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PARACOMPACTNESS OF BOX PRODUCTS AND THEIR SUBSPACES

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Abstract. Interested in $\Box(\omega+1)^\omega$, we explore the related space $\nabla(\omega+1)^\omega$. We find a number of discrete subspaces of $\nabla(\omega+1)^\omega$ — for example, the set of functions which are non-decreasing on their finite parts — and use this to find many paracompact subspaces of both $\nabla(\omega+1)^\omega$ and $\Box(\omega+1)^\omega$. We also explore some other questions relating to the paracompactness of box products of countably many compact first countable spaces.

1. Background

Definition 1. $\Box_{i\in I}X_i$ is the topology τ on $\Pi_{i\in I}X_i$ in which $u\in\tau$ iff each $\pi_i[u]$ open in X_I. I.e., a base for τ consists of all $\Pi_{i\in I}u_i$ where each u_i open in X_i.

If all $X_i = X$, we write $\Box X^I$.

In a combinatorial tour de force published in 1996 [11], L. Brian Lawrence proved that $\Box(\omega+1)^{\omega_1}$ fails to be normal, much less paracompact. So, as far as normality and paracompactness go, only the countable index case is interesting.

Conjecture 1. (a) $\Box(\omega+1)^\omega$ is paracompact (normal).

(b) $\Box_{n<\omega}X_n$ is paracompact (normal) if each X_n is compact metrizable.

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(c) $\Box_{n<\omega}X_n$ is paracompact (normal) if each X_n is compact first countable.

Clearly (c) $\Rightarrow$ (b) $\Rightarrow$ (a). When we refer to Conjecture 1, we will mean the strong form, with "paracompact," not "normal."

The first major result in this area was Mary Ellen Rudin's 1972 result that, under CH, the box product of countably many compact metrizable spaces is paracompact [14]. Rudin required only that the metrizable spaces were locally compact and σ-compact. In 1975, Eric K. van Douwen showed that $\mathbb{P}\times\Box(\omega+1)^\omega$ is not normal, thus showing the necessity of some kind of compactness requirement [2]. Over the next two decades, a number of results generalized these results in various directions: tweaking the kind of spaces, tweaking the set theoretic context, or both. Positive results tended to have the form "... is paracompact" and negative results the form "... is not normal." Negative results tended to be ZFC results; positive results – at least the ones directly relevant to the conjecture – were consistency results. In particular, $\mathfrak{d}=\mathfrak{c}$ is sufficient to prove conjecture 1(c) (Judy Roitman [13]) and $\mathfrak{b}=\mathfrak{d}$ is sufficient to prove conjecture 1(b) (van Douwen [4]); Scott W. Williams [19] had earlier proved that $\mathfrak{b}=\mathfrak{d}$ is sufficient to prove conjecture 1(a).

Other results on box products have also appeared, focused on other properties. Sometimes they were part of a more general study, e.g., William G. Fleissner and Adrienne M. Stanley's 2001 paper [5] in which the authors show that the box product of scattered spaces of height 1 is a D space. More rarely the results squarely focused on box products, e.g., Louis Wingers' 1995 [21] exploration of the effect of the Hurewicz property on whether $X\times\Box(\omega+1)^\omega$ is Baire, and Wingers' 1994 [20] proof that the countable box product of σ-compact spaces is pseudonormal and ω_1-collectionwise Hausdorff.

This paper asks what we can do with no set theoretic axioms. Our main results are that various subspaces of $\Box(\omega+1)^\omega$ are paracompact. We also look at subspaces and weaker topologies of $\Box_{n<\omega}X_n$ where each X_n is compact first countable, or compact metrizable.

The superscript * (as in $=^*, \leq^*, \subset^*$ etc.) means *mod finite.*

2. The basics

2.1 $\Box,\nabla$ and their bases

We assume all X_n are first countable, and for $z \in X_n$ let $\{u_{z,j} : j < \omega\}$ denote a countable base for z so that each cl $u_{z,j+1} \subset u_{z,j}$.

If X_n is metrizable with metric d, $X_n \neq \omega + 1$, we require that $u_{z,j} = \{w : d(z,w) < 2^j\}$.

If $X_n = \omega + 1$, we require that $u(z,j) = \begin{cases} \{z\} & \text{if } z < \omega \\ (j, \omega] & \text{if } z = \omega. \end{cases}$

Suppose $x \in \Box_{n<\omega} X_n$. We define $N(x,f) = \Pi_{n<\omega} u_{x(n),f(n)}$. $\{N(x,f) : x \in \Box_{n<\omega} X_n, f \in \omega^\omega\}$ is a base for $\Box_{n<\omega} X_n$.

For $x \in \Box_{n<\omega} X_n$, we write $\bar{x} = \{y \in \Box_{n<\omega} X_n : \forall^\infty n\ y(n) = x(n)\}$. For $S \subset \Box_{n<\omega} X_n$, we write $\bar{S} = \{\bar{y} : y \in S\}$, and if $N = N(x,f)$, we write $\bar{N} = N(\bar{x}, f)$. $\nabla_{n<\omega} X_n$ is the quotient topology under the relation $x \equiv y$ iff $\bar{x} = \bar{y}$.

In $\nabla_{n<\omega} X_n$, we define $N^*(\bar{x}, f) = \bigcap_{n<\omega} N(\bar{x}, n \cdot f)$. Each $N^*(\bar{x}, f)$ is clopen. So $\nabla_{n<\omega} X_n$ is 0-dimensional.

2.2 Reduction to ∇ and translations back to $\Box$

The following two theorems simplify the problem greatly.

Theorem 1 (Kunen [8]). *If each X_n is compact first countable, then $\Box_{n<\omega} X_n$ is paracompact iff $\nabla_{n<\omega} X_n$ is paracompact.*

Theorem 2 (Kunen [8]). *If each X_n is compact first countable and $\nabla_{n<\omega} X_n$ is paracompact, then it is ultraparacompact; i.e., every open cover has a pairwise disjoint covering refinement.*

Since we will work exclusively with the ∇-product, here we make explicit how to translate back into the $\Box$-product.

Lemma 1. (a) *If $x \in N(y,f)$, then $\bar{x} \in N(\bar{y}, f)$.*

(b) *If $N(x,f) \cap N(y,g) \neq \emptyset$, then $N(\bar{x}, f) \cap N(\bar{y}, g) \neq \emptyset$.*

(c) *$N(\bar{x}, f) \cap N(\bar{y}, g) \neq \emptyset$ iff $\{n : u(x(n), f(n)) \cap u(y(n), g(n)) = \emptyset\}$ is finite.*

From this obvious lemma, we have another obvious lemma.

