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CONCERNING PRESERVATION OF INDECOMPOSABILITY UPON TAKING A PREIMAGE UNDER $z \mapsto z^n$

R. PATRICK VERNON

Abstract. In 2005, David P. Bellamy (*Certain analytic preimages of pseudocircles are pseudocircles*, Topology Proc. **29** (2005), no. 1, 19–25) proved that if X is a pseudo-circle in the complex plane which separates 0 from ∞, and if $n \in Z^+$, then the preimage of X under $z \mapsto z^n$ is also a pseudo-circle. He ended his paper with two questions. The first question asks whether the preimage under $z \mapsto z^n$ of a hereditarily indecomposable continuum which is irreducible with respect to separating 0 from ∞ is necessarily hereditarily indecomposable. The second question asks whether the preimage under $z \mapsto z^n$ of a continuum which properly contains a pseudocircle can ever be hereditarily indecomposable. In this paper, the author provides affirmative answers to both questions. In addition, the author explores the general behavior of indecomposability and hereditary indecomposability under the operation of taking preimages under $z \mapsto z^n$.

1. Introduction

Let $\overline{\mathbb{C}}$ denote the one-point compactification of the complex plane, and let ∞ denote the point of $\overline{\mathbb{C}}$ which is not in the complex plane. A *continuum* is a compact, connected, nonempty metric space. Except where stated otherwise, this paper will discuss continua which

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are subsets of the complex plane. A continuum X *separates* 0 *from* ∞ if 0 and ∞ belong to different complementary domains of X; that is, if 0 and ∞ are in different components of $\overline{\mathbb{C}} \setminus X$. For each $n \in \mathbb{N}$ such that $n > 1$, let $\phi_n \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the rotation of $\overline{\mathbb{C}}$ by $\frac{2\pi}{n}$ radians, and let $f_n \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be defined by $f_n(z) = z^n$. Define f_n to be *trivial* over a continuum X if $f_n^{-1}(X) = X_1 \cup \cdots \cup X_n$, a disjoint union of n continua such that $f_n|_{X_i} \colon X_i \to X$ is a homeomorphism for $i = 1, \ldots, n$.

Theorem 1. *Let X be a continuum with* 0 *in its unbounded complementary domain. Then f_n is trivial over X.*

Proof: Suppose 0 is in the unbounded complementary domain of X. Let δ be an arc from 0 to ∞ that does not intersect X. Then $f_n^{-1}(\delta)$ separates the plane into n open sets S_i, and $f_n|_{S_i} \colon S_i \to (\overline{\mathbb{C}} \setminus \delta)$ is a homeomorphism for each i. Then each S_i contains exactly one copy X_i of X such that X_i is homeomorphic to X via f_n. □

Theorem 2. *Let X be a continuum which separates* 0 *from* ∞*. Then $f_n^{-1}(X)$ is a continuum.*

This was proven by David P. Bellamy in [1].

2. First Question

Bellamy showed that the preimage under $z \mapsto z^n$ of a pseudo-circle which separates 0 from ∞ is again a pseudo-circle [1]. One property used in the proof was the fact that every proper subcontinuum of a pseudo-circle which separates 0 from ∞ is a pseudo-arc with 0 in its unbounded complementary domain. Bellamy asked whether this can be generalized in the following fashion: If X is a hereditarily indecomposable continuum which is irreducible with respect to separating 0 from ∞, is $f_n^{-1}(X)$ necessarily hereditarily indecomposable?

Before beginning, we make a few notes.

Lemma 1. *If X is a decomposable continuum, then $f_n^{-1}(X)$ is decomposable.*

Proof: Suppose $X = A \cup B$, each proper subcontinuum of X. Then $f_n^{-1}(A)$ is a continuum if A contains 0 or separates 0 from ∞, and $f_n^{-1}(A)$ is the disjoint union of n continua otherwise. Similarly, $f_n^{-1}(B)$ is either a continuum or the disjoint union of n continua. Then $f_n^{-1}(X)$ can be written as the union of 2, $n+1$, or $2n$ proper subcontinua, and is thus decomposable. □

Corollary 1. *If X is not hereditarily indecomposable, then $f_n^{-1}(X)$ is not hereditarily indecomposable.*

We will give examples later in the paper to show that the converses of the above statements are not true. However, we would like to see under what conditions $f_n^{-1}(X)$ can be (hereditarily) indecomposable.

Definition 1. *Define f_n to be n-crisp over a continuum X if*

(1) *$f_n^{-1}(X)$ is a continuum, and*
(2) *for any proper subcontinuum X' of X, we have f_n is trivial over X'.*

The notion of an n-crisp map is based on Jo Heath's original concept of a "crisp" 2-to-1 map [3].

