# Topology Proceedings

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
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**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

**TOPOLOGY PROCEEDINGS**
Volume 31, No. 1, 2007
Pages 163-179
http://topology.auburn.edu/tp/

# SOME APPLICATIONS OF SEMI-SEQUENCES TO EXTENSION THEORY

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Abstract. Suppose that $K$ is a CW-complex, $\mathbf{X}$ is an inverse sequence of stratifiable spaces, and $X = \lim \mathbf{X}$. One says that $X$ is an absolute co-extensor for $K$; i.e., $K$ is an absolute extensor for $X$ if each map $f : A \to K$ from a closed subset $A$ of $X$ extends to a map $F : X \to K$. This is often written, $X \tau K$. Previously, using the concept of semi-sequence, we have provided necessary and sufficient conditions for $X \tau K$ in terms of the inverse sequence $\mathbf{X}$ and without recourse to any specific properties of its limit. Using such a theorem, we now prove a sufficiency result that is stronger than the ones formerly known for detecting that $X \tau K$. This will lead finally to a "local" version of the theorem.

## 1. Introduction

The following limit theorem of Keiô Nagami [18] has been used frequently since its first appearance in 1959 (see [19] for proof details).

**Theorem 1.1.** *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of metrizable spaces,* $X = \lim \mathbf{X}$*, and suppose that for each* $i \in \mathbb{N}$, $\dim X_i \leq n$*. Then* $\dim X \leq n$*.*

2000 *Mathematics Subject Classification.* Primary 54C55, 54B35, Secondary 54F45.

*Key words and phrases.* absolute co-extensor, absolute extensor, cohomological dimension, CW-complex, Eilenberg-Mac Lane complex, extension theory, inverse limit, inverse sequence, polyhedron, semi-limit, semi-sequence, simplicial complex, stratifiable space.

This theorem, however, can be seen as a result in extension theory ([4], [5], [6], [7], [8], [9]) for the following reasons. If $K$ is a CW-complex and $X$ is a space, then one says that $K$ is an *absolute extensor* for $X$, or that $X$ is an *absolute co-extensor* for $K$ (sometimes written $X\tau K$), if for each closed subset $A$ of $X$ and map (i.e., continuous function) $f : A \to K$, there exists a map $F : X \to K$ such that $F$ is an extension of $f$. Since it is well known that for a metrizable (or even a stratifiable) space $X$, $\dim X \leq n$ if and only if $X$ is an absolute co-extensor for $S^n$, then Theorem 1.1 can be stated in the following way.

**Theorem 1.2.** *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of metrizable spaces, let* $X = \lim \mathbf{X}$*, and suppose that for each* $i \in \mathbb{N}$*,* $X_i$ *is an absolute co-extensor for* $S^n$*. Then* $X$ *is an absolute co-extensor for* $S^n$*.*

We should also remark that cohomological dimension $\dim_G X$ over an abelian group $G$ for stratifiable spaces $X$ can be defined in terms of extension theory. One has that $\dim_G X \leq n$ if and only if $X$ is an absolute co-extensor for $K(G,n)$ [15, Theorem 26, p. 40]. Here, by $K(G,n)$, we mean any Eilenberg-Mac Lane CW-complex associated with the group $G$ and $n$. Also, one should be aware that CW-complexes are absolute neighborhood extensors for stratifiable spaces (see section 3).

A direct generalization of Theorem 1.2, which by the preceding paragraph applies to cohomological dimension over any abelian group $G$ (put $K = K(G,n)$), was given by Leonard R. Rubin and Philip J. Schapiro in [22].

**Theorem 1.3.** *Let* $K$ *be a* CW*-complex,* $\mathbf{X} = (X_i, p_i^{i+1})$ *an inverse sequence of metrizable spaces such that for each* $i \in \mathbb{N}$*,* $X_i$ *is an absolute co-extensor for* $K$*, and* $X = \lim \mathbf{X}$*. Then* $X$ *is an absolute co-extensor for* $K$*.*

Observe that in this result there is no weight requirement on the spaces $X_i$; the only requirement is that they be metrizable. Similarly, there is no weight restriction on the CW-complex $K$. Theorem 1.3 has been applied already in [3] and [16].

Another step was taken in [20]. Instead of requiring that $X_i$ be an absolute co-extensor for $K$ for each $i \in \mathbb{N}$, a condition was placed on the bonding maps $p_i^{i+1}$. The requirement was that for each $i \in \mathbb{N}$, closed subset $A$ of $X_i$, and map $f : A \to K$, there are to exist $j \geq i$ and a map $F : X_j \to K$ having the property that $F(x) = f(p_i^j(x))$ for each $x \in (p_i^j)^{-1}(A)$. The reader will find this concept in Definition 5.2 below.

Sibe Mardešić [17] extended the work in [20] to the class of stratifiable spaces – we shall speak more about these spaces in section 3. But we mention that the classes of metrizable spaces and of CW-complexes are proper subclasses of the class of stratifiable spaces. More recently, in [10], the notion of semi-sequence was introduced and a limit theorem in extension theory was proved for a semi-limit within the inverse limit of an inverse sequence of stratifiable spaces and for arbitrary CW-complexes. The main theorem of that paper generalizes all the previously mentioned propositions involving inverse sequences of metrizable spaces or even stratifiable spaces.

The Edwards-Walsh cell-like resolution theorem was established for metrizable spaces in [21] and the $\mathbb{Z}/p$-resolution theorem for metrizable spaces has been provided in [14]. During the International Conference and Workshops on Geometric Topology held at Będlewo, Poland, during 3–10 July 2005, Michael Levin introduced the question of whether the $G$-acyclic resolution theorems (see [23]) for metrizable compacta could be extended to arbitrary metrizable spaces. Given an abelian group $G$, $n \in \mathbb{N}$, and a metrizable space $X$ with $\dim_G X \leq n$, the problem is to prove the existence of a metrizable space $Z$ with $\dim Z \leq n+1$, $\dim_G Z \leq n$, and a $G$-acyclic map of $Z$ onto $X$. But Rubin and Schapiro had considered this question much earlier. Moreover, they had several communications about this problem with Akira Koyama and Katsuya Yokoi after the publications of [21] and [14]. No concrete new developments arose from that.

The difficulty is that the only known way of obtaining such a resolution comes from the techniques used in [21]. The resolving space $Z$ in that approach is constructed as a subspace of the inverse limit of an inverse sequence of metrizable polyhedra. In cases (depending on the group $G$) when $Z$ can have dimension $\leq n$, then $\dim_G Z \leq n$ is “automatically” true. But for those groups $G$ where it is not possible to have $\dim Z \leq n$, it is not certain that this property would be enjoyed by $Z$ under such a construction. One could phrase the question this way: Is there a property of the inverse sequence whose limit is $Z$ that could be used to detect that $Z$ is an absolute co-extensor for $K(G, n)$?[1] The results in [22], [20], [17], [10], [11], and [12] give us steps in this direction.