Lemma 2. *Suppose $S \subset \Box_{n<\omega} X_n$ and $x \neq^* y$ for all $x, y \in S$.*

(a) *If $\bar{S}$ is discrete, then S is discrete.*

(b) *If $p \in \Box_{n<\omega} < X)n \setminus S, p \neq^* x$ for all $x \in S$, and $\bar{S}$ is closed discrete in $\nabla_{n<\omega} X_n \setminus \{\bar{p}\}$, then S is closed discrete in $\Box_{n<\omega} X_n \setminus \{p\}$.*

Finally, we have a lemma which is a direct corollary of K. Kunen's proof of Theorem 1.

Lemma 3. *Suppose $S \subset \square_{n<\omega} X_n$ where if $x, y \in S$, then $x \neq^* y$, and each X_n is compact first countable. If $\tau_{\mathcal{F}}$ is the topology on $\square_{n<\omega} X_n$ generated by $\{N(x,f) : f \in \mathcal{F}\}$ and $\bar{\tau}_{\mathcal{F}}$ is the topology on $\nabla_{n<\omega} X_n$ generated by $\{N(\bar{x},f) : f \in \mathcal{F}\}$, and if G_δ's in $\bar{\tau}_f$ are open, then $\tau_{\mathcal{F}}$ is paracompact iff $\bar{\tau}_{\mathcal{F}}$ is paracompact.*

2.3 $\mathfrak{b}$, $\mathfrak{d}$, and their effects

Recall that $\mathfrak{b}$ is the least κ so that there is an unbounded family of functions in ω^ω of size κ, and $\mathfrak{d}$ is the least κ so that there is a dominating family of functions in ω^ω of size κ. There is always an unbounded family of order type $\mathfrak{b}$ which is well-ordered under $\leq^*$, and if $\mathfrak{b} = \mathfrak{d}$, there is a dominating family which is well-ordered under $\leq^*$ (called a scale).

If each X_n is first countable, $\nabla_{n<\omega} X_n$ has π-weight $\mathfrak{d}$ and is $\mathfrak{b}$-open (= every intersection of fewer than $\mathfrak{b}$ open sets is open).

The following is ancient folklore.

Lemma 4. (a) *If $\mathcal{F}$ is unbounded in ω^ω, $A \in [\omega]^\omega$, and $g \in \omega^\omega$, then for some $f \in \mathcal{F}\{n \in A : g(n) < f(n)\}$ is infinite.*

(b) *If $F \in [\omega^\omega]^{<\mathfrak{d}}$ and $\mathcal{A} \in [[\omega]^\omega]^{<\mathfrak{d}}$, then there is $g \in \omega^\omega$ so that $\forall A \in \mathcal{A}$ $\{n \in A : g(n) > f(n)\}$ is infinite.*

Let $\mathcal{N}_f = \{N^*(\bar{x}, f) : x \in \square_{n<\omega} X_n\}$. The following was proved by van Douwen [4].

Lemma 5. *If each X_n is metrizable, then $\bar{x} \in N^*(\bar{y}, f)$ iff $\bar{y} \in N^*(\bar{x}, f)$, and if $g \geq^* f$, then $\mathcal{N}_g$ refines $\mathcal{N}_f$.*

Hence, $\mathcal{N}_f$ is a pairwise disjoint cover of $\nabla_{n<\omega} X_n$.

This gives us a weaker topology than $\nabla_{n<\omega} X_n$, which is paracompact and Hausdorff.

Proposition 1. If $\mathcal{F} \subset \omega^\omega$ is unbounded and each X_n is first countable, then $\nabla_{\mathcal{F}} X_n$ is Hausdorff.

Proof: Suppose $\bar{x} \neq \bar{y}$. There is $g \in \omega^\omega$, so $N(\bar{x}, g) \cap N(\bar{y}, g) = \emptyset$. Let $A = \{n : u_{x(n),g(n)} \cap u_{y(n),g(n)} = \emptyset\}$. A is infinite, so there is $f \in \mathcal{F}$ with $\{n \in A : g(n) < f(n)\}$ infinite. Hence, $N(\bar{x}, f) \cap N(\bar{y}, f) = \emptyset$. $\square$

Theorem 3. *Suppose each X_n is metrizable. If $\mathcal{F} \subset \omega^\omega$ is unbounded and well-ordered by $\leq^*$ and $|\mathcal{F}| = \mathfrak{b}$, then $\nabla_{\mathcal{F}} X_n$ is paracompact.*

Proof: We list $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\}$ in increasing $\leq^*$ order. If $\mathcal{U} \subset \bar{\tau}_\mathcal{F}$ is a cover, by induction, we construct a pairwise disjoint covering refinement by defining $\mathcal{V}_\alpha = \{\bar{N} \in \mathcal{N}_{f_\alpha} : \exists \bar{U} \in \mathcal{U}\ \bar{N} \subset \bar{U}$ and $\forall \beta < \alpha\ \bar{N} \cap \bigcup \mathcal{V}_\beta = \emptyset\}$; $\mathcal{V} = \bigcup_{\alpha<\mathfrak{b}} \mathcal{V}_\alpha$.

$\bigcup_{\alpha<\mathfrak{b}} V_\alpha$ is a pairwise disjoint refinement. Here is why it is a cover: Consider an arbitrary point $\bar{x}$. Let α be least so $N(\bar{x}, f_\alpha) \subset \bar{U}$ for some $\bar{U} \in \mathcal{U}$. Since $\mathcal{N}_\alpha$ refines $\mathcal{N}_\beta$ for all $\beta < \alpha$, $N(\bar{x}, f_\alpha) \in \mathcal{V}_\alpha$. □

This is essentially van Douwen's proof that a κ-open, κ-metrizable space is compact; hence, if $\mathfrak{b} = \mathfrak{d}$, and each X_n is compact metrizable, $\square_{n<\omega} X_n$ is paracompact.

The following definitions are key to the next section.

Definition 2. A set S is strongly separated iff there is an open discrete family $\mathcal{U} = \{U_x : x \in S\}$ with $S \cap U_x = \{x\}$ for all $x \in S$.

We say that $\mathcal{U}$ strongly separates S.

Definition 3. Let $S \subset X$. We say that S is X-paracompact iff every open cover of X has a pairwise disjoint open (in X) refinement covering S.

We write ∇-paracompact when $X = \nabla_{n<\omega} X_n$.

