Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
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Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 31, No. 1, 2007
Pages 243-256
http://topology.auburn.edu/tp/

AN m-DIMENSIONAL HEREDITARILY INDECOMPOSABLE CONTINUUM WITH EXACTLY n CONTINUOUS MAPPINGS ONTO ITSELF

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Abstract. We show that for every $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, there exists a hereditarily indecomposable m-dimensional continuum X which has exactly n continuous surjections onto itself (each one being a homeomorphism).

Moreover, we construct a family of cardinality $2^{\aleph_0}$ of continua of this type such that no two different continua from this family are comparable either by continuous mappings or by embeddings.

1. Introduction

Our terminology follows [5] and [8]. We assume that all our spaces are separable metrizable. By *dimension*, we mean the covering dimension dim and by a *continuum*, we mean a compact connected space. A continuum X is *hereditarily indecomposable*, abbreviated HI, if for any two intersecting subcontinua K, L of X, either $K \subset L$ or $L \subset K$.

The first HI continuum, now called the *pseudo-arc*, was constructed by Bronisław Knaster [7]. The pseudo-arc, which will be denoted by P, is an HI one-dimensional chainable continuum

2000 *Mathematics Subject Classification.* Primary 54F15, 54F45.

Key words and phrases. continuous surjections, Cook's continua, dimension, hereditarily indecomposable continua, pseudo-arc, Waraszkiewicz spirals.

The author was supported in part by MNiSW Grant Nr. N201 034 31/2717.

(unique, up to a homeomorphism), and every non-trivial subcontinuum of P is homeomorphic to P. (For more information and references concerning the pseudo-arc see [12].)

The first examples of HI continua of dimension m, where $m = 2, 3, \ldots, \infty$, were constructed by R. H. Bing [3].

We say that two continua are *comparable by continuous mappings* (by embeddings, respectively) if there exists a continuous mapping (an embedding, respectively) of one of those continua onto (into, respectively) the other. By a *Cook continuum*, we understand a non-trivial continuum X such that no two different nondegenerate subcontinua of X are comparable by continuous mappings. The first example of a hereditarily indecomposable Cook continuum was constructed in [4]. In the same paper, H. Cook constructed for every $n \in \mathbb{N} = \{1, 2, \ldots\}$, a continuum H_n which has exactly n continuous mappings onto itself, each one being a homeomorphism. The continuum H_n is decomposable and admits an atomic mapping onto a simple closed curve. Applying the ideas from [17], [19], and [10], we will prove the following theorem.

Theorem 1.1. *For each $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, there exists a hereditarily indecomposable continuum X_{nm} of dimension m which has exactly n continuous mappings onto itself, each one being a homeomorphism. Moreover, X_{nm} admits an atomic mapping onto the pseudo-arc P and the group of autohomeomorphisms of X_{nm} onto X_{nm} is the cyclic group of order n.*

In the special cases when $m = 1$ or $n = 1$, these results were obtained in [19]. Any 1-dimensional HI Cook continuum satisfies the condition of Theorem 1.1 for $m = n = 1$.

Moreover, we will prove the following theorem.

Theorem 1.2. *For every $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, there exists a family $\{X_{nm}(s) : s \in S\}$, where S is a set of cardinality $2^{\aleph_0}$ of topologically different HI m-dimensional continua such that every $X_{nm}(s)$ has exactly n continuous surjections onto itself and admits an atomic mapping p_s onto the pseudo-arc P. Moreover,*

- (i) *if $s \neq t$, then there is no continuous mapping of $X_{nm}(s)$ onto $X_{nm}(t)$;*
- (ii) *if $s \neq t$, then $X_{nm}(s)$ does not embed into $X_{nm}(t)$.*

Our construction is a modification of the ones given in [17] and [19] and applies a method of condensation of singularities. As before, we exploit an HI Cook continuum and we use a theorem of Wayne Lewis stating that for each $n \in \mathbb{N}$ there exists an embedding of the pseudo-arc P in the plane such that the restriction r of a period n rotation of the plane around $(0,0)$ to P is a homeomorphism of P onto P of period n [11]. To raise the dimension of the space obtained in [19], we construct our space in such a way that it contains a certain m-dimensional continuum Y_1. The new idea in the proof lies in Lemma 2.2 below. Roughly speaking, this lemma states that one can "replace" one point of a given continuum X by a special continuum in such a way that the resulting space can be mapped onto any given Waraszkiewicz spiral. In this way, we can "improve" a given continuum X so that a given continuum Y_1 does not map onto the whole X.