In this paper, we are going to obtain new, significantly sharper results for detecting the absolute co-extensor property, with respect to a given CW-complex $K$, for the limit of an inverse sequence of stratifiable (and hence for metrizable) spaces. These are the main outcomes of this work; they will be found as theorems 5.6 and 6.3. The latter provides the first "local" version of such a theorem. Since each of the resolving spaces $Z$ mentioned in the previous paragraph is constructed as the limit of an inverse sequence of metrizable spaces, then these new results might be helpful in the theory of resolutions.

For the reader unfamiliar with concepts such as stratifiable space, semi-sequence, maps of semi-sequences, and semi-limits, we provide all the necessary background in sections 2–4 below.

## 2. Semi-sequences

We shall now provide a portion of the theory of semi-sequences. For the remainder of this section, $\mathbf{X} = (X_i, p_i^{i+1})$ will denote an inverse sequence of spaces and $X = \lim \mathbf{X}$. Let us repeat Definition 1.3 of [10].

**Definition 2.1.** Let $\mathbb{N}^*$ be an infinite subset of $\mathbb{N}$, and for each $i \in \mathbb{N}^*$, let $M_i$ be a subset of $X_i$. We shall refer to $\mathbf{M} = (M_i, \mathbb{N}^*)$ as a semi-sequence of $\mathbf{X}$ and define $\operatorname{slim} \mathbf{M}$ to be those $x \in X$ having the property that there exists $i \in \mathbb{N}^*$ such that $x_j \in M_j$ for all $j \in \mathbb{N}^*$ and $j \geq i$. We call $M = \operatorname{slim} \mathbf{M}$ the semi-limit of $\mathbf{M}$.

In this paper, however, we shall always use $\mathbb{N}^* = \mathbb{N}$, so let us just write $\mathbf{M} = (M_i)$ instead of $(M_i, \mathbb{N})$. We may always treat $\mathbf{X}$ as $(X_i)$, i.e., we may think of $\mathbf{X}$ as a semi-sequence of $\mathbf{X}$. As usual, $p_i : X = \operatorname{slim} \mathbf{X} \to X_i$ will denote the $i$th coordinate projection.

Whenever $x \in \operatorname{slim} \mathbf{M}$, then there exists a first $i \in \mathbb{N}$ such that $x_j \in M_j$ for all $j \geq i$. We shall denote this by $i = \phi_{\mathbf{M}}(x)$ and call it the $\mathbf{M}$-birth index of $x$.

**Definition 2.2.** Let $\mathbf{M} = (M_i)$, $\mathbf{H} = (H_i)$ be semi-sequences of $\mathbf{X}$.

[1] It is known that if a metrizable space $Z$ is an absolute co-extensor for $K$ then so is every subspace of $Z$.

(1) We shall call $\mathbf{M}$ a subsemi-sequence of $\mathbf{H}$ and denote this $\mathbf{M} \subset \mathbf{H}$ if for each $i \in \mathbb{N}$, $M_i \subset H_i$.

(2) Define the union of $\mathbf{M}$ and $\mathbf{H}$, $\mathbf{M} \cup \mathbf{H}$, to be the semi-sequence $(M_i \cup H_i)$ of $\mathbf{X}$. The intersection $\mathbf{M} \cap \mathbf{H}$ is defined as $(M_i \cap H_i)$.

(3) Let us say that $\mathbf{M}$ is open (closed, respectively) in $\mathbf{X}$ if for each $i \in \mathbb{N}$, $M_i$ is open (closed, respectively) in $X_i$.

(4) Designate that $\mathbf{M}$ is an expanding semi-sequence of $\mathbf{X}$ if $(p_i^{i+1})^{-1}(M_i) \subset M_{i+1}$ for each $i \in \mathbb{N}$.

(5) Assume that $K$ is a space, and for each $i \in \mathbb{N}$, $g_i : H_i \to K$ is a map. We shall then say that $\mathbf{g} = (g_i)$ is a map of $\mathbf{H}$ to $K$ and denote this by $\mathbf{g} : \mathbf{H} \to K$ if the consistency equation, $g_{i+1}(x) = g_i \circ p_i^{i+1}(x)$, is true whenever $x \in (p_i^{i+1})^{-1}(H_i) \cap H_{i+1}$ and $i \in \mathbb{N}$.

(6) Let $\mathbf{g} = (g_i) : \mathbf{H} \to K$, $\mathbf{h} = (h_i) : \mathbf{M} \to K$ be maps, $\mathbf{M} \subset \mathbf{H}$, and $h_i = g_i | M_i : M_i \to K$ for each $i \in \mathbb{N}$. Then we say that $\mathbf{h}$ is the restriction of $\mathbf{g}$ to $\mathbf{M}$, written $\mathbf{h} = \mathbf{g}|\mathbf{M}$, and $\mathbf{g}$ is an extension of $\mathbf{h}$ to $\mathbf{H}$.

(7) We denote by $\overline{\mathbf{M}}$ the semi-sequence $(\mathrm{cl}_{X_i} M_i)$ of $\mathbf{X}$.

The following lemmas appear in section 2 of [11].

**Lemma 2.3.** *Let* $\mathbf{M} = (M_i)$ *and* $\mathbf{H} = (H_i)$ *be semi-sequences of* $\mathbf{X}$. *If* $\mathbf{M} \subset \mathbf{H}$, *then* $\operatorname{slim} \mathbf{M} \subset \operatorname{slim} \mathbf{H} \subset \operatorname{slim} \mathbf{X} = X$.

**Lemma 2.4.** *Let* $\mathbf{M}$, $\mathbf{H}$ *be semi-sequences of* $\mathbf{X}$. *Then*

(1) $\mathbf{M} \cap \mathbf{H} \subset \mathbf{M}$ *and* $\mathbf{M} \subset \mathbf{M} \cup \mathbf{H}$;

(2) *if both* $\mathbf{M}$, $\mathbf{H}$ *are open (closed, respectively), then* $\mathbf{M} \cup \mathbf{H}$ *and* $\mathbf{M} \cap \mathbf{H}$ *are open (closed, respectively); and*

(3) *if both* $\mathbf{M}$ *and* $\mathbf{H}$ *are expanding, then so are* $\mathbf{M} \cup \mathbf{H}$ *and* $\mathbf{M} \cap \mathbf{H}$; *moreover,* $\operatorname{slim}(\mathbf{M} \cup \mathbf{H}) = \operatorname{slim} \mathbf{M} \cup \operatorname{slim} \mathbf{H}$ *and* $\operatorname{slim}(\mathbf{M} \cap \mathbf{H}) = \operatorname{slim} \mathbf{M} \cap \operatorname{slim} \mathbf{H}$.