Lemma 6. *Suppose each X_n is first countable. If $\mathcal{N}$ is a pairwise disjoint collection of sets of the form $N^*(\bar{x}, f)$ and $|\mathcal{N}| < \mathfrak{d}$, then $\mathcal{N}$ is discrete.*

Proof: If $\bar{z} \notin \bigcup \mathcal{N}$, then $\forall N^*(\bar{x}, f) \in \mathcal{N}\ \{n : \exists m_{\bar{x},n}\ z(n) \notin \text{cl}\ u_{x(n), m_{\bar{x},n} \cdot f(n)}\}$ is infinite. By Lemma 4(b), let h be defined so that for each $N^*(\bar{x}, f) \in \mathcal{N}$, $\{n : u_{z(n),h} \cap u_{x(n), m_{\bar{x},n} \cdot f(n)} = \emptyset\}$ is infinite. Then $N^*(\bar{z}, h) \cap N^*(\bar{x}, f) = \emptyset$ for all $N^*(\bar{x}, f) \in \mathcal{N}$. □

Theorem 4. (a) *If each X_n is first countable, $S \subset \nabla_{n<\omega} X_n$, and $|S| < \mathfrak{d}$, then S is strongly separated.*

(b) *If each X_n is first countable, $S \subset \nabla_{n<\omega} X_n$, and $|S| = \mathfrak{d}$, then S is ∇-paracompact.*

Proof: (a) Fix $\bar{x} \in S$. For each $\bar{y} \neq \bar{x}$ with $\bar{y} \in S$, let $g_{\bar{x},\bar{y}}$ be a function so that $N(\bar{x}, g_{\bar{x},\bar{y}}) \cap N(\bar{y}, g_{\bar{x},\bar{y}}) = \emptyset$, and let $A_{\bar{x},\bar{y}} = \{n : u_{x(n), g_{\bar{x},\bar{y}}(n)} \cap u_{y(n), g_{\bar{x},\bar{y}}(n)} = \emptyset\}$. By Lemma 4(b), there is $k_{\bar{x}} \in \omega^\omega$ with $k_{\bar{x}}|_{A_{\bar{x},\bar{y}}} \not\leq^* g_{\bar{x},\bar{y}}|_{A_{\bar{x},\bar{y}}}$ for all $\bar{y} \in S \setminus \{\bar{x}\}$. So $\mathcal{N} = \{N^*(\bar{x}, k_{\bar{x}}) : \bar{x} \in S\}$ is pairwise disjoint.

$\mathcal{N}$ is discrete by Lemma 6.

(b) Using Lemma 6, we construct a pairwise disjoint refinement covering S by induction. □

The theorem that $\nabla_{n<\omega} X_n$ is paracompact if each X_n is first countable and $\mathfrak{d} = \mathfrak{c}$ is a corollary of Theorem 4(b).

The next section is concerned with showing that many sets are strongly separated. We end this section by showing how this automatically will prove that many subspaces are ∇-paracompact.

Theorem 5. *If a space X is κ-open and $X = \bigcup_{\alpha<\kappa} S_\alpha$ where each S_α is strongly separated in $X \setminus \bigcup_{\beta<\alpha} S_\beta$, then X is paracompact.*

Proof: Let $S = \bigcup_{\alpha<\kappa} S_\alpha$ where $\mathcal{W}_\alpha$ is an open family which strongly separates S_α in $X \setminus \bigcup_{\beta<\alpha} S_\beta$. By induction, we construct a pairwise disjoint open family $\mathcal{R} = \bigcup_{\alpha<\kappa} \mathcal{R}_\alpha$ where each $\mathcal{R}_\alpha$ refines a subset of $\mathcal{W}_\alpha$ in a 1-1 fashion; hence, $\mathcal{R}_\alpha$ is discrete in $X \setminus \bigcup_{\beta<\alpha} S_\beta$. Note that, by κ-open, if $\alpha < \kappa$, then $\bigcup_{\beta<\alpha} \mathcal{R}_\alpha$ will be discrete.

Let $\mathcal{U}$ be an open cover of X. At stage α, we consider $E_\alpha = S_\alpha \setminus \bigcup_{\beta<\alpha} \mathcal{R}_\beta$. For each $x \in E_\alpha$, we let V_x be an open neighborhood of x with $x \in V_x \subset U \cap W$ where $U \in \mathcal{U}$ and $W \in \mathcal{W}_\alpha$. Let $\mathcal{R}_\alpha = \{V_x : x \in E_\alpha\}$. □

Corollary 6. *If a space X is κ-open, then the union of at most κ many strongly separated sets is X-paracompact.*

Thus, when we prove that various sets are strongly separated, we will automatically be proving that the union of at most $\mathfrak{b}$ many of them are ∇-paracompact. Similarly, if $\delta \leq \mathfrak{b}$, and we have a collection of sets $\{S_\alpha : \alpha < \delta\}$ where each S_α is strongly separated in $X \setminus \bigcup_{\alpha<\delta} S_\alpha$, then $\bigcup_{\alpha<\delta} S_\alpha$ is paracompact.

3. $\nabla(\omega+1)^\omega$

For this section, $X = \square(\omega+1)^\omega \setminus \{\infty\}$, where ∞ is the function which is constantly ω.

This section focuses on $\nabla(\omega+1)^\omega$ and finds discrete subsets of $\nabla^- = \bar{X}$ which are strongly separated in certain subspaces of $\bar{X}$ (some even in $\bar{X}$ itself). Taking unions as in Theorem 5 or Corollary 6 will give ∇^- paracompact spaces. Hence, since we have left out only one point from ∇, such a union is ∇-paracompact.

3.1 A simple example

Working in ∇^-, we note that $N(\bar{g},h) \cap N(\bar{f},k) \neq \emptyset$ if and only if $f|_{F(f)\cap F(g)} =^* g|_{F(f)\cap F(g)}$ and $f|_{F(f)\cap I(g)} >^* h|_{F(f)\cap I(g)}$ and $g|_{F(g)\cap I(f)} >^* k|_{F(g)\cap I(f)}$. Hence, there are easily described combinatorial properties which guarantee that no neighborhood of a point meets a given neighborhood of another point.

Lemma 7. *Let $f, g \in (\omega+1)^\omega, k \in \omega^\omega$. For all $h \in \omega^\omega$ $N(\bar{g},h) \cap N(\bar{f},k) = \emptyset$ iff one of the following holds.*

(i) $f|_{F(g)\cap F(f)} \neq^* g|_{F(g)\cap F(f)}$, or
(ii) $g|_{I(f)\cap F(g)} \not>^* k|_{I(f)\cap F(g)}$.

In particular,

Lemma 8. *Let $f, g \in (\omega+1)^\omega, k \in \omega^\omega$ and suppose $g \supset f, \bar{g} \notin N(\bar{f},k)$. Then $\forall h \in \omega^\omega$ $\forall g' \supset g$ $N(\bar{g}',h) \cap N(\bar{f},k) =^* \emptyset$.*

Proof: Condition (ii) of Lemma 7 holds for g, hence for g'. □

Definition 4. Let $f \in (\omega+1)^\omega$. f is non-decreasing (strictly increasing, respectively) iff $\forall n < m$, if $n, m \in F(f)$, then $f(n) \leq f(m)$ ($f(n) < f(m)$, respectively).