2. Preliminaries

A continuum Y is a *common model* for a family of continua $\mathcal{W}$, if every member of $\mathcal{W}$ is a continuous image of Y (we do not assume that $Y \in \mathcal{W}$).

By the *ray*, we will understand a space homeomorphic to the half-line $[0, +\infty)$. In [22], Z. Waraszkiewicz constructed a family $\mathcal{W}$ of planar continua without a common model. By a *Waraszkiewicz spiral*, we mean a member of this family. Every Waraszkiewicz spiral W is a compactification of the ray L with the remainder S homeomorphic to the circle. We have

(1) for every continuum A there exists a Waraszkiewicz spiral W such that A cannot be mapped onto W.

The *composant* of a point x in a continuum X is the union of all proper subcontinua of X containing x. If X is a non-degenerate HI continuum, then X has $2^{\aleph_0}$ different composants, which are pairwise disjoint and are connected F_σ-subsets of X, both dense and a boundary set in X (see [8, §48, VI]).

A mapping $f : X \to Y$ between continua is *confluent* (*weakly confluent*, respectively), if for each subcontinuum Q of Y each (some, respectively) component of $f^{-1}(Q)$ is mapped by f onto Q. As proved by Cook in [4], each mapping of a continuum onto an HI continuum is confluent.

A subcontinuum K of a continuum X is *terminal* if every subcontinuum of X which intersects both K and its complement must contain K. A continuous mapping from a continuum X onto Y is *atomic* if every fiber of f is a terminal subcontinuum of X.

Lemma 2.1 ([1], cf. [14] and [20])**.** *Let X and Y be two continua and $a \in X$. Then there exists a continuum $M(X, Y, a)$ and an atomic mapping $p : M(X, Y, a) \to X$ onto X such that $p^{-1}(a)$ is homeomorphic to Y and $p \mid p^{-1}(X \setminus \{a\}) : p^{-1}(X \setminus \{a\}) \to X \setminus \{a\}$ is a homeomorphism.*

Every continuum $M(X, Y, a)$ with the properties described in this lemma will be called a *pseudosuspension of Y over X at the point a* (cf. [14, 1.13]) and the mapping p will be called a *natural projection* from $M(X, Y, a)$ onto X.

Since $p^{-1}(a)$ is a terminal continuum in $M(X, Y, a)$, then (see [13, Proposition 11])

(2) if X and Y are HI, then so is $M(X, Y, a)$.

By the countable sum theorem (see [5], Theorem 1.5.3), we get

(3) $\dim M(X, Y, a) = \max\{\dim X, \dim Y\}$.

The following lemma, which was suggested by the referee of [16] (see Remark 5.2), was proved in detail in [10, Lemma 5.1].

Lemma 2.2. *Let X be any continuum, let a be any point of X, and let $W = L \cup S$ be a Waraszkiewicz spiral, being a compactification of the ray L with the remainder S homeomorphic to the circle. Let Y be a continuum satisfying the following condition:*

(4) *There exists a mapping $f : Y \to W$ of Y onto W and a sequence $M_1 \subset M_2 \subset \ldots$ of subcontinua of Y contained in $f^{-1}(L)$ such that the union $\bigcup_{i=1}^{\infty} M_i$ is dense in Y.*

Then there exists a pseudosuspension $M(X, Y, a)$ which admits a mapping $\tilde{f} : M(X, Y, a) \to W$ onto W.

Lemma 2.3. *For every Waraszkiewicz spiral W there exists an HI continuum Y of dimension ≤ 2 which satisfies condition* (4) *of Lemma 2.2. Moreover, Y can be chosen as a subcontinuum of any given HI continuum Z with $2 \leq \dim Z < \infty$.*

Proof: Let Z be any given HI continuum of finite dimension ≥ 2 and W be a Waraszkiewicz spiral. Let $Z' \subset Z$ be a 2-dimensional subcontinuum of Z. By a theorem of Mazurkiewicz [15], there exists a weakly confluent mapping of Z' onto the square $\mathbb{I}^2$. Since $W \subset \mathbb{I}^2$, there exists a subcontinuum $X \subset Z'$ which is mapped by f onto W. By Lemma 5.2 of [10], X contains a subcontinuum Y, which satisfies (4). □

Lemma 2.4 ([4]). *There exists a one-dimensional HI continuum H such that for any two different non-degenerate subcontinua of H, there is no mapping from one onto the other.*