Lemma 2. *Let X be a continuum which separates 0 from ∞. If f_n is an n-crisp map, then*

(1) *if $f_n^{-1}(X) = A \cup B$, two continua, then one of them maps onto X under f_n;*
(2) *if C is a proper subcontinuum of $f_n^{-1}(X)$ that maps onto X via f_n, then $f_n^{-1}(X) = \bigcup_{i=0}^{n-1} \phi_n^i(C)$, each a proper subcontinuum of $f_n^{-1}(X)$.*

Proof: (1) If $f_n(A)$ is a proper subcontinuum of X, then $f_n^{-1}(f_n(A))$ is the union of n disjoint homeomorphic continua $\phi_n^i(A)$, $i = 1, \ldots, n$. Then $\phi_n(A)$ is contained in B, so $f_n(B)$ contains $f_n(\phi_n(A)) = f_n(A)$. Hence, $f_n(B) = X$.

(2) Each $\phi_n^i(C)$ is a continuum since ϕ_n is a homeomorphism. Let $x \in f_n^{-1}(X) \setminus C$. Then $\phi_n^i(x) \in f_n^{-1}(X) \setminus \phi_n^i(C)$, so each $\phi_n^i(C)$ is a proper subset of $f_n^{-1}(X)$. If $f_n(C) = X$, then every point of $f_n^{-1}(X)$ is in at least one of the $\phi_n^i(C)$. □

Theorem 3. *Let Y be a hereditarily indecomposable continuum which separates 0 from ∞, and let X be a continuum such that $f_n(X) = Y$. Then X separates 0 from ∞.*

Proof: Let $\hat{Y}$ be the union of Y with all of its complementary domains which do not contain 0 or ∞. Let $\hat{X}$ be the union of X with all of its complementary domains which do not contain 0 or ∞. Let $i_1 \colon Y \to \hat{Y}$ be the inclusion map of Y into $\hat{Y}$, and let $i_2 \colon X \to \hat{X}$ be the inclusion map of X into $\hat{X}$. Let $f_{n,1} = f_n|_X$ and let $f_{n,2} = f_n|_{\hat{X}}$. Then $f_{n,1} = f_{n,2}|_X$. A. Lelek has shown in [5] that since $f_{n,1}$ is confluent, $f^*_{n,1} \colon H^1(Y) \to H^1(X)$ is a monomorphism, where H^1 denotes the first Čech cohomology.

By Alexander duality, $H^1(\hat{Y}) \cong \tilde{H}_0(S^2 \setminus \hat{Y}) \cong \mathbb{Z}$, since $S^2 \setminus \hat{Y}$ has two components. Also, $H^1(\hat{Y}, Y) \cong 0$ since $\hat{Y}/Y$ is homotopy equivalent to the wedge of some number of copies of S^2. Then by exactness, $i^*_1 \colon H^1(\hat{Y}) \to H^1(Y)$ is a monomorphism. But $i^*_2 \circ f^*_{n,2} = f^*_{n,1} \circ i^*_1$. Thus, $H^1(\hat{X}) \ncong 0$, so $\hat{X}$ separates the plane. Then X separates 0 from ∞. □

Theorem 4. *Let X be a continuum which is irreducible with respect to separating 0 from ∞. Then $f_n^{-1}(X)$ is also irreducible with respect to separating 0 from ∞.*

Proof: Suppose that there exists a proper subcontinuum C of $f_n^{-1}(X)$ which separates 0 from ∞. Then $f_n(C)$ is a subcontinuum of X which separates 0 from ∞. Let p be a point in $f_n^{-1}(X) \setminus C$, and let ϵ be the distance from p to C. Let V be a ball of radius $\epsilon/2$ about p. Then either V is in the unbounded complement of C, or V is in a bounded complement of C. Then we have either that no point in V is accessible from 0 or that no point in V is accessible from ∞. By symmetry, it follows that either no point in $\bigcup_{i=0}^{n-1} \phi_n^i(V)$ is accessible from 0 or no such point is accessible from ∞. Then either no point in $f_n(V)$ is accessible from 0 or no point in $f_n(V)$ is accessible from ∞. Let X_1 and X_2 be the closures of the sets of points in X which are accessible from 0 and ∞, respectively. Then V cannot be contained in both of these subcontinua, so either $X_1 \neq X$ or $X_2 \neq X$. But X_1 and X_2 both separate 0 from ∞ by their construction. So there exists a proper subcontinuum of X which separates 0 from ∞, contradicting our initial assumption. Thus, $f_n^{-1}(X)$ is irreducible with respect to separating 0 from ∞. □

Lemma 3. *Let X be a continuum which is irreducible with respect to separating 0 from ∞. Then f_n is n-crisp over X.*