Definition 5. (a) Let $A \in [\omega]^\omega, n \in \omega$. $n^+_A = \inf a \setminus (n+1)$.

(b) Let $f \in (\omega+1, A \subset F(f)$. $f^+_A(n) = 1 + f(n^+_A)$. If $A = F(f)$, we just write f^+.

Note that $f < f^+_A$ for all non-decreasing f.

Proposition 2. Let $k \in \omega^\omega, k$ non-decreasing. Let $X_k = \{g \in X : g|_{F(g)} \leq^* k|_{F(g)}\}$. Then $\bar{X}_k$ is strongly separated in ∇^-.

Proof: Let $\mathcal{N} = \{N(\bar{g}, k^+_{F(g)}) : \bar{g} \in \bar{X}_k\}$. $\mathcal{N}$ is disjoint: if $\bar{g} \neq \bar{g}'$ and $g, g' \in \bar{X}_k$, then without loss of generality $F(g) \setminus F(g')$ is infinite. But if $n \in F(g) \setminus F(g')$, then $g(n) < k(n) \leq k(n^+_{F(g')}) < k^+_{F(g')}(n)$.

$\mathcal{N}$ is discrete: if $g' \notin \bigcup \mathcal{N}$, then for all $g \in X_k$, either property (i) of Lemma 7 holds or property (ii) of Lemma 7 holds vis-a-vis $k^+_{F(g)}$. □

By Proposition 2, Corollary 6, and the fact that ∇^- is $\mathfrak{b}$-open, we have the theorem that if $\mathfrak{b} = \mathfrak{d}$, then $\square(\omega+1)^\omega$ is paracompact.

By Theorem 6, we would know that $\nabla(\omega+1)^\omega$ is paracompact if $\forall k \in \omega^\omega$ $\{g \in (\omega+1)^\omega : g|_{F(g)} \not\geq^* k|_{F(g)}\}$ were strongly separated, but this is false; $\{g \in (\omega+1)^\omega : g|_{F(g)} \not\geq^* k|_{F(g)}\}$ is, in fact, open and not discrete.

The goal of the rest of this section is to find other strongly separated subsets of ∇^- or various subspaces thereof.

3.2 The machinery

Let $f, g \in (\omega+1)^\omega$. We present the following definitions.

Definition 6. $f^\perp(n) = \begin{cases} f(n) & \text{if } f(n) \leq f(m)\ \forall m \in F(f) \setminus n \\ \omega & \text{otherwise.} \end{cases}$

Definition 7. (a) $g \subset f$ iff $g|_{F(g)} = f|_{F(g)}$.

(b) If $g \subset f$, then $(f \setminus g)(n) = \begin{cases} f(n) & \text{if } n \in F(f) \setminus F(g) \\ \omega & \text{otherwise.} \end{cases}$

(c) If $f|_{F(f)\cap F(g)} = g|_{F(f)\cap F(g)}$,

$$\text{then } (f \cup g)(n) = \begin{cases} f(n) & \text{if } n \in F(f) \\ g(n) & \text{if } n \in F(g) \\ \omega & \text{if otherwise.} \end{cases}$$

Definition 8. $f_0 = f^\perp$; $f_{n+1} = (f \setminus f_n)^\perp$.

Definition 9. $\nabla_n = \{\bar{f} \in \nabla^- : f_{n+1} = \infty\}$; $\nabla_\omega = \nabla^- \setminus \bigcup_{n<\omega} \nabla_n$.

Thus, f is non-decreasing (mod finite) iff $\bar{f} \in \nabla_0$; $\nabla_0 = \{\bar{f} \in \nabla^- : f =^* f_0\}$; and $\bar{f} \in \nabla_n$ iff $f =^* \bigcup_{j\leq n} f_j$.

We will show that if $n < \omega$, then ∇_n is discrete in ∇ and strongly separated in $\nabla^- \setminus \bigcup_{i<n} \nabla_i$. While ∇_ω is not discrete, we will also show that some combinatorially defined subsets are discrete in ∇ and strongly separated in ∇_ω.

Definition 10. (a) If $f \in (\omega+1)^\omega$ and π is a permutation of ω, then we define the function πf as $\pi f(n) = f(\pi(n))$; we define $\pi\bar{f} = (\overline{\pi f})$.

(b) Given $\mathcal{F} \subset \nabla^-$ and π a permutation of ω, we define $\pi\mathcal{F} = \{\pi\bar{f} : \bar{f} \in \mathcal{F}\}$.

Fleissner pointed out in conversation that if $\mathcal{F}$ is strongly separated in ∇^-, so is $\pi\mathcal{F}$ for every permutation π of ω. Since $\bar{f} \in \bigcup_{i<\omega} \nabla_i$ iff $\pi f \in \bigcup_{i<\omega} \nabla_i$, we also have $\mathcal{F}$ is strongly separated in

$\nabla^- \setminus \bigcup_{i<j} \nabla_i$ iff $\pi\mathcal{F}$ is, for all $j \leq \omega$. So the results below automatically give us many more strongly separated spaces of ∇^- or its subspaces.

3.3 ∇_n, n finite

Theorem 7. *∇_0 is strongly separated in ∇^-. In particular, let $\mathcal{N} = \{N(\bar{f}, f^+) : f \in \nabla_0\}$. Then $\mathcal{N}$ strongly separates ∇_0 in ∇^-.*

Proof: Suppose $f \neq^* g, f, g \in \nabla_0$.

Claim 7.1. $N(\bar{f}, f^+) \cap N(\bar{g}, g^+) = \emptyset$.

Proof of Claim: We may assume that $f|_{F(f)\cap F(g)} = g|_{F(f)\cap F(g)}$. Without loss of generality $F(f) \setminus F(g)$ is infinite, $f|_{F(f)\setminus F(g)} > g^+|_{F(f)\setminus F(g)}$, and $g|_{F(g)\setminus F(f)} > f^+|_{F(g)\setminus F(f)}$. Let $n \in F(f) \setminus F(g), m_n = n^+_{F(g)}$. Then $f(n) > g^+(n) = g^+(m_n) = 1 + g(m_n) > g(m_n)$. Since $f \in \nabla_0, m_n \notin F(f)$. Let $j_n = (m_n)^+_{F(f)}$. By the same argument, $g(m_n) > f(j_n)$. This happens infinitely often, which contradicts $f \in \nabla_0$. □

Hence, $\mathcal{N}$ is pairwise disjoint and separates ∇_0. Below, we show that it is discrete.

Claim 7.2. If $g \notin N(\bar{g}, g^+), g' \in \nabla_0$, and $g^\perp = g'$, then all $N(\bar{g}, h) \cap N(\bar{g}', (g')^+) = \emptyset$.

