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k-SPACES, AND PRODUCTS OF WEAK TOPOLOGIES

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Abstract. We investigate conditions for weak topologies to be productive and consider countable products of k-spaces.

1. Introduction and preliminaries

We assume that all spaces are regular and T_1, and all maps are continuous and onto.

For a cover $\mathcal{P}$ of a space X, we recall that X *is determined by* $\mathcal{P}$ [7] if X has the *weak topology* with respect to $\mathcal{P}$ [3]; that is, $G \subset X$ is open in X if $G \cap P$ is open in P for each $P \in \mathcal{P}$. Here, we can replace "open" by "closed." We call such a cover $\mathcal{P}$ a *determining cover* in [30].

A space X is a *sequential space* (respectively, *k-space*; *quasi-k-space* [14]) if X has a determining cover by compact metric sets (respectively, compact sets; countably compact sets). Then a space X is sequential if X has a determining cover by metric sets (or convergent sequences). Sequential spaces are k-spaces, and k-spaces are quasi-k-spaces.

As is well-known, every sequential space (respectively, k-space; quasi-k-space) is characterized as a quotient image of a locally compact metric space (respectively, locally compact paracompact space;

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M-space (i.e., space which admits a quasi-perfect map onto a metric space); for example, see [4] (respectively, [3]; [14]).

Similarly, as is known, every space with a countable determining cover by compact sets is precisely a quotient image of a locally compact Lindelöf space. Also, every space with a point-countable (respectively, point-finite) determining cover by compact sets is precisely an image of a locally compact paracompact space under a quotient map with each point-inverse Lindelöf (respectively, compact). Here, a cover $\mathcal{P}$ of X is *point-countable* (respectively, *point-finite*) if every $x \in X$ is in, at most, countably (respectively, finitely) many $P \in \mathcal{P}$.

For a collection $\mathcal{P}$ of sets of a space X, $\mathcal{P}$ is *closure-preserving* (*CP*), if for any subfamily $\mathcal{P}'$ of $\mathcal{P}$, $cl(\bigcup\{P : P \in \mathcal{P}'\}) = \bigcup\{clP : P \in \mathcal{P}'\}$, and $\mathcal{P}$ is *hereditarily closure-preserving* (*HCP*), if for any subcollection $\mathcal{P}' = \{P_\alpha : \alpha\}$ of $\mathcal{P}$, and any $\{A_\alpha : \alpha\}$ with $A_\alpha \subset P_\alpha$, the collection $\{A_\alpha : \alpha\}$ is CP.

For a closed cover $\mathcal{F}$ of a space X, we recall that X *is dominated by* $\mathcal{F}$ [9] if $\mathcal{F}$ is a CP cover such that any $\mathcal{P} \subset \mathcal{F}$ is a determining cover of the union of $\mathcal{P}$. (Sometimes we say that X has the *Whitehead weak topology*, *Morita weak topology* (in the sense of [12]), or *hereditarily weak topology*, with respect to $\mathcal{F}$). We call such a closed cover $\mathcal{F}$ a *dominating cover* in [30]. A space X with an increasing determining cover $\{X_n : n \in N\}$ is called the *inductive limit* of $\{X_n : n \in N\}$. When the X_n are closed in X, $\{X_n : n \in N\}$ is a dominating cover of X. As is well-known, every CW-complex has a dominating cover by compact metric sets. For some properties and questions on determining or dominating covers, see [8].

Open covers $\Rightarrow$ *Determining covers* $\Leftarrow$ *Dominating covers* $\Leftarrow$ *HCP closed covers* $\Leftarrow$ *Locally finite closed covers.*

Notation. For a cover $\mathcal{P}$ of a space, we will use symbols $[\mathcal{P}] = \{A : A \text{ is a finite union of elements of } \mathcal{P}\}$ and $\mathcal{P}^\circ = \{intP : P \in \mathcal{P}\}$.

In this paper, we shall consider the following question on products of determining or dominating covers and products of k-spaces in terms of weak topologies. Question 1.1(1) is mainly considered in [29]. Related to (3), note that, for a space $X = F_1 + F_2$,

$\mathcal{P} = \{F_1,\ F_2\}$ is a binary closed cover of X, but a countable product $\mathcal{P}^\omega(= \mathcal{P} \times \mathcal{P} \times \cdots)$ is not a determining cover of X^ω. So, we consider the products of type $[\mathcal{P}]^\omega$ (instead of $\mathcal{P}^\omega$) in (3).