Lemma 2.5 (see [4] and [21], cf. [19], Lemma 2.2). *If $f : P \to H$ is a continuous mapping of the pseudo-arc into a Cook continuum H, then f is a constant mapping.*

Lemma 2.6 (see Lemma 5.1 of [9] and its proof). *For any proper subcontinuum M of a 1-dimensional HI Cook continuum H and for every $m = 1, 2, \ldots, \infty$, there exists an m-dimensional HI continuum M_m such that every map from a subcontinuum of M into M_m is constant.*

3. Proofs

Proof of Theorem 1.1: For every $n \in \mathbb{N}$, a 1-dimensional continuum X_{n1} with the required properties was constructed in [17] and [19]. We shall modify this construction in order to raise the dimension of such a space. Fix $n \in \mathbb{N}$ and $m \in \{2, 3, \ldots, \infty\}$. Inductively, let us define a sequence $Y_1, Y_2, \ldots$ of HI continua and a sequence $W_1, W_2, \ldots$ of Waraszkiewicz spirals such that

(5) $\dim Y_1 = m$ and $\dim Y_l \leq 2$ for $l = 2, 3, \ldots$;

(6) condition (4) is satisfied for $Y = Y_l$ and $W = W_{l-1}$, for every $l = 1, 2, \ldots$;

(7) Y_l cannot be mapped onto W_l for $l = 1, 2, \ldots$.

Let Y_1 be any HI m-dimensional continuum. By (1), there exists a Waraszkiewicz spiral W_1 such that Y_1 cannot be mapped onto W_1. Suppose now that $Y_1, Y_2, \ldots, Y_{l-1}$ and $W_1, W_2, \ldots, W_{l-1}$ are already defined for some $l \geq 2$. For Y_l, we take a continuum Y of dimension ≤ 2 from Lemma 2.3, where we put $W = W_{l-1}$. Thus, Y_l can be mapped onto W_{l-1} and satisfies (4) for $W = W_{l-1}$. Again by condition (1), there exists a Waraszkiewicz spiral W_l such that Y_l does not map onto W_l.

By a theorem of Lewis [11], there exists a pseudo-arc P in the Euclidean plane and a homeomorphism $r : P \to P$ onto P of period n, which is the restriction of the rotation of the plane about the point $(0,0)$ through the angle $\frac{2\pi}{n}$. Note that $(0,0) \in P$, since the pseudo-arc P has the fixed point property (see [6]). Let $P_0 = \{(x_1, x_2) \in P : x_1 = \lambda \cos\alpha \text{ and } x_2 = \lambda \sin\alpha \text{ for some } 0 < \lambda < \infty \text{ and } 0 < \alpha < \frac{2\pi}{n}\}$, and $P_k = r^k(P_0)$ for $k = 0, 1, \ldots, n-1$. Let $\{b_1, b_2, \ldots\}$ be a countable dense subset of P_0 such that b_i and b_j are in the same composant of P if and only if $i = j$.

There exists a composant C in P which does not contain any b_i. In $C \cap P_0$, we choose a point c_0, a sequence Q_i of continua containing c_0 and converging to $\{c_0\}$, and a sequence $c_1, c_2, \ldots$ of points such that $c_i \in Q_i$ and $c_i \neq c_j$ for $i \neq j$.

Now, let $\{a_1, a_2, \ldots\}$ be a sequence such that $a_{2l-1} = c_l$ and $a_{2l} = b_l$ for $l = 1, 2, \ldots$. Put $B_0 = \bigcup\{b_l\}_{l=1}^{\infty}$, $C_0 = \bigcup\{c_l\}_{l=1}^{\infty}$, and $A_0 = B_0 \cup C_0$.

Then

(8) the set $B_0 \setminus F$, where F is any finite subset of B_0, is dense in P_0.

Let $B = \bigcup_{k=0}^{n-1} r^k(B_0)$, $C = \bigcup_{k=0}^{n-1} r^k(C_0)$, and $A = B \cup C$. Since a homeomorphic image of a composant of P is a composant of P, then

(9) every composant of P contains at most n points from B,

and

(10) C intersects at most n composants of P.

Finally, let $K_1, K_2, \ldots$ be a sequence of disjoint non-degenerate subcontinua of the hereditarily indecomposable Cook continuum H from Lemma 2.4. Thus,

(11) for every $j \neq i$, every continuous mapping from a subcontinuum of K_j into K_i is constant.

