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ON SOME EQUIVALENT AND NON-EQUIVALENT NOTIONS OF HOMOGENEITY

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Abstract. We discuss various types of homogeneity and their relationships. Included are homogeneity, local homogeneity, strong local homogeneity, uniform local homogeneity, even homogeneity, and micro-homogeneity. In addition, the relation of Effros' Theorem to these different types of homogeneity is presented.

1. Introduction

Many variations of homogeneity have been defined since Wacław Sierpinski formally introduced homogeneity to the world of topology around 1920. A topological space X is *homogeneous* provided that for any two points $p, q \in X$ there exists a homeomorphism h on X such that $h(p) = q$.

The purpose of this paper is to look at various types of homogeneity and their connections. The paper is divided into several sections. In section 2, we discuss homogeneity and two different types of local homogeneity and their relationships with each other. Section 3 focuses on even homogeneity and Effros' Theorem, while

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section 4 is a discussion of some new results involving local homogeneity and even homogeneity. In the last section, connections between uniform local homogeneity, micro-homogeneity, and even homogeneity are highlighted. We assume throughout this paper that (X, T) is a topological space with more than one point. We denote the set of all homeomorphisms on X by $H(X)$.

2. Local homogeneity

Although there are several definitions of local homogeneity, we will be using the definition that Ali Ahmad Fora [7] gave in 2000: (X, T) is *locally homogeneous* (LH) provided that for any $p \in X$ there exists an open set O containing p such that for each $q \in O$ there is an $h \in H(X)$ with $h(p) = q$.

The following definition of strong local homogeneity was developed by Lester R. Ford, Jr. [8] in 1954. A topological space (X, T) is *strongly locally homogeneous* (SLH) provided for any $p \in X$ and any $U \in T$ containing p there exists an open set O with $p \in O \subseteq U$ such that if $q \in O$ there is an $h \in H(X)$ with $h(p) = q$ and $h|_{X \setminus O} = e|_{X \setminus O}$ where e is the identity map.

The proofs of the following two theorems can be easily seen from the definitions of homogeneity, local homogeneity, and strong local homogeneity.

Theorem 2.1 ([7])**.** *If X is homogeneous, then X is locally homogeneous.*

Note that the converse of Theorem 2.1 is false as shown by the following example.

Example 2.2. Let $X = (0, 1) \cup \{2\}$ with the subspace topology inherited from the usual topology on the reals. Then X is not homogeneous but is locally homogeneous.

Theorem 2.3 ([7])**.** *If X is SLH, then X is LH.*

The space X above in Example 2.2 is SLH so that it also follows that strong local homogeneity does not imply homogeneity. Consider this next example which illustrates that the converse of Theorem 2.3 is not true, as well as shows that strong local homogeneity does not follow from homogeneity.

Example 2.4 ([8])**.** Let X be the product of the unit circle S^1 with the Cantor set C where the topologies on S^1 and C are the usual topologies inherited from R^2 and R, respectively. Then X is homogeneous and hence LH, but not SLH.

Although we have seen that strong local homogeneity does not imply homogeneity and that homogeneity does not imply strong local homogeneity, by adding in connectedness, Peter Fletcher [6] arrived at the following theorem whose converse is also shown to be false by Example 2.4.

Theorem 2.5 ([6])**.** *Let X be SLH and connected then X is homogeneous.*

3. Even homogeneity

A space (X, T) is *evenly homogeneous w.r.t.* $(H(X), T*)$ [11] (EH w.r.t. $T*$) provided that for all $O \in T*$ containing the identity e there exists an open cover $\mathcal{V}$ such that if $x, y \in V \in \mathcal{V}$, there is an $h \in O$ such that $h(x) = y$. Let us recall several results about even homogeneity.