Question 1.1. (1) *For each $i = 1,\ 2$, let $\mathcal{P}_i$ be a determining cover of a space X_i. Under what conditions is $\mathcal{P}_1 \times \mathcal{P}_2 (= \{P_1 \times P_2 : P_i \in \mathcal{P}_i\})$ a determining cover of $X_1 \times X_2$?*

(2) *Same as (1), but replace "determining" by "dominating."*

(3) *Let $\mathcal{P}$ be a determining cover of a space X. Under what conditions is $[\mathcal{P}]^\omega$ a determining cover of X^ω ?*

Let us recall some elementary facts which will be used often in this paper. Fact 1.2 is routinely shown (see [7], [24], [25], [28], [29], [30]). For Fact 1.3, (1) holds by the proof of [22, Lemma 6], and (2) holds by [24, Lemma 2.5]. For Fact 1.4, (1) holds by Fact 1.2(1), and (2) holds by [12, Lemma 3].

Fact 1.2. (1) *Let $\mathcal{C}$ be a determining cover of X. Let $\mathcal{P}$ be a cover (respectively, closed cover) of X. If $\mathcal{C}$ is a refinement of $\mathcal{P}$ (respectively, $[\mathcal{P}]$), then $\mathcal{P}$ is a determining cover of X.*

(2) *Let $\{P_\alpha : \alpha\}$ be a determining cover of X. If each P_α has a determining cover $\mathcal{P}_\alpha$, $\bigcup\{\mathcal{P}_\alpha : \alpha\}$ is a determining cover of X.*

(3) *Let $\mathcal{P}$ be a determining cover of X. If S is a closed or open set of X, then $\{P \cap S : P \in \mathcal{P}\}$ is a determining cover of S.*

A decreasing sequence (A_n) of non-empty sets of X is a *q-sequence* (respectively, *k-sequence*) [11], [14], if $C = \bigcap\{A_n : n \in N\}$ is countably compact (respectively, compact) in X, and each open set U with $C \subset U$ contains some A_n, equivalently, for any $x_n \in A_n$, $\{x_n : n \in N\}$ has an accumulation point in C [11].

Fact 1.3. (1) *Let $\mathcal{P}$ be a point-countable determining cover of X. Then, for each q-sequence (A_n) in X, some A_n is contained in an element of $[\mathcal{P}]$.*

(2) *Let $\mathcal{F} = \{X_\alpha : \alpha \leq \gamma\}$ be a dominating cover of X. For each $\alpha \leq \gamma$, let $L_\alpha = X_\alpha - \bigcup\{X_\beta : \beta < \alpha\}$. Then $\{clL_\alpha : \alpha \leq \gamma\}$ is a determining cover of X such that, for each q-sequence (A_n) in X, some A_n meets only finitely many L_α; hence, some A_n is contained in an element of $[\mathcal{F}]$.*

Fact 1.4. (1) *Let $\mathcal{P}$ be a determining cover of $X = \Pi X_i$ $(i \in N)$. Then $\Pi\mathcal{P}_i$ $(i \in N)$ is a determining cover of X, where $\mathcal{P}_i = P_i(\mathcal{P})$ for the projection $P_i : X \to X_i$.*

(2) *Let X be a locally compact space, and let $\mathcal{P}$ be a determining cover of Y. Then $\{X\} \times \mathcal{P}$ is a determining cover of $X \times Y$.*

For the following proposition, (1) is stated in [29] (or shown by Fact 1.2(2)), and (2) is due to [9], [13].

Proposition 1.5. (1) *Every space with a determining cover by sequential spaces (respectively, k-spaces; quasi-k-spaces) is a sequential space (respectively, k-space; quasi-k-space).*

(2) *Every space with a dominating cover by paracompact spaces (respectively, normal spaces) is paracompact (respectively, normal).*

2. Results

Let us recall two canonical quotient spaces, the *sequential fan* S_ω and the *Arens' space* S_2. For an infinite cardinal number α, let S_α be the space obtained from the disjoint union $\Sigma\{L_\beta : \beta < \alpha\}$ of convergent sequences by identifying all the limit points to a single point. Let S_2 be the space obtained from the disjoint union $\Sigma\{L_n : n = 0, 1, \cdots\}$ of copies of the sequence $\{1/n : n \in N\} \cup \{0\}$ by identifying each $1/n \in L_0$ with $0 \in L_n$ $(n \geq 1)$. For $\alpha \geq \omega$, let $\mathcal{F}_\alpha$ be the obvious HCP closed cover of the space S_α by α convergent sequences.