**Lemma 2.5.** *Let* $\mathbf{H} = (H_i)$ *be a semi-sequence of* $\mathbf{X}$, $\mathbf{g} = (g_i) : \mathbf{H} \to K$ *a map, and* $\mathbf{M} = (M_i) \subset \mathbf{H}$. *Put* $h_i = g_i | M_i : M_i \to K$ *for each* $i \in \mathbb{N}$. *Then* $\mathbf{h} = (h_i)$ *is a map of* $\mathbf{M}$ *to* $K$ *and* $\mathbf{h} = \mathbf{g}|\mathbf{M}$.

We next restate Lemma 2.12 of [11], noting that (1) of that lemma does not depend on the existence of the map $\mathbf{g}$.

**Lemma 2.6.** *Let* $\mathbf{M} = (M_i)$ *be an expanding open semi-sequence of* $\mathbf{X}$. *For each* $i \in \mathbb{N}$, *let* $U_i = p_i^{-1}(M_i)$, *and put* $U = \bigcup\{U_i \mid i \in \mathbb{N}\}$. *Then* $U = \operatorname{slim} \mathbf{M}$, *and* $U$ *is open in* $X$. *If* $K$ *is a space and* $\mathbf{g} = (g_i)$ *a map of* $\mathbf{M}$ *to* $K$, *then* $g = \operatorname{slim} \mathbf{g} : U \to K$ *is a map having the property that* $g|U_i = g_i \circ p_i|U_i$ *for each* $i \in \mathbb{N}$.

We shall need Lemma 6.4(1), (2) of [11] for the following lemma.

**Lemma 2.7.** *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of spaces,* $X = \lim \mathbf{X}$, *and* $W$ *be an open subset of* $X$. *Then there exists an open and expanding semi-sequence* $\mathbf{W} = (W_i)$ *of* $\mathbf{X}$ *such that* $W = \operatorname{slim} \mathbf{W}$.

## 3. Stratifiable spaces

In this section, we provide information about stratifiable spaces that will be sufficient for the purposes of this paper. Stratifiable spaces were introduced in 1961 by Jack G. Ceder [2] under the name $M_3$-spaces and later renamed stratifiable spaces by Carlos J. R. Borges [1]. We refer the reader to [10] for a fuller account of this subject (section 2) and for citations of related research in this area.

**Definition 3.1.** A stratification on a space $X$ is a sequence $(s_k)_{k\in\mathbb{N}}$ having the property that for each $k$, $s_k$ is a function that assigns to every open set $U \subseteq X$ an open set $s_k(U) \subseteq X$ such that

(S1) $\overline{s_k(U)} \subseteq U$;

(S2) $\bigcup_{k=1}^{\infty} s_k(U) = U$;

(S3) $U \subseteq V \Rightarrow s_k(U) \subseteq s_k(V)$.

A $T_1$-space $X$ is stratifiable provided it admits a stratification.

**Lemma 3.2.** *Every discrete space is stratifiable.*

**Lemma 3.3.** *If a space* $X$ *is the topological sum* $\sum\{X_a \mid a \in A\}$ *of spaces* $X_a$, *then* $X$ *is stratifiable if and only if* $X_a$ *is stratifiable for each* $a \in A$.

**Remark 3.4.** The properties (S1) and (S2) of a stratification show that every open subset of a stratifiable space $X$ is an $F_\sigma$-set in $X$.

**Lemma 3.5.** *Every metrizable space is stratifiable.*

The following product theorem can be extracted from page 107 of [2] or may be found as Theorem 2.7 in [10].

**Theorem 3.6.** *Suppose that* $\{X_i \mid i \in \mathbb{N}\}$ *is a collection of stratifiable spaces; then* $\prod_{i=1}^{\infty} X_i$ *is a stratifiable space.*

**Remark 3.7.** Let $X$ be a stratifiable space. Then

(1) $X$ is hereditarily stratifiable;

(2) $X$ is paracompact;

(3) $X$ is hereditarily paracompact because of (2) and (1);

(4) CW-complexes are absolute neighborhood extensors for $X$; and

(5) because of (3), Lemma 3.5, and Theorem 3.6, $X$ satisfies the homotopy extension property with respect to CW-complexes and hence to polyhedra $|K|_{\mathrm{CW}}$.

**Proposition 3.8.** *The inverse limit of any inverse sequence of stratifiable spaces is stratifiable.*

Here, we quote Theorem 3.6 of [9].

**Proposition 3.9.** *Let* $X$ *be a stratifiable space and* $K$ *a* CW-*complex. If* $X$ *is an absolute co-extensor for* $K$*, then each subspace of* $X$ *is an absolute co-extensor for* $K$*.*

## 4. Lemmas

Let us state Lemma 8.1 of [11].

**Lemma 4.1.** *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of normal spaces and* $\mathbf{W} = (W_i)$ *an expanding open semi-sequence of* $\mathbf{X}$. *Suppose that for each* $i \in \mathbb{N}$, $\{E_i^j \mid j \in \mathbb{N}\}$ *is a collection of closed subsets of* $X_i$ *and for each* $j$, $E_i^j \subset W_i$. *Then there exists an expanding open semi-sequence* $\mathbf{M} = (M_i)$ *of* $\mathbf{X}$ *such that*

(1) $\mathbf{M} \subset \mathbf{W}$*;*

(2) $\mathrm{cl}_{X_i}(M_i) \subset W_i$ *and* $(p_i^{i+1})^{-1}(\mathrm{cl}_{X_i}(M_i)) \subset M_{i+1}$ *for each* $i \in \mathbb{N}$*;*

(3) $(p_j^i)^{-1}(E_j^k) \subset M_i$ *whenever* $j,\, k \leq i$*; and*

(4) *if for each* $i \in \mathbb{N}$, $\bigcup\{E_i^k \mid k \in \mathbb{N}\} = W_i$*, then* $\operatorname{slim} \mathbf{M} = \operatorname{slim} \mathbf{W}$.

Recall that an open subset $M$ of a topological space $X$ is called *regular-open* if $M = \mathrm{int}_X \overline{M}$.

**Lemma 4.2.** *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of stratifiable spaces and* $\mathbf{W} = (W_i)$ *an expanding open semi-sequence of* $\mathbf{X}$. *Then there exists an expanding open semi-sequence* $\mathbf{M} = (M_i)$ *of* $\mathbf{X}$ *consisting of regular-open sets* $M_i$ *such that*

(1) $\overline{\mathbf{M}} \subset \mathbf{W}$*;*

(2) $(p_i^{i+1})^{-1}(\mathrm{cl}_{X_i} M_i) \subset M_{i+1}$ *for each* $i \in \mathbb{N}$*; and*

(3) $\mathrm{slim}\, \mathbf{M} = \mathrm{slim}\, \overline{\mathbf{M}} = \mathrm{slim}\, \mathbf{W}$.