Theorem 3.1 ([11])**.** *Let Λ be an index set of arbitrary cardinality. For all $\alpha \in \Lambda$, let X_α be a topological space, and let G_α be a transitive subgroup of $H(X_\alpha)$. For each $\alpha \in \Lambda$, let T_α be a topology for G_α such that X_α is evenly homogeneous w.r.t. (G_α, T_α). Set $X = \prod_{\alpha\in\Lambda} X_\alpha$ and $G = \prod_{\alpha\in\Lambda} G_\alpha$. Let T and T_P be the product topologies on X and G, respectively. Then X is evenly homogeneous w.r.t. (G, T_P).*

Theorem 3.2 ([11])**.** *If X is EH w.r.t. $(H(X), T*)$ and $T_t \neq T' \subseteq T*$, then X is EH w.r.t. $(H(X), T')$, where T_t is the trivial topology on $H(X)$.*

In 1965, algebraist Edward G. Effros [5] published a paper involving homogeneous spaces. This paper has been extensively used in the study of continuum theory after Gerald S. Ungar [13] and Charles L. Hagopian [9] discovered its importance to topology. The next theorem uses the ϵ-push property which evolved from Effros' Theorem. Theorem 3.4 is the widely used corollary of Effros' Theorem which was proven in 1983 by David P. Bellamy.

Theorem 3.3 ([9])**.** *Let (X, ρ) be a non-degenerate, homogeneous, metric continuum. Let $\epsilon > 0$ be given and let $x \in X$. Then there exists an open set, O, in X such that $x \in O$ and if $y, z \in O$, there exists $h \in H(X)$ with $h(y) = z$ and $\rho(v, h(v)) < \epsilon$ for all $v \in X$.*

Theorem 3.4 ([2])**.** *Let (X, T) be a non-degenerate, compact, homogeneous, metric space. Then for all $x \in X$, the evaluation map, $E_x : H(X) \to X$ defined by $E_x(f) = f(x)$, is an open map when $H(X)$ is given the compact-open topology.*

Kathryn F. Porter, in 1988, used even homogeneity to generalize Effros' Theorem to include non-metrizable spaces, and in the process, gave the following definition. A topology T for $H(X)$ is *RMC* provided that the map $m_g : H(X) \to H(X)$ defined by $m_g(h) = h \circ g$ is continuous for all $g \in H(X)$. Porter's generalization is stated in Theorem 3.5 below.

Theorem 3.5 ([11])**.** *Let (X, T) be a non-degenerate topological space which is homogeneous. Let $T*$ be an RMC topology for $H(X)$. If X is EH w.r.t. $(H(X), T*)$, then for each $x \in X$, the evaluation map, $E_x : H(X) \to X$ defined by $E_x(f) = f(x)$, is an open map.*

Since the compact-open topology is RMC and X is EH w.r.t. T_{co} when X is compact, homogeneous, and metrizable, Theorem 3.4 is actually a corollary of Theorem 3.5.

4. Local homogeneity and even homogeneity

The following new theorem shows that local homogeneity is equivalent to the weakest form of even homogeneity.

Theorem 4.1. *X is LH iff X is EH w.r.t. T_t.*

Proof: ($\Rightarrow$) Assume X is LH. $H(X)$ is the only open set in T_t containing e. For each $x \in X$, there is, by LH, an open set O_x containing x such that if $y \in O_x$, there exists $h \in H(X)$ such that $h(x) = y$. Set $\mathcal{V} = \{O_x | x \in X\}$; then $\mathcal{V}$ is the open cover needed for EH.

($\Leftarrow$) Assume X is EH w.r.t. T_t. Let $p \in X$. $H(X)$ contains e so there is an open cover $\mathcal{V}$ such that if $x, y \in V \in \mathcal{V}$, there exists an $h \in H(X)$ with $h(x) = y$. Since there is a $V \in \mathcal{V}$ with $p \in V$, X is LH. $\square$

The next theorem is a new result which connects strong local homogeneity to even homogeneity w.r.t. the closed-open topology.

Theorem 4.2. *If X is SLH, then X is EH w.r.t. T_{clo}.*

Proof: Assume X is SLH. Let (C, O) be a subbasis element of the closed-open topology on $H(X)$ containing the identity e. Thus, $C \subseteq O$. If $x \in C$, then $x \in O$ so there exists an SLH open set, U_x such that $x \in U_x \subseteq O$. If $x \in X \setminus C$, there exists an SLH open set V_x such that $x \in V_x \subseteq X \setminus C$. Let $\mathcal{V} = \{U_x | x \in C\} \cup \{V_x | x \in X \setminus C\}$. Then if $p, q \in U_x$ for some $x \in C$, there exist $h_1, h_2 \in H(X)$ with $h_1(x) = p, h_2(x) = q$, and $h_i|_{X-U_x} = e|_{X-U_x}$ for $i = 1, 2$. So $h_2 \circ h_1^{-1}(p) = q$ and $h_2 \circ h_1^{-1} \in (C, O)$. The same argument can be used in the case that $p, q \in V_x$. Hence, X is EH w.r.t. T_{clo}. □