As for products of determining covers, we have Example 2.1 below. For (1), (a) holds by Fact 1.4(2), and (b) holds by [10, (7.5)] (or Corollary 2.5 below). For (2), as is well-known, $Q \times S_\omega$ is not a k-space; thus, (a) holds by means of Proposition 1.5(1). (b) is essentially given in [2] (see also [3, Example 5, p. 132]), and (c) holds by [6, Lemma 5]. For products of point-finite determining covers by compact metric sets, the similar examples also hold, using the space S_2 or K^*_α (in [26]), instead of S_ω or S_α, where $\alpha = \omega_1$ or c $(= 2^\omega)$.

Example 2.1. (1) (a) $\{R\} \times \mathcal{F}_\alpha$ is a determining cover of $R \times S_\alpha$, where R is the space of real numbers.

(b) $\mathcal{F}_\omega \times \mathcal{F}_\omega$ is a determining cover of $S_\omega \times S_\omega$.

(2) (a) $\{Q\} \times \mathcal{F}_\omega$ is not a determining cover of $Q \times S_\omega$, where Q is the space of rational numbers.

(b) $\mathcal{F}_\omega \times \mathcal{F}_c$ is not a determining cover of $S_\omega \times S_c$.
(c) $\mathcal{F}_{\omega_1} \times \mathcal{F}_{\omega_1}$ is not a determining cover of $S_{\omega_1} \times S_{\omega_1}$.

Theorem 2.2. *Let $\mathcal{P}_i$ be a determining cover of X_i $(i = 1, 2)$.*
(1) *Let X_1 be locally compact. Then $\mathcal{P}_1 \times \mathcal{P}_2$ is a determining cover of $X_1 \times X_2$ if one of the following (a) $\sim$ (e) holds.*
(a) *$\mathcal{P}_1$ is an open cover.*
(b) *$\mathcal{P}_1$ is a countable increasing cover.*
(c) *$\mathcal{P}_1$ is a point-countable closed cover.*
(d) *$\mathcal{P}_1$ is a dominating cover.*
(e) *Elements of $\mathcal{P}_2$ are k-spaces.*

(2) *$\mathcal{P}_1 \times \mathcal{P}_2$ is a determining cover of $X_1 \times X_2$ if (a) or (b) below holds* [29].
(a) *$X_1 \times X_2$ is a quasi-k-space with X_1 sequential.*
(b) *$X_1 \times X_2$ is a k-space, and the elements of $\mathcal{P}_1$ are k-spaces.*

Proof: We prove (1) holds. For (a), $X_1 \times X_2$ has an open cover (hence, a determining cover) $\{P_1 \times X_2 : P_1 \in \mathcal{P}_1\}$. Each $P_1 \in \mathcal{P}_1$ is locally compact, then $P_1 \times X_2$ has a determining cover $\{P_1 \times P_2 : P_2 \in \mathcal{P}_2\}$ by Fact 1.4(2). Thus, the result for (a) holds by Fact 1.2(2). For (b) $\sim$ (e), X_1 has an open cover $\mathcal{V} = \{V : clV$ is compact$\}$. Then $X_1 \times X_2$ has a determining cover $\mathcal{Q} = \{clV \times P_2 : V \in \mathcal{V}, P_2 \in \mathcal{P}_2\}$ by (a) and Fact 1.2(1). Thus, the result for (b), (c), or (d) holds, using Fact 1.2 and Fact 1.3. For (e), each $P_2 \in \mathcal{P}_2$ is a k-space (hence, it has a determining cover by compact sets); therefore, each $clV \times P_2 \in \mathcal{Q}$ has a determining cover $\{clV \times K : K$ is compact in $X_2\}$ by Fact 1.4(2), but each $clV \times K$ has a determining cover $\{(clV \cap P_1) \times (K \cap P_2) : P_i \in \mathcal{P}_i\}$ by Fact 1.2(3). Thus, the result for (e) holds by Fact 1.2. $\square$

Theorem 2.3. *Let $\mathcal{F}_i$ be a dominating cover of a space X_i $(i = 1, 2)$ such that $\mathcal{F}_1$ is HCP or increasing. Then $\mathcal{F}_1 \times \mathcal{F}_2$ is a dominating cover of $X_1 \times X_2$ if X_1 is locally compact, or $X_1 \times X_2$ is a quasi-k-space.*