Let us define an inverse sequence $\{L_i, p_j^i, \{0\} \cup \mathbb{N}\}$ in the following way. Put $L_0 = P$. Let $L_1 = M(P, Y_1, a_1)$ be a pseudosuspension of an m-dimensional HI continuum Y_1 over P at $a_1 = c_1$ and let p_0^1 be the natural projection. Suppose that L_i and p_j^i are already defined for $j \le i \le s$, where $s \in \mathbb{N}$. If $s = 2l$ for $l \ge 1$, then let $L_s = L_{2l} = M(L_{s-1}, K_l, (p_0^{s-1})^{-1}(a_s))$ be a pseudosuspension of a

Cook continuum K_l over L_{s-1} at $(p_0^{s-1})^{-1}(a_s) = (p_0^{s-1})^{-1}(b_l)$. If $s = 2l-1$ for some $l \geq 2$, then $a_s = c_l$, and by (6), the conditions of Lemma 2.2 are satisfied for $W = W_{l-1}$ and $Y = Y_l$, so there exists a pseudosuspension $L_s = L_{2l-1} = M(L_{s-1}, Y_l, (p_0^{s-1})^{-1}(a_s))$ of Y_l over L_{s-1} at $(p_0^{s-1})^{-1}(a_s)$ which admits a mapping onto W_{l-1}. Now, let p_{s-1}^s be the natural projection and $p_j^s = p_j^{j+1} \circ \ldots \circ p_{s-1}^s$ for $j < s$. Let L be an inverse limit of this inverse sequence and let $p_s : L \to L_s$ be the projection. In particular, let $p = p_0$ be the projection of the limit space onto $L_0 = P$.

Let us note that for every $s \in \mathbb{N}$, L_s is the union of an open subset homeomorphic to $P \setminus \bigcup_{i=0}^{s} \{a_i\}$, of a copy of the m-dimensional continuum Y_1, and of finitely many copies of at most 2-dimensional continua from the family $\{K_1, Y_2, K_2, Y_3, \ldots\}$. Thus, by the countable sum theorem, $\dim L_s = m$ for every $s \in \mathbb{N}$. By the theorem on the dimension of the limit of an inverse sequence (see [5, Theorem 1.13.4]) and since L contains a topological copy of Y_1, it follows that

(12) the dimension of the limit space L is equal to m.

Since L_{2l-1} can be mapped onto W_{l-1} for $l \geq 2$, and L projects onto L_{2l-1}, then

(13) L can be mapped onto every W_l, for $l = 1, 2, \ldots$.

Since the projection $p_j^i : L_i \to L_j$ is a composition of finitely many atomic mappings, then it is atomic (see [13, (1.4)]). Hence, p is atomic (see [2, Theorem II]).

Let us note also that by (8) and from the definition of topology of the inverse limit,

(14) for every finite subset F of B_0, every open subset of $p^{-1}(P_0)$ contains some set $p^{-1}(b)$, where $b \in B_0 \setminus F$.

We can assume additionally that $L \subset P \times \mathbb{I}^\infty$, where $\mathbb{I} = [0, 1]$, and that p is the restriction of the projection of $P \times \mathbb{I}^\infty$ onto P. Moreover, we can assume that $p^{-1}(y) = (y, (0, 0, \ldots))$ for every $y \in P \setminus P_0$.

Indeed, assume that $L \subset \mathbb{I}^\infty$ and for $x, y \in \mathbb{R}^2$ let $\rho(x, y) = \min(\rho_e(x, y), 1)$, where ρ_e is the Euclidean metric in the plane. If $f(x) = (p(x), \rho(p(x), \mathbb{R}^2 \setminus P_0) \cdot x)$ for $x \in L$, then f is continuous and one-to-one; hence, it is a homeomorphism of L onto $f(L) \subset P \times \mathbb{I}^\infty$.

Thus, we can replace L by $f(L)$ and p by the restriction of the projection of $P \times \mathbb{I}^\infty$ onto P.

From the construction, it follows that for $l \in \mathbb{N}$, $p^{-1}(a_{2l-1})$ is homeomorphic to the continuum Y_l, $p^{-1}(a_{2l})$ is homeomorphic to the Cook continuum K_l, and $p \mid p^{-1}(P \setminus A_0) : p^{-1}(P \setminus A_0) \to (P \setminus A_0)$ is a homeomorphism.