Proof: Since X separates 0 from ∞, we know by Theorem 2 that $f_n^{-1}(X)$ is a continuum. If X' is a proper subcontinuum of X, then 0 is in the unbounded complementary domain of X', so by Theorem 1, it follows that f_n is trivial over X'. Then by Definition 1, we have that f_n is n-crisp over X. □

Theorem 5. *Let X be a hereditarily indecomposable continuum which is irreducible with respect to separating 0 from ∞. Then $f_n^{-1}(X)$ is hereditarily indecomposable.*

Proof: Suppose C is a subcontinuum of $f_n^{-1}(X)$ which maps onto X via f. By Theorem 3, we have that C separates 0 from ∞, and thus, by Theorem 4, we have that $C = f_n^{-1}(X)$. But f_n is n-crisp over X by Lemma 3, so by Lemma 2, we have that $f_n^{-1}(X)$ is indecomposable. Moreover, if C is a proper subcontinuum of $f_n^{-1}(X)$, then by the above argument, $f_n(C)$ is a proper subcontinuum of X. Then C is homeomorphic to $f_n(C)$, so C is indecomposable since X is hereditarily indecomposable. Thus, $f_n^{-1}(X)$ is hereditarily indecomposable. □

Theorem 5 tells us that the answer to Bellamy's first question is affirmative. Before answering Bellamy's second question, we will construct a continuum which will help us in obtaining our results.

3. A Symmetric Pseudo-arc

A *chain* $C = \{C_1, \ldots, C_n\}$ is a nonempty finite indexed collection of open sets C_j such that

$$\overline{C_j} \cap \overline{C_k} \neq \emptyset \iff |j-k| \leq 1.$$

Each C_j is called a *link* of C. The *length* of C is the number of links in C. The first and last links are called the *end-links* of C. A chain C is a *chain from p to q* if p and q are each in a different end-link of C. The *mesh* of a chain is the supremum of the diameters of the links in the chain. A continuum X is called *chainable* if for any $\epsilon > 0$ there exists a chain of mesh less than ϵ covering X. A point $p \in X$ is called an *endpoint of X* if for any $\epsilon > 0$ we can find an ϵ-chain covering X in which p is in an end-link. A *pseudo-arc* is a hereditarily indecomposable chainable continuum, and a *pseudo-circle* is a hereditarily indecomposable circularly chainable continuum which is not chainable.

One method of constructing a pseudo-arc uses the notion of crooked chains [2]. A chain E *refines* a chain D if for each link e_i in E, the closure of e_i is a subset of a link of D. If the chain $E = \{e_1, e_2, \ldots, e_n\}$ refines the chain $D = \{d_1, \ldots, d_m\}$, E is called *crooked* in D provided that if $k - h > 2$ and e_i and e_j are links of E in links d_h and d_k of D, respectively, then there are links e_r and e_s of E in links d_{k-1} and d_{h+1}, respectively, such that either $i > r > s > j$ or $i < r < s < j$.

We can construct a pseudo-arc using a sequence of chains C_i such that C_{i+1} is crooked in C_i for each $i \in \mathbb{N}$, with the sequence of meshes of the C_i going to 0 as i goes to ∞.

Define a chain C to be *symmetric* if for each link $C_i \in C$, we have $\phi_2(C_i)$ is also a link of C. Define a continuum X to be *symmetric* if $\phi_2(X) = X$. Define a map $g\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to be *symmetric* if $g(-z) = -g(z)$ for every $z \in \overline{\mathbb{C}}$.

Lemma 4. *If n is an odd positive integer, and C is a symmetric chain from p to $-p$ of length n with round balls for links, then for any $\epsilon > 0$, we can create a symmetric ϵ-chain C' from p to $-p$ of odd length, with round balls for links, which is crooked in C.*

Proof: For $n = 1$, let C_1 be the sole symmetric convex link of C containing 0. Let α be a straight line segment whose endpoints are in the boundary of C_1, which contains the origin, and which does not contain p or $-p$. Then α separates C_1 into two components C_1^a and C_1^b. Let D be an ϵ-ball around p which is contained in C_1 and does not intersect the origin, and let E be an ϵ-ball around the origin which is contained in C_1 and does not intersect D. Let C'_a be a chain of ϵ-balls whose first link is E, last link is D, and every other link is contained in $C_1 \setminus -D$. Let C' be the chain whose end links are D and $-D$ and every other link is a link of C' or $-C'$. Then C' is a symmetric ϵ-chain from p to $-p$. Notice also that if k is the number of links of C'_a, then C' has $2k - 1$ links, so has odd length. For $n \geq 3$, notice that any chain that is crooked in a chain of length n can be broken into three chains: a chain that is crooked in a chain of length $n - 1$, one that is crooked in a chain of length $n - 2$, and a final one that is crooked in a chain of length $n - 1$.