Proof: Let $\mathcal{F} = \{A_\alpha \times B_\beta : (\alpha, \beta) \in \Gamma\} \subset \mathcal{F}_1 \times \mathcal{F}_2$. We will show that $S = \bigcup \mathcal{F}$ is closed in $X_1 \times X_2$, and $\mathcal{F}$ is a determining cover of S. For each α, let $\Gamma_\alpha = \{\beta : (\alpha, \beta) \in \Gamma\}$, and $C_\alpha = \bigcup\{B_\beta : \beta \in \Gamma_\alpha\}$. Then $S = \bigcup\{A_\alpha \times C_\alpha : \alpha\}$, and S is closed in $X_1 \times X_2$ (because, for $(a, b) \notin S$, let $V = X_1 - \cup\{A_\alpha : A_\alpha \not\ni a\}$,

$W = X_2 - \cup\{C_\alpha : A_\alpha \ni a\}$), then $V \times W$ is a nbd of (a, b) which doesn't meet S). We will show that $\mathcal{F}$ is a determining cover of S.

First, let X_1 be a locally compact space. Then $X_1 \times X_2$ has a determining cover $\{(C \times X_2) : C$ is compact in $X_1\}$. Since S is closed in $X_1 \times X_2$, S has a determining cover $\mathcal{C}_S = \{(C \times X_2) \cap S : C$ is compact in $X_1\}$ by Fact 1.2(3).

Next, let $X_1 \times X_2$ be a quasi-k-space. Then, by Fact 1.4(1), $X_1 \times X_2$ has a determining cover $\{(K_1 \times K_2) : K_i$ is countably compact in $X_i\}$. Then S has a determining cover $\mathcal{K}_S = \{(K_1 \times K_2) \cap S : K_i$ is countably compact in $X_i\}$.

Now, for $A \subset S$, assume (*) $F \cap A$ is closed for each $F \in \mathcal{F}$. Then A is closed in S. To see this, let us show that $C' \cap A$ is closed in C' for each $C' \in \mathcal{C}_S$ (or $\mathcal{K}_S$) for case (i) or (ii) below.

(i) $\mathcal{F}_1$ is HCP: For X_1 being locally compact, let C be a compact set in X_1. Then each $C \cap A_\alpha$ is compact; thus, each $(C \cap A_\alpha) \times C_\alpha$ has a determining cover $\{(C \cap A_\alpha) \times B_\beta : \beta \in \Gamma_\alpha\}$ by Fact 1.4(2). Hence, by the assumption (*), each $((C \cap A_\alpha) \times C_\alpha) \cap A$ is closed in $(C \cap A_\alpha) \times C_\alpha$, so is closed in $(C \times X_2) \cap S$. For $X_1 \times X_2$ being quasi-k, let K_i be a countably compact set in X_i. Then each countably compact set $K_2 \cap C_\alpha$ is contained in a finite union of B_β's ($\beta \in \Gamma_\alpha$) by Fact 1.3(2). Thus, each $(K_1 \cap A_\alpha) \times (K_2 \cap C_\alpha)$ has a determining cover $\{(K_1 \cap A_\alpha) \times (K_2 \cap B_\beta) : \beta \in \Gamma_\alpha\}$. Hence, by the assumption (*), each $((K_1 \cap A_\alpha) \times (K_2 \cap C_\alpha)) \cap A$ is closed in $(K_1 \times K_2) \cap S$. While $\mathcal{F}_1$ is HCP, then for each countably compact L in X_1, L meets only finitely many elements F of $\mathcal{F}_1$ with $F \cap L$ infinite; otherwise, $L \cap \bigcup \mathcal{F}_1$ is finite. Thus, since $S = \bigcup\{A_\alpha \times C_\alpha : \alpha\}$, for each $C' = (C \times X_2) \cap S \in \mathcal{C}_S$ (or $(K_1 \times K_2) \cap S \in \mathcal{K}_S$), $C' \cap A$ can be expressed as a finite union of closed sets in C' (hence, it is closed in C'). Here, for $L^* = L \cap \bigcup \mathcal{F}_1$ ($L = C$ or K_1) being finite, let F_i ($i \leq k$) be all finite sets of L^*, and $G_i = \bigcup\{C_\alpha : F_i \cap A_\alpha \neq \emptyset\}$. Then $(C' \cap A) \cap (L^* \times X_2)$ is a finite union of $((F_i \times G_i) \cap A) \cap C'$ ($i \leq k$), and these sets are closed in C' as in the above, for each F_i is compact and $\mathcal{F}_2$ is a dominating cover of X_2.