Let $\overline{r}(y,t) = (r(y),t)$ for $(y,t) \in P \times \mathbb{I}^\infty$. Let $\overline{P_0}$ be the closure of P_0 in P. For $k = 0,1,\ldots,n-1$, let $\tilde{P}_k = \overline{r}^k(p^{-1}(\overline{P_0}))$ and $X_{nm} = \bigcup_{k=0}^{n-1} \tilde{P}_k$.

Note that X_{nm} admits a continuous mapping g onto L, being the identity on $\tilde{P}_0$, such that $g \mid \bigcup_{k=1}^{n-1} \tilde{P}_k$ is the restriction of the projection of $P \times \mathbb{I}^\infty$ onto P. Thus, by (13),

(15) X_{nm} can be mapped onto W_l for every $l = 1,2,\ldots$.

Let $\tilde{p} : X_{nm} \to P$ be the restriction of the projection of $P \times \mathbb{I}^\infty$ onto the first axis. The mapping $\tilde{r} = \overline{r} \mid X_{nm}$ is a period n homeomorphism of X_{nm} onto X_{nm}, such that

(16) $\tilde{p} \circ \tilde{r}^k = r^k \circ \tilde{p}$ for every $k = 0,1,\ldots,n-1$,

and

(17) $p(x) = \tilde{p}(x)$ for $x \in \tilde{P}_0$.

As in [17], we check that $\tilde{p}$ is atomic (cf. [17, Lemma 2.7]).

Note that

(18) $\tilde{p} \mid \tilde{p}^{-1}(P \setminus A) : \tilde{p}^{-1}(P \setminus A) \to P \setminus A$ is a homeomorphism.

Moreover, for $k \in \{0,1,\ldots,n-1\}$, we have that

(19) if $x = r^k(a_1) = r^k(c_1)$, then $\tilde{p}^{-1}(x)$ is a copy of the m-dimensional HI continuum Y_1;

(20) if $x = r^k(a_{2l-1}) = r^k(c_l)$ for some $l \geq 2$, then the set $Y_l^k = \tilde{p}^{-1}(x)$ is a copy of the HI continuum Y_l with $\dim Y_l \leq 2$;

and

(21) if $x = r^k(a_{2l}) = r^k(b_l)$ for some $l \geq 1$, then $K_l^k = \tilde{p}^{-1}(x)$ is a copy of the HI Cook continuum K_l.

From (14), it follows that

(22) if F is a finite subset of B, and $x(b) \in \tilde{p}^{-1}(b)$ for $b \in B \setminus F$, then the set $\{x(b) : b \in B \setminus F\}$ is dense in X_{nm}.

Since X_{nm} is the union of n closed subspaces which embed into L, then, by (12) and (19), $\dim X_{nm} = m$.

The space X_{nm} is HI, since it is the preimage of an HI continuum P under the atomic mapping $\tilde{p}$ with HI fibers.

Since $\tilde{p}$ is an atomic mapping, every composant of X_{nm} is equal to $\tilde{p}^{-1}(L)$ for some composant L of P (see [20, Lemma 2.8]). By (9),

(23) every composant of X_{nm} contains at most n copies of continua from the family $\mathcal{K} = \{K_l^k : l \in \mathbb{N},\ k = 0, 1, \ldots, n-1\} = \{\tilde{p}^{-1}(b) : b \in B\}$.

Let $f : X_{nm} \to X_{nm}$ be an arbitrary continuous mapping of X_{nm} onto X_{nm}. We will show that $f = \tilde{r}^k$ for some $k \in \{0, 1, \ldots, n-1\}$.

For every k and l, Y_l^k cannot be mapped onto W_l, while X_{nm} admits a mapping onto W_l by (15), so $f(Y_l^k) \neq X_{nm}$. Thus,

(24) for every $k \in \{0, 1, \ldots, n-1\}$ and $l \in \mathbb{N}$, $f(Y_l^k)$ is contained in one of the composants of X_{nm}.

Recall that Q_i is a sequence of continua in P containing c_i converging to c_0, with diameters tending to 0. In every L_s, the sequence of continua $\{(p_0^n)^{-1}(Q_i)\}_{i=1}^{\infty}$ converges to the point $(p_0^n)^{-1}(c_0)$, so in the inverse limit space L, the sequence of continua $\{p^{-1}(Q_i)\}_{i=1}^{\infty}$ converges to the point $p^{-1}(c_0)$.