*Proof:* For each $i \in \mathbb{N}$, using Remark 3.4, write $W_i = \bigcup\{E_i^k \mid k \in \mathbb{N}\}$ where $E_i^k$ is closed in $X_i$ for each $k$. Since stratifiable spaces are paracompact (Remark 3.7(2)) and hence normal, we may apply Lemma 4.1 to obtain an expanding open semi-sequence $\mathbf{M} = (M_i)$ of $\mathbf{X}$ satisfying (1)–(4) of that lemma. To obtain a semi-sequence in which each $M_i$ is regular-open, simply replace each $M_i$ by $\mathrm{int}_{X_i} \overline{M}_i$ and use (2) of Lemma 4.1 to check that this new semi-sequence meets the stated requirements. □

## 5. Limit theorem

In this section, we shall prove a limit theorem, Theorem 5.6, and later state some of its corollaries. In our proof below, we shall apply the characterization theorem, Theorem 4.3, of [12]. Since we do not need that theorem in its entirety, let us state the form of it that we shall use.

**Theorem 5.1.** *Let* $\mathbf{X} = (X_i, p_i^{i+1})$ *be an inverse sequence of stratifiable spaces,* $X = \lim \mathbf{X}$*, and* $K$ *be a* CW*-complex. Then* $X$ *is an absolute co-extensor for* $K$ *if for any expanding open semi-sequences* $\mathbf{M}$ *and* $\mathbf{H}$ *of* $\mathbf{X}$ *and map* $\mathbf{g} : \mathbf{M} \to K$*, there exist expanding open subsemi-sequences,* $\mathbf{M}^*$ *of* $\mathbf{M}$ *and* $\mathbf{H}^*$ *of* $\mathbf{H}$*, and a map* $\mathbf{g}^* : \mathbf{M}^* \cup \mathbf{H}^* \to K$ *such that*

(1) $\mathrm{slim}\, \mathbf{M}^* \cup \mathrm{slim}\, \mathbf{H}^* = \mathrm{slim}\, \mathbf{M} \cup \mathrm{slim}\, \mathbf{H}$*, and*

(2) $\mathbf{g}^*|\mathbf{M}^* = \mathbf{g}|\mathbf{M}^*$.

**Definition 5.2.** Let $\mathbf{X} = (X_i, p_i^{i+1})$ be an inverse sequence of spaces and $K$ a CW-complex. Then we write $\mathbf{X}\tau K$ to mean that for each $i \in \mathbb{N}$, closed subset $A$ of $X_i$, and map $g : A \to K$, there exists $j \geq i$ such that the map $h : (p_i^j)^{-1}(A) \to K$, defined by $t \mapsto g \circ p_i^j(t)$, extends to a map of $X_j$ to $K$.

**Definition 5.3.** Let $\mathbf{X} = (X_i, p_i^{i+1})$ be an inverse sequence of spaces, $X = \lim \mathbf{X}$, and $K$ a CW-complex. Then we write $\mathbf{X}\tau_\sigma K$ to mean that for each $i \in \mathbb{N}$, there exists a sequence $(X_i^q)$ of closed subsets of $X_i$ such that

(1) for each $q \in \mathbb{N}$, $s \geq i$, closed subset $A$ of $(p_i^s)^{-1}(X_i^q)$, and map $g : A \to K$, there exists $j \geq s$ such that the map $h : (p_s^j)^{-1}(A) \to K$, defined by $t \mapsto g \circ p_s^j(t)$, extends to a map of $(p_i^j)^{-1}(X_i^q)$ to $K$, and

(2) for each $x \in X$, there exists a finite set of pairs, $\{(i_j, q_j) \mid 1 \leq j \leq b\} \subset \mathbb{N} \times \mathbb{N}$ such that for some $i \geq \max\{i_j \mid 1 \leq j \leq b\}$, $x_i \in \operatorname{int}_{X_i} \bigcup\{(p_{i_j}^i)^{-1}(X_{i_j}^{q_j}) \mid 1 \leq j \leq b\}$.

**Lemma 5.4.** *Let* $\mathbf{X}$ *be an inverse sequence of spaces and* $K$ *a* CW*-complex. If* $\mathbf{X}\tau K$*, then* $\mathbf{X}\tau_\sigma K$*.*

**Remark 5.5.** If (1) in the statement of Definition 5.3 is replaced by the following, then we will have an equivalent definition.

(1) For each $q \in \mathbb{N}$, $s \geq i$, closed subset $A$ of $(p_i^s)^{-1}(X_i^q)$, and map $g : A \to K$, there exists $j \geq s$ such that for all $k \geq j$, the map $h : (p_s^k)^{-1}(A) \to K$, defined by $t \mapsto g \circ p_s^k(t)$, extends to a map of $(p_i^k)^{-1}(X_i^q)$ to $K$.

**Theorem 5.6.** *Let* $K$ *be a* CW*-complex,* $\mathbf{X} = (X_i, p_i^{i+1})$ *an inverse sequence of stratifiable spaces such that* $\mathbf{X}\tau_\sigma K$*, and* $X = \lim \mathbf{X}$*. Then* $X$ *is an absolute co-extensor for* $K$*.*

*Proof:* To avoid excessive notation, whenever $i \in \mathbb{N}$ and $A \subset X_i$, we shall use $\overline{A}$ to designate $\operatorname{cl}_{X_i} A$.

We shall apply Theorem 5.1 to prove that $X$ is an absolute co-extensor for $K$. Let $\mathbf{M} = (M_i)$ and $\mathbf{H} = (H_i)$ be expanding open semi-sequences of $\mathbf{X}$ and $\mathbf{g} = (g_i) : \mathbf{M} \to K$ a map. Using Lemma 4.2, we obtain expanding open semi-sequences $\mathbf{M}^1 = (M_i^1)$, $\mathbf{H}^1 = (H_i^1)$ of $\mathbf{X}$ such that

(SE) $\operatorname{slim} \mathbf{M}^1 = \operatorname{slim} \mathbf{M}$, $\operatorname{slim} \mathbf{H}^1 = \operatorname{slim} \mathbf{H}$,
$\mathbf{M}^1 \subset \overline{\mathbf{M}}^1 \subset \mathbf{M}$, $\mathbf{H}^1 \subset \overline{\mathbf{H}}^1 \subset \mathbf{H}$,

and so that for each $i \in \mathbb{N}$,

(M) $(p_i^{i+1})^{-1}(\overline{M}_i^1) \subset M_{i+1}^1$,

(H) $(p_i^{i+1})^{-1}(\overline{H}_i^1) \subset H_{i+1}^1$.

Let $\theta = (\theta_1, \theta_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a surjective function such that

(TH1) for each $i \in \mathbb{N}$, $\theta_1(i) \leq i$, and

(TH2) each fiber of $\theta$ is infinite.