(ii) $\mathcal{F}_1$ is increasing: For each α, let $D_\alpha = A_\alpha - \bigcup\{A_\gamma : \gamma < \alpha\}$, and $E_\alpha = \bigcup\{C_\gamma : \gamma \geq \alpha\}$. Since $\mathcal{F}_1$ is increasing, $S = \bigcup\{D_\alpha \times E_\alpha : \alpha\}$. Each countably compact set of X_1 meets only finitely many D_α by Fact 1.3(2). Then, for each $C' \in \mathcal{C}_S$ (respectively, $\mathcal{K}_S$), C' is contained in a finite union of sets $(A_{\alpha_i} \times E_{\alpha_i}) \cap C'$. But each

$(A_{\alpha_i} \times E_{\alpha_i}) \cap C'$ has a determining cover $\{(A_{\alpha_i} \times B_\beta) \cap C' : \beta \in \Gamma_\alpha, \alpha \geq \alpha_i\}$ by Fact 1.4(2) (respectively, Fact 1.3(2)). As in (i), we see that $C' \cap A$ is a finite union of closed sets in C' (hence, it is closed in C'); here note that $\mathcal{F}_1$ is increasing. $\square$

Corollary 2.4 below holds by Theorem 2.2(2); here every CW-complex has a determining cover by the closures of all the cells. Corollary 2.5 below holds by Theorem 2.2(2), for $X_1 \times X_2$ is a k-space by [27, Theorem 6]. A space X is a *singly bi-quasi-k-space* [11] if, whenever $x \in clA$, there exists a q-sequence (B_n) with $x \in cl(A \cap B_n)$. Every k'-space (i.e., the B_n are the same compact set) is a singly bi-quasi-k-space. Corollary 2.6 below holds by Theorem 2.3 and Proposition 1.5(2).

Corollary 2.4. *Let X and Y be CW-complexes. Then $X \times Y$ is a CW-complex iff it is a quasi-k-space.*

Corollary 2.5. *Let X_i $(i = 1, 2)$ have a determining cover $\mathcal{C}_i$ by locally compact sets such that $\mathcal{C}_i$ are countable decreasing, countable closed, or point-countable closed with X_i singly bi-quasi-k-spaces. Then, for any determining cover $\mathcal{P}_i$ of X_i, $\mathcal{P}_1 \times \mathcal{P}_2$ is a determining of $X_1 \times X_2$ if $\mathcal{P}_1$ consists of k-spaces. Here, this condition can be omitted when X_1 or X_2 is sequential.*

Corollary 2.6. *Let $\mathcal{F}_i$ be a dominating cover of a space X_i $(i = 1, 2)$ such that $\mathcal{F}_1$ is HCP or increasing. If $X_1 \times X_2$ is a quasi-k-space, then $X_1 \times X_2$ is paracompact (respectively, normal) iff each element of $\mathcal{F}_1 \times \mathcal{F}_2$ is paracompact (respectively, normal).*

Following [11], a space X is a *countably bi-quasi-k-space* if, whenever (A_n) is a decreasing sequence x with $x \in clA_n$, there exists a q-sequence (B_n) such that $x \in cl(A_n \cap B_n)$; in particular, X is *countably bi-sequential* (= *strongly Fréchet* [18]) when there exist $x_n \in A_n$ such that $\{x_n : n \in N\}$ converges to the point x.

As a generalization of countably bi-quasi-k-spaces, let us consider the following property (P), changing "compact" to "countably compact" in [7, (3.1)].

(P): For each decreasing sequence (A_n) in X with $\bigcap\{clA_n : n \in N\} \neq \emptyset$, there exists a countably compact set K of X such that $K \cap A_n \neq \emptyset$ for all $n \in N$.

A space X has *countable tightness*, $t(X) \leq \omega$, if whenever $a \in clA$, $a \in clC$ for some countable $C \subset A$ (equivalently, X has a determining cover by countable sets; see [11]). A sequential space or a hereditarily separable space has countable tightness.