Proof of Claim: By Lemma 8. □

Claim 7.3. Let $g \notin \nabla_0, g' \in \nabla_0$. If $g' \neq g^\perp$, then $N(\bar{g}, k_{g^\perp}) \cap N(\bar{g}', (g')^+) = \emptyset$.

Proof of Claim: If H is an infinite subset of $F(g) \setminus F(g^\perp)$, then $g|_H$ is not non-decreasing. So consider $H = \{n \in F(g') \cap I(g^\perp) : g'(n) \leq k_{g^\perp}(n)\}$. If $H \cap F(g)$ is infinite, then $g|_H \neq^* g'|_H$, and we are done. So we may assume that $H \subset^* I(g)$. If H is infinite, then $\{n \in F(g') \cap I(g) : g'(n) \leq k_{g^\perp}(n)\}$ is infinite, and we are done. So we may assume $H =^* \emptyset$.

We may assume that $g^\perp|_{F(g^\perp)\cap F(g')} =^* g'_{F(g^\perp)\cap F(g')}$ (or again, we are done). All that is left to consider is $g^\perp|_{F(g^\perp)\cap I(g')}$. Since $N(\bar{g}^\perp, k_{g^\perp}) \cap N(\bar{g}', (g')^+) = \emptyset$, and H is finite, we must have $g^\perp|_{F(g^\perp)\cap I(g')} \not>^* (g')^+|_{F(g^\perp)\cap I(g')}$, and we are done. □

Let $g \notin \bigcup \mathcal{N}$. By Claim 7.3, if $N(\bar{g}, k_{g^\perp}) \cap N(\bar{f}, f^+) \neq \emptyset$ for $\bar{f} \in \nabla_0$, then $f =^* g^\perp$. Since $g \notin N(\bar{f}, f^+)$, by Lemma 7 all $N(\bar{g}, h) \cap N(\bar{f}, f^+) = \emptyset$. And this concludes the proof. □

Now we consider $\nabla_n, n > 0$.
If $f \in \nabla_n$, we define $k_f = \sup\{(f_i)^+ : i \leq n\}$.

Lemma 9. *Suppose $f \in \nabla_n, g \in \nabla \setminus \bigcup_{i \leq n} \nabla_i$. If $N(\bar{f}, f^+) \cap N(\bar{g}, (g_n)^+) \neq \emptyset$, then $f =^* \bigcup_{i \leq n} g_i$.*

Proof: Imitate the proof of Claim 7.3 on the space $\nabla(\omega + 1)^E$ where $E = F(g_n) \cup I(g)$. □

Note that $f \in \nabla$ is finite-to-one iff there is some π a permutation of ω with $\pi f \in \nabla_0$. Hence, as previously noted, if $\{\pi_\alpha : \alpha < \mathfrak{b}\}$ is a collection of permutations of ω, then $\bigcup_{\alpha < \mathfrak{b}} \pi_\alpha \nabla_0$ is paracompact.

Theorem 8. *Each ∇_n is discrete. In particular, $\mathcal{N}_n = \{N(\bar{g}, k_g) : g \in \nabla_n\}$ strongly separates ∇_n in $\nabla^- \setminus \bigcup_{i<n} \nabla_i$.*

Proof: Assume that for each $i \leq n, \mathcal{N}_i$ strongly separates ∇_i in $\delta^- \setminus \bigcup_{j<i} \nabla_j$. We show that for each $f \in \nabla_n, \{\bar{g} \in \nabla_{n+1} : g_n = f\}$ is strongly separated in $\nabla^- \setminus \bigcup_{i<n+1} \nabla_i$ by $\{N(\bar{g}, k_g) : g \in \nabla_{n+1}, g_n = f\}$.

Let $\bar{g}, \bar{g}' \in \nabla_{n+1}, f = \bigcup_{i \leq n} g_i = \bigcup_{i \leq n} g'_i$. Let $E = \omega \setminus F(f)$. $g|_E$ and $g'|_E$ are non-decreasing, $k_g > (g|_E)^{F(g_{n+1})}$, and $k_{g'} > (g'|_E)^{F(g'_{n+1})}$. The proof of Claim 7.1 shows that $\{N(\bar{g}, k_g) : g \in \nabla_{n+1}, g_n = f\}$ separates $\{\bar{g} \in \nabla_{n+1} : \bigcup_{i \leq n} g_i = f\}$.

Hence, by induction, $\mathcal{N}_n$ separates ∇_n.

To show that $\mathcal{N}_n$ is discrete in $\nabla^- \setminus \bigcup_{i<n} \nabla_i$, consider $g \notin \bigcup \mathcal{N}_n$. If $N(\bar{g}, (\bigcup_{i \leq n} g_i)^+) \cap N(\bar{f}, k_f) \neq \emptyset$ for $\bar{f} \in \nabla_n$, then $f =^* \bigcup_{i \leq n} g_i$. Since $g \notin N(\bar{f}, k_f)$, then $\{n \in F(g) \setminus F(f) : g(n) < k_f(n)\}$ is infinite, and all $N(\bar{g}, h) \cap N(\bar{f}, k_f) = \emptyset$. □

3.4 ∇_ω

Discrete subsets of ∇_ω are harder to describe.

Definition 11. (a) Let $f \in \nabla_\omega$. $L_f(n) = \inf F(f_n)$.
(b) $\nabla_{\text{si}} = \{f \in \nabla_\omega : L_f$ is strictly increasing$\}$.
(c) If $f \in \nabla_{\text{si}}$, then $k_f(n) = 1 + \sup\{(f_i)^+(n) : i \leq L_f(n)\}$.

Note that L_f is 1-1.

Lemma 10. *Suppose $f \in \nabla_{\mathrm{si}}$. Then the following hold.*
(a) $\forall n\ L_f(n) \geq n$;
(b) $\forall n (f_n)^+(n) < k_f(n)$.

Proof: (a) follows because $f \in \nabla_{\mathrm{si}}$.
(b) is immediate from (a) and the definition of k_f. □

Lemma 11. *Suppose $f, g \in \nabla_{\mathrm{si}}$ and $N(\bar{f}, k_f) \cap N(\bar{g}, k_g) \neq \emptyset$. $\forall n\ g_n =^* f_n$.*

Proof: By induction, using the fact that $k_f >^* (f_n)^+$ for all n. □

Theorem 9. *∇_{si} is discrete and is strongly separated in ∇_ω by $\mathcal{N} = \{N(\bar{f}, k_f) : f \in \nabla_{\mathrm{si}}\}$.*

Proof: We first prove that $\mathcal{N}$ separates ∇_{si}. So consider $\bar{f}, \bar{g}$ distinct elements of ∇_{si}.