We shall make a construction by induction on $k \in \mathbb{N}$. Let $\{X_i^q\}$ be chosen as in Definition 5.3. Put $l_1 = 1$, $A_{l_1} = \overline{M}^1_{l_1} \setminus H^1_{l_1}$, and $B_{l_1} = \emptyset$. Certainly by (TH1), $\theta_1(1) = 1$, so $i \leq l_i$ for $i = 1$. Let $\tilde{g}_{l_1} : A_{l_1} \cup B_{l_1} \to K$ be the unique map having the property that $\tilde{g}_{l_1}|A_{l_1} = g_{l_1}|A_{l_1}$.

Suppose that $k \in \mathbb{N}$, and we have chosen $l_i \in \mathbb{N}$ for each $1 \leq i \leq k$ in such a manner that

(I0) $i \leq l_i$, and if $i < k$, then $l_i < l_{i+1}$.

For $1 \leq j \leq l_k$, assume that we have chosen closed subsets $A_j$ and $B_j$ of $X_j$ and a map $\tilde{g}_j : A_j \cup B_j \to K$. This will be done so that

(I1) $A_j \subset \overline{M}^1_j$;

(I2) $B_j \subset H^1_j \subset \overline{H}^1_j$;

(I3) $\tilde{g}_j(t) = g_j(t),\ t \in A_j$.

We require that if $i < k$ and $l_i < j < l_{i+1}$, then

(I4) $A_j = (p^j_{l_i})^{-1}(A_{l_i})$,

(I5) $B_j = (p^j_{l_i})^{-1}(B_{l_i})$.

Also, we must have that for any $u < l_k$,

(I6) if $t \in (p^{u+1}_u)^{-1}(A_u \cup B_u)$, then $\tilde{g}_{u+1}(t) = \tilde{g}_u \circ p^{u+1}_u(t)$.

On the other hand, for each $1 < i \leq k$, we put $l_i^* = l_i - 1$ and ordain that

(I7) $A_{l_i} = (p^{l_i}_{l_{i-1}})^{-1}(A_{l_{i-1}}) \cup (\overline{M}^1_{l_i} \setminus H^1_{l_i})$;

(I8.1) $B^*_{l_i} = (p^{l_i^*}_{\theta_1(i-1)})^{-1}(X^{\theta_2(i-1)}_{\theta_1(i-1)}) \cap \overline{H}^1_{l_i^*}$;

(I8.2) $B_{l_i} = (p^{l_i}_{l_{i-1}})^{-1}(B_{l_{i-1}}) \cup (p^{l_i}_{l_i^*})^{-1}(B^*_{l_i})$.

Let $l^*_{k+1} > l_k$ be selected from the fact that $\mathbf{X}\tau_\sigma K$ and for the map $\tilde{g}_{l_k}|A$ where $A = (A_{l_k} \cup B_{l_k}) \cap (p^{l_k}_{\theta_1(k)})^{-1}(X^{\theta_2(k)}_{\theta_1(k)})$, which is a closed subset of $(p^{l_k}_{\theta_1(k)})^{-1}(X^{\theta_2(k)}_{\theta_1(k)})$. The correspondence with Definition 5.3 is that $(q, i, s, j) = (\theta_2(k), \theta_1(k), l_k, l^*_{k+1})$. Using Remark 5.5, one may also require that $k + 1 \leq l^*_{k+1}$. Let $l_{k+1} = l^*_{k+1} + 1$. Hence, (I0) is satisfied when $k$ is replaced by $k + 1$. Also, using

(TH1), one sees that $\theta_1(k) \leq l_k < l^*_{k+1} < l_{k+1}$, which we shall need shortly when defining $B^*_{l_{k+1}}$ and $B_{l_{k+1}}$.

For each $l_k < j < l_{k+1}$, we choose $A_j = (p^j_{l_k})^{-1}(A_{l_k})$ and $B_j = (p^j_{l_k})^{-1}(B_{l_k})$. The inductive assumptions for $l_k$ in (I1) and (I2), along with (M) and (H), imply that for such $j$, (I1) and (I2) hold true, and, clearly, (I4) and (I5) are also true. We define $\tilde{g}_j : A_j \cup B_j \to K$ to be the map such that

(A0) $\tilde{g}_j(t) = \tilde{g}_{l_k} \circ p^j_{l_k}(t)$, $t \in A_j \cup B_j$, $l_k < j < l_{k+1}$.

Suppose now that $j = l_k + 1$. Note that $j < l_{k+1}$. Using (I1), (SE), and the fact that $\mathbf{M}$ is expanding, we get $A_j = (p^j_{l_k})^{-1}(A_{l_k}) \subset (p^j_{l_k})^{-1}(\overline{M}^1_{l_k}) \subset (p^j_{l_k})^{-1}(M_{l_k}) \subset M_j$. Thus, if $t \in A_j$, then $p^j_{l_k}(t) \in A_{l_k}$, so by (I3) for $l_k$, $\tilde{g}_{l_k} \circ p^j_{l_k}(t) = g_{l_k} \circ p^j_{l_k}(t)$. Since $\mathbf{g} : \mathbf{M} \to K$ is a map, then $g_j(t) = g_{l_k} \circ p^j_{l_k}(t) = \tilde{g}_{l_k} \circ p^j_{l_k}(t) = \tilde{g}_j(t)$, which establishes (I3) for $j = l_k + 1$. This argument can be employed recursively (think of $l_k = j-1$) to show that (I3) holds true as long as $l_k < j < l_{k+1}$.

Now, to satisfy (I7), (I8.1), and (I8.2), put

$$A_{l_{k+1}} = (p^{l_{k+1}}_{l_k})^{-1}(A_{l_k}) \cup (\overline{M}^1_{l_{k+1}} \setminus H^1_{l_{k+1}});$$

$$B^*_{l_{k+1}} = (p^{l^*_{k+1}}_{\theta_1(k)})^{-1}(X^{\theta_2(k)}_{\theta_1(k)}) \cap \overline{H}^1_{l^*_{k+1}};$$

$$B_{l_{k+1}} = (p^{l_{k+1}}_{l_k})^{-1}(B_{l_k}) \cup (p^{l_{k+1}}_{l^*_{k+1}})^{-1}(B^*_{l_{k+1}}).$$

Of course, $B^*_{l_{k+1}} \subset \overline{H}^1_{l^*_{k+1}}$. Because of this, (H), (I2) for $l_k$, and the fact that $\mathbf{H}^1$ is expanding, one can see that $B_{l_{k+1}} \subset H^1_{l_{k+1}}$. Therefore, (I2) for $j = l_{k+1}$ is satisfied. The fact that (I1) is true for $j = l_{k+1}$ can be done by a similar, but simpler argument involving $\mathbf{M}^1$.