Proposition 2.7. (1) *Let X be a Fréchet space, or be a sequential space which is hereditarily normal or points are G_δ-sets. Then X is countably bi-sequential iff X has property (P).*

(2) *Let X^ω be a quasi-k-space with $t(X) \leq \omega$ (in particular, let X^ω be sequential). Then X has property (P).*

(3) *Under the same assumption in (1), if X^ω is sequential, then X is countably bi-sequential.*

Proof: For (1), let X have property (P). Then X contains no closed copy of S_ω and no S_2. Thus, X is countably bi-sequential by [23, Theorem 1.5 & Theorem 3.1]. For (2), in view of [21, Theorem 4.13], for each $x \in X$ and each decreasing sequence (A_n) with $x \in cl(A_n - \{x\})$, there exist $x_n \in A_n$ such that $\{x_n : n \in N\}$ is not closed in X. Then X has property (P), for X is a quasi-k-space. (3) holds by (1) and (2). □

Remark 2.8. (1) In Proposition 2.7, the assumption in (1) is essential. Indeed, for the compact sequential, non-Fréchet space Ψ^* in [5, Example 7.1], $\Psi^{*\omega}$ is sequential by Lemma 2.14(1) below.

(2) In Proposition 2.7(2), "$t(X) \leq \omega$" is essential. Indeed, under (CH), there exists a k'-space X with a countable HCP (determining) cover by compact sets such that X^ω is a k-space, but X is not locally compact [1]; hence, X doesn't have property (P) by Theorem 2.10(1) below.

Lemma 2.9. *Let $\mathcal{P}$ be a point-countable cover of X with $t(X) \leq \omega$. Then the following are equivalent.*

(a) *$[\mathcal{P}]^\circ$ is an open cover of X.*

(b) *For each countably compact set K of X, $K \subset P$ for some $P \in [\mathcal{P}]^\circ$.*

(c) *For each decreasing sequence (A_n) in X with $\bigcap\{clA_n : n \in N\} \neq \emptyset$, there exists $P \in \mathcal{P}$ with $P \cap A_n \neq \emptyset$ for all $n \in N$.*

Proof: Obviously, (b) $\rightarrow$ (a) $\rightarrow$ (c) holds. To see that (c) $\rightarrow$ (b) holds, suppose (b) doesn't hold for some countably compact set K. For each countable set C, let $\{P \in \mathcal{P} : P \cap C \neq \emptyset\} = \{P_i(C) : i \in N\}$. Since $t(X) \leq \omega$, by induction, there exist points x_n and countable sets C_n $(n \in N)$ such that $x_n \in K \cap clC_n$, but

$C_n \cap P_i(C_j) = \emptyset$ if $i, j < n$. The sequence $\{x_n : n \in N\}$ in K has an accumulation point x. Let $A_n = \bigcup\{C_k : k \geq n\}$, then (A_n) is a decreasing sequence with $x \in clA_n$. Then, by (c), there exists $P \in \mathcal{P}$ such that P meets infinitely many C_n, a contradiction. □

Theorem 2.10. (1) *Let X have property (P), and let $\mathcal{P}$ be a determining cover of X. Then $[\mathcal{P}]^\circ$ is an open cover of X (in fact, (b) in Lemma 2.9 holds) if (a), (b), or (c) below holds.*

(a) *$\mathcal{P}$ is a countable cover.*

(b) *$\mathcal{P}$ is a point-countable cover, and $t(X) \leq \omega$.*

(c) *$\mathcal{P}$ is a dominating cover, and $t(X) \leq \omega$.*

(2) *Let X be a countably bi-quasi-k-space, and let $\mathcal{P}$ be a determining cover of X. Then $[\mathcal{P}]^\circ$ is an open cover of X if $\mathcal{P}$ is a dominating or point-countable closed cover of X.*

Proof: For (1), since X has property (P), (c), and therefore (b), in Lemma 2.9 holds. Here, for (a) and (b) in (1), use Fact 1.3(1). For (c) in (1), let $\mathcal{P} = \{X_\alpha : \alpha \leq \gamma\}$ be a dominating cover of X, and let $L_\alpha = X_\alpha - \bigcup\{X_\beta : \beta < \alpha\}$ for each $\alpha \leq \gamma$. Since X has property (P), $\mathcal{F} = \{clL_\alpha : \alpha \leq \gamma\}$ is a point-finite determining closed cover of X. Indeed, suppose not for some $x \in X$. Then $x \in clL_{\alpha_i}$ for some α_i ($i \in N$), and let $A_n = \bigcup\{L_{\alpha_i} : i \geq n\}$. Then (A_n) is a decreasing sequence with $x \in clA_n$. By property (P), some countably compact set meets infinitely many L_{α_i}, a contradiction to Fact 1.3(2). Thus, (c) is reduced to (b). For (2), X is countably bi-quasi-k, and the point-countable determining cover $\mathcal{P}$ or $\mathcal{F}$ (in the above) is closed; therefore, (2) holds by means of Fact 1.3(1). □

