-reversing planar homeomorphism, the sequence of indices of odd iterations is constant [7]; thus, $\mathrm{ind}(P^n, x_0) = \mathrm{ind}(P, x_0) = 0$.

Assume $n > 0$ is even. Under the assumption that $\mathrm{ind}(P, x_0) = 0$, $\mathrm{ind}(P^2, x_0) = 2l$, where l is an integer. This statement is a consequence of Dold relations (see Theorem 2.2) for $n = 2$: $\mathrm{ind}(P^2, x_0) - \mathrm{ind}(P, x_0) \equiv 0 \pmod 2$.

As P^2 is an orientation-preserving local homeomorphism, (3.1) gives that for every even n $\mathrm{ind}(P^n, x_0) = 2l = \mathrm{ind}(P^2, x_0)$.

Finally, $\mathrm{ind}(P^2, x_0) = 0$ implies $\mathrm{ind}(P^n, x_0) = 0$ for every natural n, but this contradicts the assumption $A(v, \gamma; z) \not\equiv 0$. □

The theorem for 4-dimensional manifolds needs an additional assumption on $A(v, \gamma; z)$.

Theorem 3.2. *Let v be a smooth vector field on a 4-dimensional manifold M^4 and $\gamma(t)$ be an isolated periodic orbit. Assume that $A(v, \gamma; z) \not\equiv 0$ and $A(v, \gamma; z)$ is not a monomial of the degree greater than 2. Then $\gamma(t)$ has the 2-period property.*

Proof: As in the proof of Theorem 3.1, we show that either $\mathrm{ind}(P, x_0) \neq 0$ or $\mathrm{ind}(P^2, x_0) \neq 0$. The Poincaré map P is now a local diffeomorphism in $\mathbb{R}^3$. We use the description of local indices of iterations of smooth maps in $\mathbb{R}^3$ given in [8]. It appears that $\{\mathrm{ind}(P^n, 0)\}_{n=1}^{\infty}$ has one of the following forms:

(A) $c_A(n) = a_1\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n)$;
(B) $c_B(n) = \mathrm{reg}_1(n) + a_k\mathrm{reg}_k(n)$;
(C) $c_C(n) = -\mathrm{reg}_1(n) + a_k\mathrm{reg}_k(n)$;
(D) $c_D(n) = a_k\mathrm{reg}_k(n)$;
(E) $c_E(n) = \mathrm{reg}_1(n) - \mathrm{reg}_2(n) + a_k\mathrm{reg}_k(n)$;
(F) $c_F(n) = \mathrm{reg}_1(n) + a_k\mathrm{reg}_k(n) + a_{2k}\mathrm{reg}_{2k}(n)$, where k is odd;
(G) $c_G(n) = \mathrm{reg}_1(n) - \mathrm{reg}_2(n) + a_k\mathrm{reg}_k(n) + a_{2k}\mathrm{reg}_{2k}(n)$, where k is odd.

In all cases, $k \geq 3$ and $a_i \in \mathbb{Z}$.

As we assume that $A(v,\gamma;z)$ is not a monomial of the degree greater than 2, (D) is excluded. $A(v,\gamma;z) \not\equiv 0$ straightforwardly implies that $\operatorname{ind}(P,x_0) \neq 0$ in (B), (C), (E), (F), and (G).

Let us consider (A). Notice that $a_1 = \operatorname{ind}(P,x_0)$ and $a_2 = \frac{1}{2}(\operatorname{ind}(P^2,x_0) - \operatorname{ind}(P,x_0))$. If both $\operatorname{ind}(P,x_0)$ and $\operatorname{ind}(P^2,x_0)$ are equal to zero, then $A(v,\gamma;z) \equiv 0$. This ends the proof. $\square$

Remark 3.3. The assumption in Theorem 3.2, which states that $A(v,\gamma;z)$ is not a monomial of the degree greater than 2, cannot be omitted. It results from the fact that every sequence of integers which is of one of the forms (A) - (G) can be realized as a sequence of indices of iterations of a local diffeomorphism of $\mathbb{R}^3$ (cf. [8]). In particular, for every $k \geq 3$, we may find a diffeomorphism P with $a_k \neq 0$ (cf. the realization of the type (D) in [8] subchapter 4.4.2) such that

$$\operatorname{ind}(P^n,x_0) = a_k \operatorname{reg}_k(n) = \begin{cases} 0 & \text{if } k \nmid n, \\ a_k k & \text{if } k | n. \end{cases}$$

Thus, $A(P,x_0;z) = a_k z^k$, but x_0 is an inessential, (i.e., removable by a small perturbation), fixed point for P and P^2. Taking the suspension of P, we get a flow with $A(v,\gamma;z) \not\equiv 0$ for which the periodic orbit $\gamma(t)$ does not have the 2-period property.

References

[1] Kathleen T. Alligood and James A. Yorke, *Families of periodic orbits: virtual periods and global continuability*, J. Differential Equations **55** (1984), no. 1, 59–71.

[2] I. K. Babenko and S. A. Bogatyĭ, *Behavior of the index of periodic points under iterations of a mapping*, Math. USSR-Izv. **38** (1992), no. 1, 1–26.

[3] Morton Brown, *On the fixed point index of iterates of planar homeomorphisms*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 1109–1114.

[4] Marc Bonino, *Lefschetz index for orientation reversing planar homeomorphisms*, Proc. Amer. Math. Soc. **130** (2002), no. 7, 2173–2177 (electronic).

[5] Shui-Nee Chow, John Mallet-Paret, and James A. Yorke, *A periodic orbit index which is a bifurcation invariant*, in Geometric Dynamics (Rio de Janeiro, 1981). Lecture Notes in Mathematics, 1007. Berlin: Springer, 1983. 109–131.

[6] A. Dold, *Fixed point indices of iterated maps*, Invent. Math. **74** (1983), no. 3, 419–435.

[7] Grzegorz Graff and Piotr Nowak-Przygodzki, *Fixed point indices of iterations of planar homeomorphisms*, Topol. Methods Nonlinear Anal. **22** (2003), no. 1, 159–166.

[8] ———, *Fixed point indices of iterations of C^1 maps in $\mathbb{R}^3$*, Discrete Contin. Dyn. Syst. **16** (2006), no. 4, 843–856.

[9] Jerzy Jezierski and Wacław Marzantowicz, *Homotopy Methods in Topological Fixed and Periodic Points Theory*. Topological Fixed Point Theory and Its Applications, 3. Dordrecht: Springer, 2006.

[10] John Mallet-Paret and James A. Yorke, *Snakes: oriented families of periodic orbits, their sources, sinks, and continuation*, J. Differential Equations **43** (1982), no. 3, 419–450.

[11] Wacław Marzantowicz and Piotr Maciej Przygodzki, *Finding periodic points of a map by use of a k-adic expansion*, Discrete Contin. Dynam. Systems **5** (1999), no. 3, 495–514.

[12] M. Shub and D. Sullivan, *A remark on the Lefschetz fixed point formula for differentiable maps*, Topology **13** (1974), 189–191.

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