We next prepare for defining $\tilde{g}_{l_{k+1}}$. By an application of Definition 5.3(1) with respect to our choice of $l^*_{k+1}$ to $(p^{l^*_{k+1}}_{l_k})^{-1}(A) = (p^{l^*_{k+1}}_{l_k})^{-1}(A_{l_k} \cup B_{l_k}) \cap (p^{l^*_{k+1}}_{\theta_1(k)})^{-1}(X^{\theta_2(k)}_{\theta_1(k)})$, there exists a map $h : A^* = (p^{l^*_{k+1}}_{l_k})^{-1}(A_{l_k} \cup B_{l_k}) \cup B^*_{l_{k+1}} \to K$ such that $h(t) = \tilde{g}_{l_k} \circ p^{l^*_{k+1}}_{l_k}(t)$ for $t \in (p^{l^*_{k+1}}_{l_k})^{-1}(A_{l_k} \cup B_{l_k})$. We could extend the map $h$ to include all of $(p^{l^*_{k+1}}_{\theta_1(k)})^{-1}(X^{\theta_2(k)}_{\theta_1(k)})$, but that would not serve our purposes here.

We do, however, use the fact that $B^*_{l_{k+1}} \subset (p^{l^*_{k+1}}_{\theta_1(k)})^{-1}(X^{\theta_2(k)}_{\theta_1(k)})$. Next, define $\overline{g} : (p^{l_{k+1}}_{l^*_{k+1}})^{-1}(A^*) \to K$ by $\overline{g}(t) = h \circ p^{l_{k+1}}_{l^*_{k+1}}(t)$. Then,

(AB) $\overline{g}(t) = \tilde{g}_{l_k} \circ p^{l_{k+1}}_{l_k}(t),\ \ t \in (p^{l_{k+1}}_{l_k})^{-1}(A_{l_k} \cup B_{l_k})$.

Because of (I1), if $t \in (p^{l_{k+1}}_{l_k})^{-1}(A_{l_k})$, then $t \in (p^{l_{k+1}}_{l_k})^{-1}(\overline{M}^1_{l_k}) \subset M_{l_{k+1}}$. Using (I1) again, $p^{l_{k+1}}_{l_k}(t) \in \overline{M}^1_{l_k} \subset M_{l_k}$. From (AB) and (I3) for $j = l_k$, we see that $\overline{g}(t) = g_{l_k} \circ p^{l_{k+1}}_{l_k}(t)$ for such $t$. Since $\mathbf{g}$ is a map, we have

(A1) $\overline{g}(t) = g_{l_k} \circ p^{l_{k+1}}_{l_k}(t) = g_{l_{k+1}}(t),\ t \in (p^{l_{k+1}}_{l_k})^{-1}(A_{l_k})$.

The reader may check that the other part of the domain of $\overline{g}$, besides $(p^{l_{k+1}}_{l_k})^{-1}(A_{l_k}) \subset M_{l_{k+1}}$, is $B_{l_{k+1}}$. But $B_{l_{k+1}} \subset H^1_{l_{k+1}}$ and $\overline{M}^1_{l_{k+1}} \setminus H^1_{l_{k+1}} \subset M_{l_{k+1}}$, the latter being the domain of $g_{l_{k+1}}$. Using this and (A1), one extends $\overline{g}$ to a map $\tilde{g}_{l_{k+1}} : A_{l_{k+1}} \cup B_{l_{k+1}} \to K$ by setting $\tilde{g}_{l_{k+1}}(t) = g_{l_{k+1}}(t)$ for all $t \in \overline{M}^1_{l_{k+1}} \setminus H^1_{l_{k+1}}$. With this definition, (I3) is established for $j = l_{k+1}$. Hence, (I3) is true for all $1 \le j \le l_{k+1}$.

As a result of the preceding construction and (AB), we get

(A2) $\tilde{g}_{l_{k+1}}(t) = \tilde{g}_{l_k} \circ p^{l_{k+1}}_{l_k}(t),\ t \in (p^{l_{k+1}}_{l_k})^{-1}(A_{l_k} \cup B_{l_k})$.

We need to satisfy (I6) for $l_k \le u < l_{k+1}$. From (I4) and (I5), $(p^{u+1}_u)^{-1}(A_u \cup B_u) = (p^{u+1}_{l_k})^{-1}(A_{l_k} \cup B_{l_k})$. Hence, if $t \in (p^{u+1}_u)^{-1}(A_u \cup B_u)$, then from (A0) and (A2), $\tilde{g}_{u+1}(t) = \tilde{g}_{l_k} \circ p^{u+1}_{l_k}(t) = \tilde{g}_{l_k} \circ p^u_{l_k} \circ p^{u+1}_u(t) = \tilde{g}_u \circ p^{u+1}_u(t)$. So (I6) is established.

This completes the induction.

Let $\mathbf{A} = (A_i)$ and $\mathbf{B} = (B_i)$; then $\mathbf{A}$ and $\mathbf{B}$ are semi-sequences of $\mathbf{X}$. By (I4) and (I7), $\mathbf{A}$ is expanding. On the other hand, one can use (I5) and (I8.2) to see that $\mathbf{B}$ is expanding. From (I6), $\tilde{\mathbf{g}} = (\tilde{g}_i)$ is a map of $\mathbf{A} \cup \mathbf{B}$ to $K$. Employing (I1) and (SE), we see that $\mathbf{A} \subset \overline{\mathbf{M}}^1 \subset \mathbf{M}$, and from (I3), $\tilde{\mathbf{g}}|\mathbf{A} = \mathbf{g}|\mathbf{A}$. Using (I2) and (SE), one also sees that $\mathbf{B} \subset \overline{\mathbf{H}}^1 \subset \mathbf{H}$.

For each $i \in \mathbb{N}$, put $M^*_i = (\operatorname{int}_{X_i} A_i) \cap M^1_i$ and $H^*_i = (\operatorname{int}_{X_i} B_i) \cap H^1_i$. Define $\mathbf{M}^* = (M^*_i)$, $\mathbf{H}^* = (H^*_i)$, and note that $\tilde{\mathbf{g}}|\mathbf{M}^* = \mathbf{g}|\mathbf{M}^*$. Clearly, $\mathbf{M}^*$ and $\mathbf{H}^*$ are open, $\mathbf{M}^* \subset \mathbf{M}^1$, and $\mathbf{H}^* \subset \mathbf{H}^1$. Let us show that $\mathbf{M}^*$ is expanding. First, $(p^{i+1}_i)^{-1}((\operatorname{int}_{X_i} A_i) \cap M^1_i) = (p^{i+1}_i)^{-1}(\operatorname{int}_{X_i} A_i) \cap (p^{i+1}_i)^{-1}(M^1_i)$. Now, one has $(p^{i+1}_i)^{-1}(\operatorname{int}_{X_i} A_i)$

$\subset \operatorname{int}_{X_{i+1}}(p_i^{i+1})^{-1}(A_i) \subset \operatorname{int}_{X_{i+1}} A_{i+1}$, as well as $(p_i^{i+1})^{-1}(M_i^1) \subset (p_i^{i+1})^{-1}(\overline{M}_i^1) \subset M_{i+1}^1$. So $\mathbf{M}^*$ is an expanding open subsemi-sequence of $\mathbf{M}$. One similarly shows that $\mathbf{H}^*$ is an expanding open subsemi-sequence of $\mathbf{H}$. Define $\mathbf{g}^* = \tilde{\mathbf{g}}|(\mathbf{M}^* \cup \mathbf{H}^*) : \mathbf{M}^* \cup \mathbf{H}^* \to K$.

