Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

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TOPOLOGY PROCEEDINGS
Volume 31, No. 1, 2007
Pages 145-150
http://topology.auburn.edu/tp/

SPACES OF DENSELY CONTINUOUS FORMS

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ABSTRACT. When X is a locally compact space, the space $D_k^*(X)$ of locally bounded densely continuous real-valued forms on X (= minimal usco maps on X), under the topology of uniform convergence on compact sets, is a locally convex linear topological space. (See R. A. McCoy, *Spaces of semicontinuous forms*, Topology Proc. **23** (1998), Summer, 249–275 (2000).) We give a partial answer to Question 3.1 in the above referenced paper whether X must be locally compact if addition is continuous on $D_k^*(X)$. In fact, we prove that if X is a first countable regular space, the answer is positive.

1. INTRODUCTION

In what follows, let X be a Hausdorff topological space and R be the space of real numbers with the usual metric.

We define the set of densely continuous real-valued functions on X to be the set, $DC(X)$, of all real-valued functions f on X such that $C(f)$, the set of points of continuity of f, is dense in X.

The set $D(X)$ of densely continuous real-valued forms [8], [15] is defined by

$$D(X) = \{\overline{f \restriction C(f)} : f \in DC(X)\},$$

where $\overline{f \restriction C(f)}$ is the closure of the graph of $f \restriction C(f)$ in $X \times R$.

2000 *Mathematics Subject Classification.* 54C35; Secondary 54C60.

Key words and phrases. densely continuous form, locally convex linear topological space, topology of uniform convergence on compact sets.

This work was supported by the Science and Technology Assistance Agency under contract No. APVT-51-006904.

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The densely continuous forms from X to R are not, in general, functions mapping X into R. They may be considered as multifunctions (set-valued maps). For each $x \in X$ and $\Phi \in D(X)$, define

$$\Phi(x) = \{t \in R : (x,t) \in \Phi\}.$$

If $\Phi \in D(X)$ and $A \subset X$, we say that Φ is bounded on A [15] provided that the set $\Phi(A) = \cup\{\Phi(x) : x \in A\}$ is a bounded subset of R. Then Φ is locally bounded provided that each point of X has a neighborhood on which Φ is bounded. Now define $D^*(X)$ to be the set of members of $D(X)$ that are locally bounded.

The topology of $D^*_k(X)$ can be defined using the Hausdorff metric, H, on the space of nonempty compact subsets of R. This metric is defined for nonempty compact subsets A and B of R by

$$H(A,B) = max\{max\{d(a,B) : a \in A\}, max\{d(b,A) : b \in B\}\},$$

where $d(s,T) = inf\{\mid s - t \mid : s, t \in T\}$.

Then for each Φ in $D^*(X)$, compact set A in X, and real $\epsilon > 0$, define $W(\Phi, A, \epsilon)$ to be the set of all Ψ in $D^*(X)$ such that

$$sup\{H(\Phi(a), \Psi(a)) : a \in A\} < \epsilon.$$

The family of all $W(\Phi, A, \epsilon)$ is a base for the topology of $D^*_k(X)$ [8].

The metrizability and complete metrizability of $D^*_k(X)$ were studied in [9] and [15], and the cardinal function properties of $D^*_k(X)$, such as the cellularity, the density, the netweight, and the weight of $D^*_k(X)$, were studied in [11].

There is now enough rich literature concerning densely continuous forms [8], [10], [12], [11], [13], [14], [16].

2. Minimal USCO maps and the space $D^*(X)$

Following Jens Peter Reus Christensen [1], we say that Φ is USCO, if it is an upper semicontinuous set-valued map with nonempty compact values.

A set-valued map Φ is said to be minimal USCO [2], [1] if it is a minimal element in the family of all USCO set-valued maps (with domain X and range Y), that is, if it does not contain properly any other USCO set-valued map from X to Y. By an easy application of the Kuratowski-Zorn Principle, we can guarantee that every USCO set-valued map Φ from X to Y contains a minimal USCO set-valued map from X to Y.