It follows that for every $k \in \{0, 1, \ldots, n-1\}$, the sequence of continua $\{\tilde{p}^{-1}(r^k(Q_i))\}_{i=1}^{\infty}$ converges to the one-point set $\{\tilde{p}^{-1}(r^k(c_0))\}$, so the sequence of continua $\{f(\tilde{p}^{-1}(r^k(Q_i)))\}_{i=1}^{\infty}$ converges to the one-point set $\{f(\tilde{p}^{-1}(r^k(c_0)))\}$. Thus, for a fixed k, almost all continua $f(\tilde{p}^{-1}(r^k(Q_i)))$, where $i \in \mathbb{N}$, are contained in the same composant of X_{nm} and thus, almost all continua $f(Y_l^k)$, where $l \in \mathbb{N}$, are contained in the same composant of X_{nm}. From this and (24), it follows that the union of all sets $f(Y_l^k)$, for $k = 0, 1, \ldots, n-1$ and $l = 0, 1, \ldots$, is contained in finitely many composants of X_{nm}.

Thus, by (23), only finitely many continua from the family $\{K_l^k : l \in \mathbb{N},\ k = 0, 1, \ldots, n-1\}$ can intersect the image under f of the union $\bigcup\{Y_l^k : l \in \mathbb{N},\ k = 0, 1, \ldots, n-1\} = \tilde{p}^{-1}(C)$. It follows that

(25) there exists l_0 such that for $l \geq l_0$ and every k, $K_l^k \cap f(\tilde{p}^{-1}(C)) = \emptyset$.

Let $B' = \{r^k(b_l) : l \geq l_0,\ k = 0, 1, \ldots, n-1\}$. We will show that

(26) for every $b \in B'$, there is a nontrivial subcontinuum Q of $\tilde{p}^{-1}(b)$ and $t \in \{1, 2, \dots, n-1\}$ such that $f(\tilde{r}^t(x)) = x$ for every $x \in Q$.

Fix $b \in B'$. Then $\tilde{p}^{-1}(b)$ is equal to the Cook continuum K_l^k for some $l \geq l_0$ and $k \in \{1, 2, \dots, n-1\}$. Since X_{nm} is HI, then f is confluent and thus, there exists a proper subcontinuum T of X_{nm} such that $f(T) = K_l^k$. Then T is disjoint with $\tilde{p}^{-1}(C)$ by (25) and is contained in some composant of X_{nm}. From this and (23), it follows that either T is contained in $\tilde{p}^{-1}(b')$ for some $b' \in B$, or T is the union of a non-empty subset of $\tilde{p}^{-1}(P \setminus A)$ and of finitely many continua $\tilde{p}^{-1}(b(1)), \dots, \tilde{p}^{-1}(b(r))$, where $b(i) \in B$. In the second case, there exists i such that $f(\tilde{p}^{-1}(b(i)))$ is a nondegenerate subcontinuum of $\tilde{p}^{-1}(b)$. For otherwise, the set $K_l^k \setminus \bigcup_{i=1}^r f(\tilde{p}^{-1}(b(i)))$ would contain a non-degenerate subcontinuum, which is the image of a subcontinuum $T' \subset \tilde{p}^{-1}(P \setminus A) \cap T$. However, by (18), each non-degenerate subcontinuum T' of X_{nm} contained in $\tilde{p}^{-1}(P \setminus A)$ is homeomorphic to P; therefore, by Lemma 2.5, T' admits only constant mappings into the Cook continuum K_l^k, which gives a contradiction.

Therefore, in both cases, there exists $b' \in B$ such that $f(T \cap \tilde{p}^{-1}(b'))$ is a nondegenerate subcontinuum of K_l^k. By (11), $\tilde{p}^{-1}(b')$ must be equal to K_l^t for some t, and, for $Q = T \cap \tilde{p}^{-1}(b)$, condition (26) is satisfied, because Q and $f(Q)$ must be topological copies of the same nondegenerate subcontinuum of the Cook continuum K_l. By choosing a point $x(b) \in \tilde{p}^{-1}(b) \cap K$, we get the result that

(27) for every $b \in B'$, there is a point $x(b) \in \tilde{p}^{-1}(b)$ such that $f(\tilde{r}^t(x(b))) = x(b)$ for some $t \in \{1, 2, \dots, n-1\}$.

By (22), the set $Y = \{x(b) : b \in B'\}$ is dense in X_{nm}.