Since $\mathbf{g}^*|\mathbf{M}^* = \tilde{\mathbf{g}}|\mathbf{M}^* = \mathbf{g}|\mathbf{M}^*$, then it remains only to prove that $\operatorname{slim}\mathbf{M}^* \cup \operatorname{slim}\mathbf{H}^* = \operatorname{slim}\mathbf{M} \cup \operatorname{slim}\mathbf{H}$. Note from (SE) that the latter is the same as $\operatorname{slim}\mathbf{M}^1 \cup \operatorname{slim}\mathbf{H}^1$, and it is clear that $\operatorname{slim}\mathbf{M}^* \cup \operatorname{slim}\mathbf{H}^* \subset \operatorname{slim}\mathbf{M}^1 \cup \operatorname{slim}\mathbf{H}^1$, so we simply need to prove the reverse of this inclusion.

First, we will show that if $x \in \operatorname{slim}\mathbf{M}^1 \setminus \operatorname{slim}\mathbf{H}^1$, then $x \in \operatorname{slim}\mathbf{M}^*$. Let $k = \Phi_{\mathbf{M}^1}(x)$, the $\mathbf{M}^1$-birth index of $x$. Hence, $x_{l_{k+1}} \in M_{l_{k+1}}^1$. Since $\operatorname{slim}\mathbf{H}^1 = \operatorname{slim}\overline{\mathbf{H}}^1$ and $\overline{\mathbf{H}}^1$ is expanding, then $x_{l_{k+1}} \notin \overline{H}_{l_{k+1}}^1$. Hence, $x_{l_{k+1}} \in M_{l_{k+1}}^1 \setminus \overline{H}_{l_{k+1}}^1 \subset \overline{M}_{l_{k+1}}^1 \setminus H_{l_{k+1}}^1 \subset A_{l_{k+1}}$. Since $M_{l_{k+1}}^1 \setminus \overline{H}_{l_{k+1}}^1 \subset A_{l_{k+1}}$ is an open subset of $X_{l_{k+1}}$, then, using (I7), $x_{l_{k+1}} \in (\operatorname{int}_{X_{l_{k+1}}} A_{l_{k+1}}) \cap M_{l_{k+1}}^1 = M_{l_{k+1}}^*$. Since $\mathbf{M}^*$ is expanding, then $x \in \operatorname{slim}\mathbf{M}^*$.

Now, let $x \in \operatorname{slim}\mathbf{H}^1$. We shall prove that $x \in \operatorname{slim}\mathbf{H}^*$. Indeed, since $\mathbf{H}^*$ is expanding, it is sufficient to show that for some $r$, $x_{l_r} \in (\operatorname{int}_{X_{l_r}} B_{l_r}) \cap H_{l_r}^1$.

Put $k_0 = \Phi_{\mathbf{H}^1}(x)$, the $\mathbf{H}^1$-birth index of $x$. Thus, $x_k \in H_k^1$ for all $k \geq k_0$. Using (2) of Definition 5.3, select $\{(i_j, q_j) \mid 1 \leq j \leq b\}$ and $i \geq \max\{i_j \mid 1 \leq j \leq b\}$, such that $x_i \in \operatorname{int}_{X_i} \bigcup\{(p_{i_j}^i)^{-1}(X_{i_j}^{q_j}) \mid 1 \leq j \leq b\}$. Let $k = \max\{k_0, i\}$ and define

(F1) $F = \bigcup\{(p_{i_j}^k)^{-1}(X_{i_j}^{q_j}) \mid 1 \leq j \leq b\}$.

Then

(F2) $x_k \in (\operatorname{int}_{X_k} F) \cap H_k^1$.

Note, however, that $(\operatorname{int}_{X_k} F) \cap H_k^1 \subset (\bigcup\{(p_{i_j}^k)^{-1}(X_{i_j}^{q_j}) \mid 1 \leq j \leq b\}) \cap H_k^1 = \bigcup\{(p_{i_j}^k)^{-1}(X_{i_j}^{q_j}) \cap H_k^1 \mid 1 \leq j \leq b\}$. Our proof will be complete if we can find $r \geq k$ such that $(p_k^{l_r})^{-1}(T_j) \subset B_{l_r}$ for each $T_j = (p_{i_j}^k)^{-1}(X_{i_j}^{q_j}) \cap H_k^1$ of the preceding union. For in that case, $x_{l_r}$ would lie in the open with respect to $X_{l_r}$ subset $(p_k^{l_r})^{-1}((\operatorname{int}_{X_k} F) \cap H_k^1)$ of $B_{l_r}$ and at the same time, $x_{l_r} \in H_{l_r}^1$.

Using (TH2), we may choose elements $k_j \in \mathbb{N}$, $1 \leq j \leq b$, in such a manner that $k \leq k_1 < \cdots < k_b$ and $\theta(k_j) = (\theta_1(k_j), \theta_2(k_j)) = (i_j, q_j)$. Fix $1 \leq j \leq b$ and put $s = k_j + 1$. Notice that $l_s \geq$

$s \geq k_j + 1 \geq k + 1$, so that $l_s^* = l_s - 1 \geq k \geq i \geq i_j$. Also, $\theta_1(s-1) = \theta_1(k_j) = i_j$ and $\theta_2(s-1) = \theta_2(k_j) = q_j$.