In 1979, R. A. McCoy [10] defined the open-cover topology for function spaces: Let (X, T) be a topological space and let $\Gamma(X)$ be the collection of all open covers of X. For each $\mathcal{V} \in \Gamma(X)$ and every $f \in H(X)$, we define the set $\mathcal{V}(f) = \{g \in H(X) | \forall x \in X, \exists V \in \mathcal{V}\ s.\ t.\ (f(x), g(x)) \in V \times V\}$. The collection $\mathcal{S} = \{\mathcal{V}(f) | \mathcal{V} \in \Gamma(X),$ and $f \in H(X)\}$ is a subbasis for a topology on $H(X)$ called the *open-cover topology* which is denoted by T_{oc}. In this next theorem, we establish a relationship between strong local homogeneity and even homogeneity w.r.t. the open-cover topology.

Theorem 4.3. *If X is SLH, then X is EH w.r.t. T_{oc}.*

Proof: Assume X is SLH. Let $\mathcal{V} \in \Gamma(X)$; then $e \in \mathcal{V}(e)$. Now $\forall x \in X$ there is some $V_x \in \mathcal{V}$ such that $x \in V_x$. Since X is SLH, for each $x \in X$, there is an SLH open neighborhood U_x of x such that $x \in U_x \subseteq V_x$. Hence, $\mathcal{U} = \{U_x | x \in X\}$ is an open cover of X. If $p, q \in U_x$ for some x, there exists $h_1, h_2 \in H(X)$ with $h_1(x) = p$, $h_2(x) = q$, and $h_1|_{X \setminus U_x} = e|_{X \setminus U_x} = h_2|_{X \setminus U_x}$. Then $h = h_2 \circ h_2^{-1}(p) = q$ and $h|_{X \setminus U_x} = e|_{X \setminus U_x}$. It is then easy to show that $h \in \mathcal{V}(e)$. Therefore, X is EH w.r.t. T_{oc}. □

Theorem 4.3 would be a corollary of Theorem 4.2 under the condition that X is pseudocompact since then $T_{oc} \subseteq T_{clo}$ on $H(X)$.

When we add some separation axiom restrictions to X, we arrive at the new results below.

Theorem 4.4. *If X is SLH and T_1, then X is EH w.r.t. T_p where T_p is the point-open topology.*

Proof: Assume X is SLH and T_1. Then X is EH w.r.t. T_{clo} and $T_p \subseteq T_{clo}$ since points are closed; hence, by Theorem 3.2, X is EH w.r.t. T_p. □

Theorem 4.5. *If X is SLH and T_2, then X is EH w.r.t. T_{co} where T_{co} is the compact-open topology.*

Proof: Let X be SLH and T_2. Then every compact set in X is closed so $T_{co} \subseteq T_{clo}$. Thus, by Theorem 3.2, X is EH w.r.t. T_{co}. □

Recall Example 2.4. The space $X = S^1 \times C$ is not SLH, but both C and S^1 are SLH and T_2, so both are EH w.r.t. T_{co}. Thus, by Theorem 3.1, their product X with the product topology is EH w.r.t. (H, T_P) where $H = H(S^1) \times H(C)$ and T_P is the product topology on H. If T_s is the subspace topology on H inherited from $(H(X), T_{co}$, then, as Porter has shown [11], $T_s \subseteq T_P$ on H. Therefore, by Theorem 3.2, X is EH w.r.t. $(H(X), T_{co})$. Note that in this example, X is compact; hence, $T_{co} = T_{clo} = T_{oc}$ on $H(X)$ so that Example 2.4 illustrates that the converse of each of theorems 4.2, 4.3, and 4.5 is false. This example also shows that the converse of Theorem 4.4 is false since the point-open topology $T_p \subseteq T_{co}$ so X is EH w.r.t. $(H(X), T_p)$.