Claim 9.1. If there are infinitely many n so that there is $i_n \in [F(f) \cap F(f_n) \setminus F(g_n)] \cup [F(g) \cap F(g_n) \setminus F(f_n)]$ with $i_n \geq \sup\{L_f(n), L_g(n)\}$, then $N(\bar{f}, k_f) \cap N(\bar{g}, k_g) = \emptyset$.

Proof of Claim: Given n, by Lemma 11, there are at most finitely many such i_n, so we may assume i_n is the largest such, i.e., $f_n|_{\omega \setminus (i_n+1)} = g_n|_{\omega \setminus (i_n+1)}$. We consider the case $i_n \in F(f) \cap F(f_n) \setminus F(g_n)$. Let $m_n = (i_n)^+_{F(g_n)}$. Then $f_n(i_n) \leq f_n(m_n) = g_n(m_n)$. By Lemma 10(a), $L_g(n) \geq n$. By assumption, $i_n \geq L_g(n)$. Therefore,

$$k_g(i_n) \geq (g_n)^+(i_n) = 1 + g_n(m_n) > f_n(m_n) \geq f_n(i_n) = f(i_n),$$

so $f(i_n) < (g(i_n))^+$.

By symmetry, infinitely often either $f(i_n) < k_g(i_n)$ or $g(i_n) < k_f(i_n)$. □

By Claim 9.1, if $s_n = \sup\{L_f(n), L_g(n)\}$, we can assume that $\forall n\ \forall i \geq s_n\ f_n(i) = g_n(i)$. I.e., we can assume that for all n either f_n is a tail of g_n or g_n is a tail of f_n.

Claim 9.2. Suppose g_n is a tail of f_n and $i \in F(f) \cap F(f_n) \setminus F(g)$. Then $f(i) < k_g(i)$.

Proof of Claim: By Lemma 10(a), $n \leq i$. By Claim 9.1, $i < L_g(n)$ so $f_n(i) \leq f_n(L_g(n)) = g_n(L_g(n)) < (g_n)^+(L_g(n)) = (g_n)^+(n) < k_g(n) \leq k_g(i)$. □

Hence, if $N(\bar{f}, k_f) \cap N(\bar{g}, k_g) \neq \emptyset, \forall^\infty n\ f_n = g_n$. But then by Lemma 11, $f =^* g$.

Claim 9.3. $\mathcal{N}$ is discrete in ∇_ω.

Proof of Claim: Given $\bar{f} \in \nabla_\omega \setminus \nabla_{\rm si}$, we define the function $f^\#$:

$$f^\#(n) = \begin{cases} f(n) & \text{if } n \in F(f_i) \text{ and } \forall j < i\ n > L_f(j) \\ \infty & \text{otherwise.} \end{cases}$$

Note that $f^\# \in \nabla_{\rm si}$ and $f \supset f^\#$. By Lemma 8 and the proof that $\mathcal{N}$ separates $\nabla_{\rm si}$, if $g \in \nabla_{\rm si}$ and $N(\bar{f}, h) \cap N(\bar{g}, k_g) \neq \emptyset$, then $g =^* f^\#$. If $\bar{f} \notin N(\bar{f}^\#, k_{f^\#})$, then by Lemma 7, all $N(\bar{f}, h) \cap N(\bar{f}^\#, k_{f^\#}) = \emptyset$. □

And this concludes the proof of the theorem. □

To generalize Theorem 9, note that it was the fact that L_f is increasing and the properties of L_f and k_f in lemmas 10 and 11 which made the proof of discreteness work. For the covering family to be discrete, we needed the uniform definition of the L_f's for $f \in \nabla_{\rm si}$.

So our task is to define another function $L_f^\top$ and another k_f that satisfy lemmas 10 and 11. Then we will define a subset of ∇_ω whose $L_f^\top$'s will be similar enough so that the proof of Theorem 9 will straightforwardly generalize.

Definition 12. Fix $f \in \nabla^\omega$. We define the set A_f and then the function $L_f^\top$:

$A_f = \{n : \forall i > n\ L_f(n) < L_f(i)\}$.

We enumerate A_f in increasing order as $A_f = \{a_n^f : n < \omega\}$ and define $L_f^\top(n) = L_f(a_n^f)$.

For example, $\nabla_{\rm si} = \{\bar{f} : A_f = \omega\}$.

Note that $L_f^\top(n) \geq n$ for all n, and $L_f^\top$ is strictly increasing.

Define $k_f(n) = 1 + \sup\{(f_i)^+(n) : i \leq (L_f)^\top(n)\}$. Note that $(f_n)^+(n) < k_f(n)$ for all n.

Definition 13. Let $A \in [\omega]^\omega$. $\nabla_A = \{\bar{f} : A_f = A\}$.

Theorem 10. *For $A \in [\omega]^\omega$, ∇_A is discrete and is strongly separated by $\{N(\bar{g}, k_g) : g \in \nabla_A\}$ in ∇_ω.*

Proof: Since $L_f^\top$ is increasing and Lemma 10 holds with L_f replaced by $(L_f)^\top$, a straightforward replacement of L_f by $(L_f)^\top$ in the proofs of claims 9.1 and 9.2 proves that $\mathcal{N}$ separates ∇_A. □

So to complete the proof of Theorem 10, it suffices to prove the following claim.

CLAIM 10.1. For all $A \in [\omega]^\omega$, $\mathcal{N}$ is discrete in ∇_ω.

Proof of Claim: Given $f \in \nabla_\omega \setminus \nabla_A$, we define the function $f^{\#,A}$ as

$$f^{\#,A}(n) = \begin{cases} f(n) & \text{if } n \in F(f_i) \text{ and if } i \notin A, \text{ then } n > L_f(i_A^+) \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\bar{f}^{\#,A} \in \nabla_A$. By Lemma 8 and the proof that $\mathcal{N}$ separates ∇_A, if $g \in \nabla_A$ and $N(\bar{f}, h) \cap N(\bar{g}, k_g) \neq \emptyset$, then $g =^* f^{\#,A}$. Since $f \supset f^{\#}$, if $f \notin N(\bar{f}^{\#,A}, k_{f^{\#,A}})$, then by Lemma 7, all $N(\bar{f}, h) \cap N(\bar{f}^{\#,A}, k_{f^{\#,A}}) = \emptyset$. □

And the proof of the theorem is complete. □

4. Two stumbling blocks to settling conjecture 1

Conjecture 1 has been around for over 40 years, and the main results for a quarter of a century. This section gives two results which indicate why we have been stuck for so long:

- Many models of $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$ aren't counterexamples.
- A straightforward attempt to use inner models to imitate forcing proofs is doomed.