The remaining part of the proof repeats the arguments from the proof of Theorem 3.1 in [19]. First, let us note that

(28) for every $x \in X_{nm}$, there is $t \in \{0, 1, \dots, n-1\}$ such that $f(\tilde{r}^t(x)) = x$.

Indeed, one can find a sequence $\{x(b_j)\}_{j=1}^{\infty}$, where $b_j \in B'$, converging to x, such that for some $t \in \{0, 1, \dots, n-1\}$, $f(\tilde{r}^t(x(b_j))) = x(b_j)$ for every j. Thus, $f(\tilde{r}^t(x(b_j))) \to f(\tilde{r}^t(x))$, so $f(\tilde{r}^t(x)) = x$.

For every $x \neq \tilde{p}^{-1}((0,0))$, the set $Y(x) = \bigcup_{k=1}^{n-1} \tilde{r}^k(x)$ has n elements and every point of $Y(x)$ is the image of a point in $Y(x)$;

hence, $f(Y(x)) = Y(x)$ and $f \mid Y(x) \to Y(x)$ is one-to-one. In particular, for every $x \neq \tilde{p}^{-1}((0,0))$, there exists $k \in \{0,1,\ldots,n-1\}$ such that $f(x) = \tilde{r}^k(x)$.

For $k \in \{0,1,\ldots,n-1\}$, let $X(k) = \{x \in X_{nm} : f(x) = \tilde{r}^k(x)\}$. It is easy to see that every $X(k)$ is closed in X_{nm} and $X(k) \cap X(l) = \{\tilde{p}^{-1}((0,0))\}$ for $k \neq l$, $k,l \in \{0,1,\ldots,n-1\}$. It follows that every $X(k)$ is a continuum. Indeed, if $X(k)$ were the union of two disjoint closed subsets F_1 and F_2 with $\tilde{p}^{-1}((0,0)) \in F_2$, then X_{nm} would be the union of two sets F_1 and $F_2 \cup \bigcup_{l \neq k} X(l)$, disjoint and closed in X_{nm}. Since X_{nm} is hereditarily indecomposable, then $X_{nm} = X(k)$ for some $k \in \{0,1,\ldots,n-1\}$, and thus, $f = \tilde{r}^k$ and f is a homeomorphism.

This ends the proof that the set of all continuous mappings from X_{nm} onto X_{nm} is equal to the set $\{\tilde{r}^0, \tilde{r}^1, \ldots, \tilde{r}^{n-1}\}$ and forms the cyclic group of order n. $\square$

Proof of Theorem 1.2: Let S be a set of cardinality $2^{\aleph_0}$, H be the 1-dimensional HI Cook continuum (see Lemma 2.4), and K be a proper non-degenerate subcontinuum of H. For every $s \in S$, let us choose a sequence $\{K_1(s), K_2(s), \ldots\}$ of non-degenerate subcontinua of K in such a way that $K_i(s) \cap K_j(t) = \emptyset$ if $s \neq t$ or $i \neq j$. Such a family $\{K_i(s) : i \in \mathbb{N}, s \in S\}$ exists, because K has $2^{\aleph_0}$ composants which are pairwise disjoint. Thus,

(29) every mapping from a subcontinuum of $K_i(s)$ into $K_i(t)$ is constant.

If, in the proof of Theorem 1.1, we replace in the construction of $X_{nm}$the sequence $K_1, K_2, \ldots$ by the sequence $\{K_1(s), K_2(s), \ldots\}$, then we obtain an HI continuum $X_{nm}(s)$ with exactly n continuous surjections onto itself, which admits an atomic mapping $\tilde{p}_s : X_{nm}(s) \to P$ onto P. As we will prove below, the family $\{X_{nm}(s) : s \in S\}$ satisfies condition (i) of Theorem 1.2. In order to obtain such a family also satisfying condition (ii), we assume additionally that Y_1 is a space M_m constructed in Lemma 2.6 for $M = K$, and Y_l for $l \geq 2$ is a space Y of dimension ≤ 2 constructed in Lemma 2.3 for $W = W_{l-1}$, which is contained in the 2-dimensional continuum M_2 from Lemma 2.6 (where we put $M = K$).

Thus,

(30) every mapping from a subcontinuum of K into Y_l, for $l = 1, 2, \ldots$, is constant.