5. Uniform local homogeneity, micro-homogeneity, and even homogeneity

In 1967, Ungar wrote a paper [13] relating strong local homogeneity to a new homogeneity property: A completely regular topological space (X, T) is *uniformly locally homogeneous w.r.t. to the uniformity $\mathcal{U}$ (ULH w.r.t. $\mathcal{U}$)* provided $T = T_{\mathcal{U}}$ (uniform top.) and for each $x \in X$ and $U \in \mathcal{U}$, there exists an open neighborhood O of x such that if $y \in O$, there exists $g \in H(X)$ such that $g(x) = y$ and $\text{graph}(g) \subseteq U$.

Theorem 5.1 ([13])**.** *If X is SLH and uniformizable, then X is ULH.*

Example 2.4 shows that the converse of Theorem 5.1 is false. Both S^1 and C are ULH and, as Ungar proved [13], their product is ULH but not SLH.

Theorem 5.2 ([11])**.** *Let $(X, \mathcal{U})$ be a uniform space. Then X is ULH w.r.t. $\mathcal{U}$ if and only if X is EH w.r.t. $T_{\mathcal{U}*}$.*

Corollary 5.3 ([11]). *If X is SLH and completely regular, then X is EH w.r.t. $T_{\mathcal{U}*}$.*

Frederic D. Ancel [1], in 1987, defined a topological space (X, T) to be *micro-homogeneous w.r.t.* $(H(X), T*)$ *(MH w.r.t. $T*$)* provided that

(1) $T*$ is an admissible topology for $H(X)$; i.e., the evaluation map, $E : (H(X) \times X) \to X$, $E(h, x) = h(x)$ is jointly continuous;

(2) $(H(X), T*)$ is a topological group; i.e., the multiplication map and inverse map on $H(X)$ are continuous; and

(3) for all open neighborhoods O of the identity, e, each $x \in X$ has an open neighborhood U such that if $y \in U$, then there is an $h \in O$ such that $h(x) = y$.

We shall refer to the third property in the above definition as *Ancel's Property.*

Theorem 5.4. *If X is EH w.r.t. $T*$, then $H(X)$ has Ancel's Property (AP).*

Proof: Assume X is EH w.r.t. $T*$. Let O be an open neighborhood of e and let $x \in X$. By EH, there exists an open cover $\mathcal{V}$ such that if $p, q \in V \in \mathcal{V}$, there exists $g \in O$ with $g(p) = q$. Now there has to be some $\hat{V} \in \mathcal{V}$ with $x \in \hat{V}$. If $y \in \hat{V}$, then there exists $f \in O$ with $f(x) = y$. Therefore, $H(X)$ has AP. □

To see that the converse of Theorem 5.4 is false, consider $X = Z$ with the cofinite topology. Then $H(X)$ has AP, but X is not EH w.r.t. T_{co}.

Theorem 5.5. *If X is MH w.r.t. $T*$, then X is EH w.r.t. $T*$.*

Proof: Assume X is MH w.r.t. $T*$. Let $O \in T*$ such that $e \in O$. Since $(H(X), T*)$ is a topological group, there exists $W = W^{-1} \in T*$ such that $W \circ W \subseteq O$. By AP, for each $x \in X$, there is an open set V_x such that $x \in V_x$ and if $y \in V_x$, then there exists $j \in O$ such that $j(x) = y$. Set $\mathcal{V} = \{V_x | x \in X\}$. If $p, q \in V_x$ for some $x \in X$, there exists $h_1, h_2 \in W$ such that $h_1(x) = p$ and $h_2(x) = q$. Then $h = h_2 \circ h_1^{-1}(p) = q$ and $h \in W \circ W \subseteq O$. Therefore, X is EH w.r.t. $T*$. □

We can show that the converse of Theorem 5.5 is not true. To this end, let $X = Q$ with the subspace topology inherited from the reals with the usual topology. Give $H(Q)$ the compact-open topology. Q is SLH and uniformizable and thus EH w.r.t. T_{co}. However, $(H(Q), T_{co})$ is not an admissible topological group because T_{co} is strictly coarser than T_{clo} on $H(Q)$. So by Theorem 5.6 below, Q is not MH w.r.t. T_{co}.

Theorem 5.6 ([4]). *Any admissible group topology on $H(Q)$ is finer than the closed-open topology.*

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