An important fact concerning the space $D^*(X)$ is that every element Φ from $D^*(X)$ is a minimal USCO map, and if X is a Baire space, the set $D^*(X)$ coincides with the set of all minimal USCO real-valued maps [9].

The following example shows that the condition of Baireness is essential.

Example 2.1. Let X be the space of rational numbers with the usual topology. Enumerate X by $\{q_n : n \in \omega\}$ and define the set-valued map $\Phi : X \to R$ as $\Phi(x) = \{\Sigma_{n:q_n<x} 1/2^n, \Sigma_{n:q_n \leq x} 1/2^n\}$. Then Φ is a minimal USCO map which is not a densely continuous form, since it is nowhere single-valued.

To prove that Φ is upper semicontinuous, let $x \in X$ and $\epsilon > 0$. There is $n_0 \in \omega$ such that $\Sigma_{n \geq n_0} 1/2^n < \epsilon$. Put $\delta_x = min\{| q_i - x | : i \leq n_0, q_i \neq x\}$ and put $O_x = (x - \delta_x/2, x + \delta_x/2)$. Then, for every $z \in O_x$, we have $\Phi(z) \subset S_\epsilon[\Phi(x)]$, where $S_\epsilon[\Phi(x)] = \{s \in R : d(s, \Phi(x)) < \epsilon\}$. (Let $z \in O_x, z < x$. Then $\Sigma_{n:q_n<x} 1/2^n - \Sigma_{n:q_n<z} 1/2^n = \Sigma_{n:z \leq q_n < x} 1/2^n < \epsilon$. Let $z > x$. Then $\Sigma_{n:q_n \leq z} 1/2^n - \Sigma_{n:q_n \leq x} 1/2^n = \Sigma_{x<q_n \leq z} 1/2^n < \epsilon$.)

To prove that Φ is minimal, suppose there is an USCO map Ψ such that $\Psi \subset \Phi$ and there is q_n such that $\Phi(q_n) \neq \Psi(q_n)$. Suppose that $\Psi(q_n) = \{\Sigma_{i:q_i<q_n} 1/2^i\}$ (the other case is similar). The upper semicontinuity of Ψ implies that there is a neighborhood O of q_n such that $\Psi(z) \subset S_{1/2^{q_n}}[\Psi(q_n)]$ for every $z \in O$, a contradiction, since for every $z \in O, z > q_n$, we have $\Psi(z) > \Sigma_{i:q_i \leq q_n} 1/2^i$.

3. Main result

It was proved in [15] that if X is a Baire space, the set $D(X)$ does have a natural vector space structure defined by $\overline{f \upharpoonright C(f)} + \overline{g \upharpoonright C(g)} = \overline{(f+g) \upharpoonright C(f+g)}$ and $a\overline{f \upharpoonright C(f)} = \overline{af \upharpoonright C(af)}$ for $\overline{f \upharpoonright C(f)}, \overline{g \upharpoonright C(g)} \in D(X)$ and $a \in R$.

The above claim works even more generally for Volterra spaces.

A topological space X is Volterra [5] if, for each pair $f, g : X \to R$ of functions such that $C(f)$ and $C(g)$ are both dense in X, the set $C(f) \cap C(g)$ is dense in X. It was proved in [6] that X is Volterra if and only if, for each pair A, B of dense G_δ-subsets of X, the set $A \cap B$ is dense.

Of course, every Baire space is Volterra and there are Volterra spaces which are not of second category, hence not Baire [7]. It was proved in [4] that every metrizable Volterra space is Baire.

It is very easy to verify that the following propositions hold.

Proposition 3.1. *Let X be a topological space. The following are equivalent.*

(1) *X is Volterra;*
(2) *$DC(X)$ is a vector space (with the natural definitions of operations).*

Proposition 3.2. *Let X be a topological space. The following are equivalent.*

(1) *X is Volterra;*
(2) *$D(X)$ ($D^*(X)$) is a vector space with the above defined operations.*

In fact, it is the addition in $DC(X)$ ($D(X), D^*(X)$, respectively) which forces X to be a Volterra space.