4.1 $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$

If we are to show that Conjecture 1(b) fails, we need a model of $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$. Here, we show that in many models of conjecture $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$, Conjecture 1(c) holds.

Definition 14. A function $r \in \omega^\omega$ is semi-Cohen over a model M iff $\forall f \in M \cap \omega^\omega$ $r \not<^* f$.

If r is Cohen over M, it is of course semi-Cohen.

Lemma 12. *If r is semi-Cohen over M, then for all $f \in M \cap \omega^\omega, A \subset \omega, r^+|_A \not\leq^* f|_A$.*

Proof: Fix $f, A \in M$. We may assume f is non-decreasing. There are infinitely many n with $r(n) > f^+_A(n)$. For such $n, f(n^+_A) < f^+_A(n) < r(n) \leq r^+(n) \leq r^+(n^+_A)$. □

The following was the (implicit) basic idea in [13].

Theorem 11. *Suppose $M = \bigcup_{\alpha<\mathfrak{b}} M_\alpha$ where $\exists r_\alpha \in M_{\alpha+1}$ r_α is semi-Cohen over M_α. Then $\Box_{n<\omega} X_n$ is paracompact if each X_n is compact first countable.*

Proof: By Corollary 6, it suffices to prove that $M \models \{N^*(\bar{x}, r^+_\alpha) : x \in M_\alpha \cap (\omega+1)^\omega\}$ is discrete.

For $x \in \Box_{n<\omega} X_n$, recall that $\{u_{x(n),i} : i < \omega\}$ is a neighborhood base of open sets of $x(n)$ with cl $u_{x(n),i+1} \subset u_{x(n),i}$.

So let $x \neq^* y \in M_\alpha \cap (\omega+1)^\omega$. There is some $g \in M_\alpha$ with $N(\bar{x}, g) \cap N(\bar{y}, g) = \emptyset$. Let $A = \{n : u_{x(n),g(n)} \cap u_{y(n),g(n)} = \emptyset\}$. By Lemma 12, $r^+_\alpha|_A \not\leq^* g|_A$. So $N(\bar{x}, r^+_\alpha) \cap N(\bar{y}, r^+_\alpha) = \emptyset$.

Now suppose $x \in M_\beta \cap (\omega+1)^\omega \setminus \bigcup_{x \in M_\alpha} N^*(\bar{x}, r^+_\alpha)$. (Necessarily, $\beta > \alpha$.) For any $y \in M_\alpha \cap (\omega+1)^\omega$, there is m with $\bar{x} \notin N(\bar{y}, m \cdot r^+_\alpha)$. Let $A = \{n : x(n) \notin u_{y(n), m \cdot r^+_\alpha(n)}\}$. There are infinitely many $n \in A$ with $r^+_\beta(n) > 1 + r_\alpha(n)$. So $N(\bar{x}, r^+_\beta) \cap N(\bar{y}, (m+1) \cdot r^+_\alpha) = \emptyset$. □

Cohen reals also have a converse property (see [1, p. 100] for a proof).

Proposition 3. If r is Cohen over M, then every function in $M[r]$ is dominated infinitely often by some function in M.

Hence, a finite iteration that ends with Cohen forcing satisfies the following.

Lemma 13. *Suppose $M \models \mathfrak{b} = \mathfrak{d} = \kappa$, and suppose $\mathbb{P}$ is a forcing which does not collapse any cardinal $\leq \kappa$ so that every function in $M^{\mathbb{P}}$ is dominated infinitely often by some function in M. Then $M^{\mathbb{P}} \Vdash \mathfrak{d} = \kappa$.*

Proof: Otherwise, there is a dominating family $\dot{G} = \{\dot{g}_\alpha : \alpha < \lambda\}$ which is increasing mod finite, $\lambda < \kappa$, λ regular. Let $\{f_\alpha : \alpha < \kappa\}$ be a dominating family in M which is increasing mod finite. Define $\dot{\varphi} : \lambda \to \kappa$ by $\dot{\varphi}(\alpha) = \sup\{\beta : \dot{g}_\alpha >^* f_\beta\}$. By hypothesis, range $\dot{\varphi} \subset \kappa$, and since $\dot{G}$ is dominating, range $\dot{\varphi}$ is cofinal in κ. So $\mathbb{P}$ collapses κ, a contradiction. □

Many models meet the hypothesis of Theorem 11, e.g., any iterated ccc forcing of uncountable cofinality (see [13]). But there are others. Here we mention two.

Proposition 4. (a) For $\alpha < \lambda$, let $M \models \mathfrak{b} = \mathfrak{d} = \kappa \geq \omega_2$. Let $\mathbb{P} = Fn(\lambda, \omega)$, where $\lambda < \kappa$, λ regular. Let $M^{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is the measure algebra on 2^{κ^+}. Then $M^{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \mathfrak{b} < \mathfrak{d} < \mathfrak{c}$ and Conjecture 1(c).

(b) Let $M \models \lambda < \kappa < \mathfrak{c}, \lambda, \kappa$ regular. Let $\mathbb{H}$ be the Hechler forcing that adds a dominating family of ω^ω with order type $\kappa \times \lambda$. Then $M^{\mathbb{H}} \Vdash \mathfrak{b} < \mathfrak{d} < \mathfrak{c}$ and Conjecture 1(c).

Proof: (a) Define $\mathbb{P}_\alpha = Fn(\alpha, \omega), \dot{\mathbb{Q}_\alpha} = \dot{\mathbb{Q}} \cap M^{\mathbb{P}_\alpha}$. If G is $\mathbb{P} * \dot{\mathbb{Q}}$-generic, we define $M_\alpha = M[G \cap \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha]$. Lemma 13 and Theorem 11 complete the proof.

(b) For $\alpha < \lambda$, define $\mathbb{H}_\alpha$ to be the Hechler forcing that adds the subfamily of order type $\alpha \times \lambda$. If G is $\mathbb{H}$-generic, we define $M_\alpha = M[G \cap \mathbb{H}_\alpha]$. Theorem 11 completes the proof. □

In the 25 or so years since the major results on Conjecture 1 were produced, we've learned a lot about elementary submodels. So one might consider adapting the techniques of Theorem 11 to elementary submodels.

How might this work?

Start with an unbounded family $R = \{r_\alpha : \alpha < \mathfrak{b}\}$ well-ordered by $\leq^*$. Note that if M is an elementary submodel with $R \in M$ and $\sup \mathfrak{b} \cap M = \delta$, then $\forall \alpha \geq \delta$ r_α is semi-Cohen over M. To ensure that $\mathfrak{b} \cap M \neq \mathfrak{b}$, require $|M| < \mathfrak{b}$.