Theorem 2.11. (1) *Let X^ω be a sequential space. Let $\mathcal{P}$ be a determining cover of X, and let $\mathcal{P}^* = \{P \cup F : P \in \mathcal{P}, F \text{ is finite}\}$. Then $\mathcal{P}^{*\omega}$ (hence, $[\mathcal{P}]^\omega$) is a determining cover of X^ω.*

(2) *Let X^ω be a quasi-k-space. Let $\mathcal{P}$ be a dominating or point-countable determining cover of X. Then $[\mathcal{P}]^\omega$ is a determining cover of X^ω. When $t(X) \leq \omega$, $[\mathcal{P}]^{\circ\omega}$ is a determining cover of X^ω.*

Proof: For (1), X^ω has a determining cover $\{\Pi C_i : C_i \text{ is compact metric in } X\}$ by Fact 1.4(1). Since the C_i are metric and closed in X, each element ΠC_i has a determining cover $\{\Pi(C_i \cap P_i) : P_i \in \mathcal{P}^*\}$ by [20, Lemma 3.3] and Fact 1.2(3). Thus, $\mathcal{P}^{*\omega}$ is a determining cover of X^ω by Fact 1.2(1). Similarly for (2), the first half holds, using Fact 1.3. The latter part holds by Proposition 2.7(2) and Theorem 2.10(1). □

Remark 2.12. Let X be a space with a countable determining closed cover $\mathcal{F}$ by first countable and locally compact sets. Then, by Corollary 2.5, any finite product X^n is a k-space with a determining cover $\mathcal{F}^n$. But by Theorem 2.10(1) with Proposition 1.5(1) and Proposition 2.7(2), X^ω is not a quasi-k-space and $[\mathcal{F}]^\omega$ is not a determining cover of X^ω if X is not locally compact,

In view of theorems 2.2, 2.3, and 2.11, we pose the following.

Question 2.13. (1) *Let X_1 be a locally compact space, or let $X_1 \times X_2$ be a k-space. For a determining cover $\mathcal{P}_i$ of X_i $(i = 1, 2)$, is $\mathcal{P}_1 \times \mathcal{P}_2$ a determining cover of $X_1 \times X_2$?*

(2) *Same as (1), but replace "determining" by "dominating" twice.*

(3) *Let X^ω be a k-space. For a determining closed cover $\mathcal{F}$ of X, is $[\mathcal{F}]^\omega$ a determining cover of X^ω ?*

Following [11], a space X is a *bi-k-space* if, whenever $\mathcal{A}$ is a filterbase with $x \in clA$ for every $A \in \mathcal{A}$, there exists a k-sequence (B_n) with $x \in cl(A \cap B_n)$ for all $A \in \mathcal{A}$ (in particular, X is *bi-sequential* if, moreover $\bigcap\{B_n : n \in N\} = \{x\}$), and a space X is a *bi-quasi-k-space* (= *bi-k-space* [14]) if we replace "k-sequence" by "q-sequence." Every bi-k-space (respectively, bi-sequential space; bi-quasi-k-space) is characterized as an image of a paracompact M-space [11] (respectively, metric space [11]; M-space [14]) under a bi-quotient map. Here, a map $f : X \to Y$ is *bi-quotient* if whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open sets, then finitely many $f(U)$ with $U \in \mathcal{U}$ cover a nbd of y in Y [10].

Lemma 2.14. (1) *Every countable product of countably compact sequential spaces is (countably compact) sequential* [15].

(2) *Every countable product of M-and-k-spaces (respectively, countably compact k-spaces) is an M-space (respectively, countably compact space); see, for example,* [19].

Theorem 2.15. (1) *Let X be a sequential (respectively, paracompact) space. Then X^ω is a quasi-k-space iff it is a sequential space (respectively, k-space).*

(2) *If* X *is a bi-*k*-space, then so is* X^ω [11].

(3) *If* X *is a sequential bi-quasi-*k*-space, then so is* X^ω.