The following Theorem was proved in [15].

Theorem 3.3 ([15])**.** *If X is locally compact, $D_k^*(X)$ is a locally convex linear topological space.*

In connection with the above theorem, R. A. McCoy asked in his paper [15] the following question.

Question 3.4 ([15])**.** For any space X, if addition is continuous on $D_k^*(X)$, must X be locally compact?

We give a partial answer to this question. In fact, we prove that if X is a first countable regular space, the answer is positive.

Theorem 3.5. *Let X be a first countable regular space. The following are equivalent.*

(1) *X is locally compact;*
(2) *addition is continuous on $D_k^*(X)$.*

Proof: (1) $\Rightarrow$ (2) By Theorem 3.3.

(2) $\Rightarrow$ (1) Of course, X must be a Volterra space. Suppose X is not locally compact. Let $x_0 \in X$ fail to have a local base of compact sets. There are sequences $\{V_n\}, \{F_n\}, \{O_n\}$ of subsets of X such that

(a) $\{V_n\}$ is a base of neighborhoods of x_0, $\overline{V_n} \subset V_{n-1}$ for every $n \in \omega$;
(b) $\{F_n\}$ is a sequence of closed noncompact sets such that $x_0 \notin F_n$ for every $n \in \omega$ and $F_n \subset V_{n-1}$ for every $n \in \omega$;
(c) $\{O_n\}$ is a sequence of open sets such that $F_n \subset O_n$ for every $n \in \omega$, $\overline{O_n} \subset V_{n-1}$, and $\overline{V_n} \cap \overline{O_n} = \emptyset$ for every $n \in \omega$.

Let $I \subset \omega$ be the set of all even numbers ($I = \{2n : n \in \omega\}$). Put $U = \cup_{n\in I} O_n$ and $S = \cup_{n\in\omega\setminus I} O_n$. Then, of course, $S \cap U = \emptyset$ and $x_0 \in \overline{S} \cap \overline{U}$.

Define $f, g \in DC(X)$ as

$$f(x) = 1 \text{ if } x \in U \text{ and } f(x) = -1 \text{ if } x \in X \setminus U;$$

$$g(x) = -1 \text{ if } x \in U \text{ and } g(x) = 1 \text{ if } x \in X \setminus U.$$

Then $U \cup (X \setminus \overline{U}) \subset C(f) \cap C(g)$; i.e., $f, g \in DC(X)$ and also $f + g \in DC(X)$. Put $F = \overline{f \restriction C(f)}$ and $G = \overline{g \restriction C(g)}$. Since X is a Volterra space, we can well define the addition of $F + G$ and $F + G = f_0$, where f_0 is the zero function on X.

Let A be a compact set and $\epsilon > 0$. We show that there is $H \in D^*(X)$ such that $H \in W(G, A, \epsilon)$ and $F + H \notin W(f_0, \{x_0\}, 1)$; i.e., the addition is not continuous on $D^*_k(X)$.

For every $n \in I$, there is $y_n \in F_n \setminus A$ since F_n is a noncompact set. There is an open set $O(y_n)$ such that $y_n \in O(y_n)$, $\overline{O(y_n)} \subset O_n$, $\overline{O(y_n)} \cap A = \emptyset$, and $O_n \setminus \overline{O(y_n)} \neq \emptyset$.

It is easy to verify that $\overline{\cup_{n\in I} O(y_n)} = \{x_0\} \cup \cup_{n\in I} \overline{O(y_n)}$.

Define the function h as $h(x) = 1$ for $x \in \cup_{n\in I} O(y_n)$, and $h(x) = g(x)$, otherwise. Then, of course, $h \in DC(X)$. Put $H = \overline{h \restriction C(h)}$. Then $H \in W(G, A, \epsilon)$, since for every $x \in A$, we have $H(x) = G(x)$. However, $F + H \notin W(f_0, \{x_0\}, 1)$ since $2 \in (F + H)(x_0)$. $\square$

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