Let us show condition (i). From the construction, it follows that $X_{nm}(s)$ is the union of the set $\tilde{p}_s^{-1}(P \setminus A)$ homeomorphic to a subset of P, of continua from the family $\mathcal{K}(s) = \{\tilde{p}_s^{-1}(b) : b \in B\}$, and of continua from the family $\mathcal{Y} = \{\tilde{p}_s^{-1}(c) : c \in C\}$. Note that the family $\mathcal{K}(s)$ contains exactly n copies of every continuum $K_i(s)$, for every $i \in \mathbb{N}$, and the family $\mathcal{Y}$ contains n copies of every continuum Y_l, for $l \in \mathbb{N}$.

Let $f : X_{nm}(s) \to X_{nm}(t)$ be an arbitrary continuous surjection. Suppose that $t \neq s$. Similar to the proof of Theorem 1.1, one shows that the set $f(\bigcup \mathcal{Y})$ intersects only finitely many composants of $X_{nm}(t)$, so it intersects only finitely many continua from the family $\mathcal{K}(t)$. Hence, there exists $\tilde{p}_t^{-1}(b) \in \mathcal{K}(t)$, being a copy of some $K_i(t)$, which is disjoint with $f(\bigcup \mathcal{Y})$. Since f is confluent, there exists a nontrivial subcontinuum T of $X_{nm}(s)$, disjoint with $\bigcup \mathcal{Y} = \tilde{p}_s^{-1}(C)$, such that $f(T) = \tilde{p}_t^{-1}(b)$. Since T is a proper subcontinuum of $X_{nm}(s)$, it is contained in some composant of $X_{nm}(s)$. It follows that either T is contained in some $\tilde{p}_s^{-1}(b')$ for some $b' \in B$, or T is the union of a non-empty subset of $\tilde{p}_s^{-1}(P \setminus A)$ and of finitely many continua $\tilde{p}_s^{-1}(b(1)), \ldots, \tilde{p}_s^{-1}(b(r))$, where $b(i) \in B$. In the second case, there exists i such that $f(\tilde{p}_s^{-1}(b(i)))$ is a nondegenerate subcontinuum of $\tilde{p}_t^{-1}(b)$. For otherwise, the set $\tilde{p}_t^{-1}(b) \setminus \bigcup_{i=1}^{r} f(\tilde{p}_s^{-1}(b(i)))$ would contain a nondegenerate subcontinuum, which is the image of a subcontinuum $T' \subset \tilde{p}_s^{-1}(P \setminus A) \cap T$. However, each non-degenerate subcontinuum of $X_{nm}(s)$ contained in $\tilde{p}_s^{-1}(P \setminus A)$ is homeomorphic to P; therefore, by Lemma 2.5, it admits only constant mappings into the Cook continuum $\tilde{p}_t^{-1}(b)$, which gives a contradiction.

Therefore, in both cases, there exists $b' \in B$ such that $f(T \cap \tilde{p}_s^{-1}(b'))$ is a nondegenerate subcontinuum of $\tilde{p}_t^{-1}(b)$. But $\tilde{p}_s^{-1}(b')$ and $\tilde{p}_t^{-1}(b)$ are homeomorphic to two disjoint subcontinua of the Cook continuum H, which yields a contradiction. Thus, $s = t$.

To prove (ii), suppose that $s \neq t$, and $h : X_{nm}(s) \to X_{nm}(t)$ is an embedding. Let $K_1(s)'$ be a copy of $K_1(s)$ in $X_{nm}(s)$. Then $h(K_1(s)')$ is a copy of $K_1(s)$ in $X_{nm}(t)$, so it does not embed in $\tilde{p}_t^{-1}(P \setminus A)$ by Lemma 2.5. Thus, $h(K_1(s)')$ intersects some $\tilde{p}_t^{-1}(a_i)$. By (29) and (30), $h(K_1(s)')$ is not contained in $\tilde{p}_t^{-1}(a_i)$ for any

$i = 1, 2, \ldots$; hence, $\tilde{p}_t^{-1}(a_i) \subset h(K_1(s)')$ for some i. However, if $i = 2l$, then $\tilde{p}_t^{-1}(a_i)$ is a copy of $K_l(t)$, which gives a contradiction by (29). If $i = 2l - 1$, then $\tilde{p}_t^{-1}(a_i)$ is a copy of a continuum Y_l. Since $h : K_1(s)' \to h(K_1(s)')$ is a homeomorphism, then $Z = h^{-1}(\tilde{p}_t^{-1}(a_i))$ is a subcontinuum of $K_1(s)'$ such that $h(Z) = \tilde{p}_t^{-1}(a_i)$, which contradicts (30). This shows that $s = t$. □

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