Proof: For the "only if" part in (1), since X^ω is a quasi-k-space, it has a determining cover $\mathcal{P} = \{\Pi K_i : K_i$ is countably compact in $X\}$ by Fact 1.4. Since X is sequential, any countably compact set K of X is closed in X, because $K \cap C$ is closed in C for each compact metric set in X. Then the countably compact sets K_i are closed in X; hence, these are sequential. Thus, each element ΠK_i in $\mathcal{P}$ is sequential by Lemma 2.14(1). Then $\mathcal{P}$ consists of sequential spaces. Thus, X^ω is sequential by Proposition 1.5(1). For the parenthetic part, since X is paracompact, clK_i are also countably compact, hence compact. Then X^ω has a determining cover $\{clP : P \in \mathcal{P}\}$ by compact sets. Thus, X^ω is a k-space. For (3), since X is a bi-quasi-k-space, by [11, Theorem 3.F.3], X is a bi-quotient image of an M-space S, where $S \subset X \times M$ for some metric space M. But X is sequential countably bi-quasi-k and M is metric, and thus, $X \times M$ is a k-space by [27, Corollary 7]; therefore, $X \times M$ is sequential by (1). S is an M-space, hence a quasi-k-space by [14]; therefore, S has a determining cover by countably compact sets. But countably compact sets of the sequential space $X \times M$ are closed, hence sequential. Thus, S is sequential. Since S is sequential, S^ω is an M-space by Lemma 2.14(2); hence, S^ω is a quasi-k-space. Thus, S^ω is sequential by (1). Hence, S^ω is a sequential M-space. Any product of bi-quotient maps is bi-quotient [10]; thus, X^ω is a bi-quotient image of a sequential M-space S^ω. Therefore, X^ω is a sequential bi-quasi-k-space. □

The same result as Theorem 2.15 holds, replacing "X^ω" by "ΠX_i," and similar assertions would be valid in some other results.

For a sequential space X, the author knows no necessary and sufficient conditions on X for X^ω to be sequential (but if X has certain properties, X is a bi-quasi-k-space as a necessary and sufficient condition; see [31]). A necessary condition is given in Proposition 2.7(3) (or (2)), and so is a sufficient condition in Theorem 2.15(3). But these conditions are not necessary and sufficient in view of (2) and (3) in Example 2.16 below. Also, (1) and (2) are related to Lemma 2.14 or Theorem 2.15. (1) is due to [11, Example 10.7]. For (2), let X be the disjoint union of the countable spaces Y and Z in [17, Example 6.6] such that $X \times Y$ is not a k-space, hence not a quasi-k-space. For (3), let X be the countable space X in [16, Theorem 3.5] such that X^ω is countably bi-sequential, but X is not bi-sequential, hence not bi-quasi-k.

Example 2.16. (1) There exists a countably compact space X, but X^2 is not an M-space, not even a quasi-k-space.

(2) ($2^\omega < 2^{\omega_1}$). There exists a countable, countably bi-sequential space X, but X^2 is not a quasi-k-space.

(3) (CH). There exists a countable space X such that X^ω is countably bi-sequential, but X is not bi-quasi-k.

The following holds by means of theorems 2.2, 2.10, 2.11, and 2.15.

Corollary 2.17. (1) *Let X be a sequential bi-quasi-k-space. For a determining cover $\mathcal{P}$ of X, $\mathcal{P}^2$ (respectively, $[\mathcal{P}]^\omega$) is a determining cover of X^2 (respectively, X^ω).*

(2) *Let X be a bi-k-space, and let $\mathcal{P}$ be a determining cover of X. Then $\mathcal{P}^2$ is a determining cover of X^2 if $\mathcal{P}$ is a closed cover. Also, $[\mathcal{P}]^{\circ\omega}$ (respectively, $[\mathcal{P}]^\omega$) is a determining cover of X^ω if $\mathcal{P}$ is a dominating or point-countable closed cover, or a point-countable cover with $t(X) \leq \omega$ (respectively, $\mathcal{P}$ is a point-countable cover).*

Finally, let us pose the following question.

Question 2.18. (1) *For a k-space X, let X^2 be a quasi-k-space; in particular, let X be countably compact. Is X^2 a k-space ?*

(2) *For a k-space X, let X be a bi-quasi-k-space. Is X^2 a bi-quasi-k-space or a k-space (or quasi-k-space)?*

(1) is posed in [29]. When X is sequential, (1) and (2) are positive by Theorem 2.15. Related to (1) and (2), see Lemma 2.14(2) or Example 2.16(1).

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