From (I8.1), one sees that $(p_{i_j}^{l_s^*})^{-1}(X_{i_j}^{q_j}) \cap H_{l_s^*}^1 \subset B_{l_s}^*$. Apply $(p_{l_s^*}^{l_s})^{-1}$ to both sides of this inclusion and use (I8.2) to get that

$$(p_{i_j}^{l_s})^{-1}(X_{i_j}^{q_j}) \cap (p_{l_s^*}^{l_s})^{-1}(H_{l_s^*}^1) \subset B_{l_s}.$$

A recursive application of (I8.2) shows that for any $r \geq s$,

$$(p_{i_j}^{l_r})^{-1}(X_{i_j}^{q_j}) \cap (p_{l_s^*}^{l_r})^{-1}(H_{l_s^*}^1) \subset B_{l_r}.$$

With $T_j$ as above, we get $(p_k^{l_r})^{-1}(T_j) = (p_k^{l_r})^{-1}\big[(p_{i_j}^k)^{-1}(X_{i_j}^{q_j})\big] \cap (p_k^{l_r})^{-1}(H_k^1) = (p_{i_j}^{l_r})^{-1}(X_{i_j}^{q_j}) \cap (p_{l_s^*}^{l_r})^{-1}\big[(p_k^{l_s^*})^{-1}(H_k^1)\big]$. Since $\mathbf{H}^1$ is expanding, then $(p_k^{l_s^*})^{-1}(H_k^1) \subset H_{l_s^*}^1$. Therefore, $(p_k^{l_r})^{-1}(T_j) \subset (p_{i_j}^{l_r})^{-1}(X_{i_j}^{q_j}) \cap (p_{l_s^*}^{l_r})^{-1}(H_{l_s^*}^1) \subset B_{l_r}$. The final step is to choose $r = k_b + 1$; for then, $r \geq k_j + 1$ for all $1 \leq j \leq b$.

Our proof is complete. □

From Lemma 5.4 and Theorem 5.6, we get the next corollary. (See [17], where a proof is not actually given, and [10], where a proof is given in detail.)

**Corollary 5.7.** *Let $K$ be a* CW*-complex, $\mathbf{X} = (X_i, p_i^{i+1})$ an inverse sequence of stratifiable spaces such that $\mathbf{X}\tau K$, and $X = \lim \mathbf{X}$. Then $X$ is an absolute co-extensor for $K$.*

In turn, Corollary 5.7 implies the following corollary (see [17]).

**Corollary 5.8.** *Let $K$ be a* CW*-complex, $\mathbf{X} = (X_i, p_i^{i+1})$ an inverse sequence of stratifiable spaces such that for each $i \in \mathbb{N}$, $X_i$ is an absolute co-extensor for $K$, and $X = \lim \mathbf{X}$. Then $X$ is an absolute co-extensor for $K$.*

## 6. Local version

We shall now provide a local condition on an inverse sequence of stratifiable spaces that implies its limit is an absolute co-extensor for a given CW complex.

**Definition 6.1.** Let $\mathbf{X} = (X_i, p_i^{i+1})$ be an inverse sequence of spaces, let $X = \lim \mathbf{X}$, and let $K$ be a CW-complex. Then we say that $\mathbf{X}$ is a local absolute co-extensor for $K$ if for each $x \in X$, there exists $i \in \mathbb{N}$ and an open neighborhood $U$ of $x_i$ in $X_i$ such that

> (EXT) for each subset $D$ of $U$ where $D$ is closed in $X_i$, $s \geq i$, closed subset $A$ of $(p_i^s)^{-1}(D)$, and map $g : A \to K$, there exists $j \geq s$ such that the map $h : (p_s^j)^{-1}(A) \to K$, defined by $t \mapsto g \circ p_s^j(t)$, extends to a map of $(p_i^j)^{-1}(D)$ to $K$.

We shall make use of the next fact in our proof of Theorem 6.3; this appears as Theorem 2.2 in [13].

**Theorem 6.2.** *Let $K$ be a space and $X$ be a paracompact space. Then $X\tau K$ if and only if $X$ has an open cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, $U\tau K$.*

**Theorem 6.3.** *Let $K$ be a* CW*-complex, $\mathbf{X} = (X_i, p_i^{i+1})$ an inverse sequence of stratifiable spaces such that $\mathbf{X}$ is a local absolute co-extensor for $K$, and $X = \lim \mathbf{X}$. Then $X$ is an absolute co-extensor for $K$.*

*Proof:* Let $x \in X$. Using Definition 6.1, find $i \in \mathbb{N}$ and an open neighborhood $U_i$ of $x$ with the properties listed in (EXT). We name an inverse sequence $\mathbf{U} = (U_j, q_j^{j+1})$ as follows. For each $j \leq i$, put $U_j = p_j^i(U_i)$, and for $j > i$, let $U_j = (p_i^j)^{-1}(U_i)$. For any $j$, the bonding map $q_j^{j+1}$ is defined to equal $p_j^{j+1}|U_{j+1} : U_{j+1} \to U_j$. Observe that for each $i \in \mathbb{N}$, $U_i$ is stratifiable because of Remark 3.7(1). Let $U = \lim \mathbf{U}$. Then $U$ is topologically equivalent to the neighborhood $p_i^{-1}(U_i)$ of $x$. If we can prove that $\mathbf{U}\tau_\sigma K$, then by Theorem 5.6, $U\tau K$. This, along with Theorem 6.2 and Remark 3.7(2), would complete our proof.

Let us describe for each $j \in \mathbb{N}$ a sequence $(U_j^q)$ of closed subsets of $X_j$ such that $U_j^q \subset U_j$ for each $q$. If $j < i$, then we put $U_j^q = \emptyset$ for all $q$. For $j = i$, use Remark 3.4 to write $U_i = \bigcup\{V_i^q \mid q \in \mathbb{N}\}$ in such a manner that for each $q$, $V_i^q$ is open in $U_i$, $U_i^q = \overline{V}_i^q$, and $U_i^q \subset U_i$. For $j > i$ and $q \in \mathbb{N}$, we just define $U_j^q = (q_i^j)^{-1}(U_i^q)$.

For any $u \in U$, there exists $q \in \mathbb{N}$ such that $u_i \in V_i^q \subset U_i^q$. So $\mathbf{U}$ satisfies Definition 5.3(2). The last step is to verify that $\mathbf{U}$ satisfies (1) of that definition. So consider a fixed $E = U_j^q$. Then $E$ is closed in $X_j$ and $E \subset U_j$. If $j < i$, then $E = \emptyset$ and there is nothing to prove. So we must consider the case that $j \geq i$. Suppose that $s \geq j$, $A$ is a closed subset of $(q^s_j)^{-1}(E)$, and $g : A \to K$ is a map. Note that $(q^s_j)^{-1}(E) = (q^s_j)^{-1}((q^j_i)^{-1}(U^q_i)) = (q^s_i)^{-1}(U^q_i)$, $A$ is a closed subset of the latter, and $s \geq i$. By (EXT) of Definition 6.1, where $D = U^q_i$, there exists $k \geq s$ such that the map $h : (p^k_s)^{-1}(A) \to K$, defined by $t \mapsto g \circ p^k_s(t)$, extends to a map of $(p^k_i)^{-1}(U^q_i) = (p^k_s)^{-1}(E) = (p^k_s)^{-1}(U^q_j)$ to $K$. Hence, **U** meets the requirements of Definition 5.3(1). Our proof is complete. □

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