Given two functions, the proof of Theorem 11 requires that they are in the same model over which r_α is Cohen, so we would also need that if r_α were semi-Cohen over M, M' and $f \in M, f' \in M'$, then there is a model $M^\dagger$ with $f, f' \in M^\dagger$ and g_α semi-Cohen over $M^\dagger$.

Putting this together, the following would suffice: for each $\alpha < \mathfrak{b}$, there is a set of elementary submodels $\mathcal{M}_\alpha$ so $\omega^\omega \subset \bigcup_{\alpha<\mathfrak{b}} \bigcup \mathcal{M}_\alpha$, r_α is semi-Cohen over each $M \in \mathcal{M}_\alpha$; and if $f \in M \in \mathcal{M}_\alpha, f' \in M' \in \mathcal{M}_\alpha$, then there is a model $M^\dagger \in \mathcal{M}_\alpha$ with $f, f' \in M_\dagger$.

And for this to hold, we need the following conjecture to be true.

Conjecture 2. Let R be an unbounded family well-ordered by $\leq^*$ (hence, necessarily of order type $\mathfrak{b}$). Let $\mathfrak{A} =< H(\mathfrak{c}^+), \in, R, \Delta >$ where Δ is a well-ordering of $H(\mathfrak{c}^+)$. There is a sequence of ordinals

$D = \{\delta_\alpha : \alpha < \mathfrak{b}\}$, a sequence $\{\mathcal{M}_\alpha : \alpha < \mathfrak{b}\}$, and an assignment $\varphi : \mathfrak{d} \to D$ satisfying

(1) if $M \in M_\alpha$, then $|M| < \mathfrak{b}$,
(2) if $M \in \mathcal{M}_\alpha$, then $\sup(M \cap \mathfrak{b}) = \delta_\alpha$,
(3) if $\varphi(\eta) = \varphi(\zeta) = \alpha$, then $\exists M \in \mathcal{M}_\alpha \; \eta, \zeta \in M$.

The translation is as follows: Given a fixed dominating family $\{f_\eta : \eta < \mathfrak{d}\}, \varphi(\eta) = \alpha$ says that $f_\eta \in \bigcup \mathcal{M}_\alpha$.

Unfortunately, Conjecture 2 fails when $\mathfrak{b} < \mathfrak{d}$.

Proposition 5 (Ishiu)**.** If $\mathfrak{b} < \mathfrak{d}$, then Conjecture 2 is false.

The proof below uses a well-known set-theoretic technique. This particular application is from a personal communication from Tetsuya Ishiu.

Proof: Suppose $\Delta, \mathfrak{A}, D, \{\mathcal{M}_\alpha : \alpha < \mathfrak{b}\}, \varphi$ as in Conjecture 2. Let $E = \{\eta < \mathfrak{d} : \text{cf } \eta = \mathfrak{b}\}$. $|E| = \mathfrak{d}$. For each $\eta \in E$, let e_η be the Δ-least cofinal increasing function $e : \mathfrak{b} \to \eta$. Note that $\eta \in M \Rightarrow e_\eta \in M$.

By a counting argument, $\exists \alpha \; K = \{\eta \in E : \varphi(\eta) = \alpha\}$ is stationary.

By pressing down, $\exists \delta \; H = \{\eta \in K : e_\eta(\delta_\alpha) = \delta\}$ is stationary.

Since H is stationary, there is $\zeta \in H, \zeta$ a limit of H; hence, there is $\eta \in H, \eta \in [\delta, \zeta)$. Suppose $\eta, \zeta \in M$. $M \models \exists \gamma \; e_\zeta(\gamma) > \eta$. So $\exists \gamma \in M \cap (\delta, \mathfrak{b})$. Hence, $M \notin \mathcal{M}_\alpha$ □

References

[1] Tomek Bartoszyński and Haim Judah, *Set Theory. On the Structure of the Real Line.* Wellesley, MA: A K Peters, Ltd., 1995.

[2] Eric K. van Douwen, *The box product of countably many metrizable spaces need not be normal*, Fund. Math. **88** (1975), no. 2, 127–132.

[3] ______, *Another nonnormal box product*, General Topology and Appl. **7** (1977), no. 1, 71–76.

[4] ______, *Covering and separation properties of box products*, in Surveys in General Topology. Ed. George M. Reed. New York: Academic Press, 1980. 55–129.

[5] William G. Fleissner and Adrienne M. Stanley, *D-spaces*, Topology Appl. **114** (2001), no. 3, 261–271.

[6] Stephen H. Hechler, *On the existence of certain cofinal subsets of* ${}^{\omega}\omega$, in Axiomatic Set Theory. Part 2. Ed. Thomas J. Jech. Proceedings of Symposia in Pure Mathematics, Vol. 13. Providence, R.I.: Amer. Math. Soc., 1974. 155–173.

[7] K. Kunen, *Some comments on box products*, in Infinite and Finite Sets: To Paul Erdös on His 60th Birthday. Ed. A. Hajnal, Richard Rado, and Vera T. Soś. Colloquia Mathematica Societatis Janos Bolyai, Vol. 10. Amsterdam: North-Holland Pub. Co., 1975. 1011–1016.

[8] ______, *Paracompactness of box products of compact spaces*, Trans. Amer. Math. Soc. **240** (1978), 307–316.

[9] ______, *Set Theory. An Introduction to Independence Proofs*. Studies in Logic and the Foundations of Mathematics, 102. Amsterdam-New York: North-Holland Publishing Co., 1980.

[10] ______, *On normality of box products of ordinals*. Preprint.

[11] L. Brian Lawrence, *Failure of normality in the box product of uncountably many real lines*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 187–203.

[12] Judy Roitman, *Paracompact box products in forcing extensions*, Fund. Math. **102** (1979), no. 3, 219–228.

[13] ______, *More paracompact box products*, Proc. Amer. Math. Soc. **74** (1979), no. 1, 171–176.

[14] Mary Ellen Rudin, *The box product of countably many compact metric spaces*, General Topology and Appl. **2** (1972), 293–298.

[15] ______, *Countable box products of ordinals*, Trans. Amer. Math. Soc. **192** (1974), 121–128.

[16] ______, *Lectures on Set Theoretic Topology*. Regional Conference Series in Mathematics, No. 23. Providence, R.I.: American Mathematical Society, 1975.

[17] Scott W. Williams, *Is* ${}^{\omega}(\omega + 1)$ *paracompact?* Topology Proc. **1** (1976), 141–146 (1977).

[18] ______, *Boxes of compact ordinals*, Topology Proc. **2** (1977), no. 2, 631–642 (1978).

[19] ______, *Box products*, in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 169–200.

[20] Louis Wingers, *Box products of* σ*-compact spaces*, Topology Appl. **56** (1994), no. 2, 185–197.

[21] ______, *Box products and Hurewicz spaces*, Topology Appl. **64** (1995), no. 1, 9–21.

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