Suppose our hypothesis is true for all odd integers less than n, and consider now $C = \{C_1, \ldots, C_n\}$, a symmetric chain of length n satisfying all of our requirements. Then $C(2, n-1)$ is a symmetric chain of length $n-2$, so we form a symmetric chain C_a that is crooked inside this chain, is of odd length, and has end-links in C_2 and C_{n-1}. Suppose $p \in C_1$. Let L_1 be the end-link of C_a contained in C_2.

The boundaries of C_a and C are two disjoint simple closed curves. Let B be the bounded region whose boundary is the union of these two curves. Then B is a space which is homeomorphic to an annulus. Let α_1 be an arc from the boundary of L_1 to the boundary of C_2 which is contained in $\overline{C_2}$. Then $\alpha \cup -\alpha$ splits B into two components B_1 and B_2. Let C_b be a chain of round balls which is crooked in $C(2,n)$ whose first link intersects L_1, each subsequent link is contained in $B_1 \setminus (\alpha \cup -\alpha)$, and whose last link is in C_n. Let C_d be the rotation of C_b by π radians. Let C' be the maximal chain whose links are links of C_a, C_b, or C_d. Then C' is crooked in C and symmetric. □

Corollary 2. *There exists a symmetric pseudo-arc in the complex plane.*

Proof: Let $\{C_1, C_2, \dots\}$ be a sequence of symmetric chains from -1 to 1 such that each chain is crooked in the previous chain, each chain is of odd length so that we may apply Lemma 4, with the meshes of the chains approaching 0. Let $X = \bigcap_{i=1}^{\infty}(\bigcup_{j=1}^{n_i} \overline{C}_{i,j})$. Then X is a pseudo-arc. □

Lemma 5. *If X is a pseudo-arc as constructed above, then $f_2(X)$ is also a pseudo-arc.*

Proof: The image of each chain C_i is a chain, since by symmetry of C_i we can identify each link $C_{i,j}$ with its partner $-C_{i,j}$. So $f_2(X)$ is chainable. Furthermore, since $X = f_2^{-1}(f_2(X))$, by Corollary 1 we know that $f_2(X)$ is hereditarily indecomposable. So $f_2(X)$ is hereditarily indecomposable and chainable; hence, $f_2(X)$ is a pseudo-arc. □

Theorem 6. *There exists a chainable continuum containing* 0 *whose preimage under $z \mapsto z^2$ is hereditarily indecomposable.*

Proof: Letting X be our symmetric pseudo-arc and $Y = f_2(X)$, we see that $X = f_2^{-1}(Y)$ and so $f_2^{-1}(Y)$ is hereditarily indecomposable. □

We now have an example of a hereditarily indecomposable continuum containing 0 whose preimage under $z \mapsto z^2$ is also hereditarily indecomposable. We will use this in the next section.

4. Second question

Bellamy's second question concerns the converse of the first question. If X is a hereditarily indecomposable continuum which properly contains a pseudo-circle, can $f_n^{-1}(X)$ be hereditarily indecomposable? In this section, we provide an example of such a continuum.

Given a continuum X, define a point $p \in X$ to be *accessible* in X if there exists an arc α with p as an endpoint such that $\alpha \cap X = \{p\}$. If p is not accessible in X, then p is *inaccessible* in X. If p is inaccessible in every subcontinuum $X' \subset X$, then p is *hereditarily inaccessible* in X.

Lemma 6. *If X is a symmetric indecomposable continuum which does not separate $\overline{\mathbb{C}}$, then 0 is inaccessible in X.*

Proof: Suppose 0 is accessible in X. Let α be an arc from 0 to ∞ which only intersects X at 0. Then $\alpha \cup \phi_2(\alpha)$ separates X, showing that 0 is a cut point of X. Then X is decomposable, a contradiction. So 0 is inaccessible in X. □

Corollary 3. *If X is a symmetric hereditarily indecomposable continuum which does not separate $\overline{\mathbb{C}}$, then 0 is hereditarily inaccessible in X.*

One method of constructing a hereditarily indecomposable continuum which properly contains a pseudo-circle can be attributed to R. H. Bing in [2]. To do so, let M_1 be a pseudo-arc in $\overline{\mathbb{C}}$ and let p be a hereditarily inaccessible point in M_1. Let M_2 be a pseudo-circle in $\overline{\mathbb{C}}$ and let T be a map which takes M_2 and its bounded complementary domain B to the point p, such that T is one-to-one elsewhere. Then $M_3 = T^{-1}(M_1) \setminus B$ is a hereditarily indecomposable continuum which properly contains a pseudo-circle.

We will use a variation of this method which will provide us with such a continuum, with the added feature that our continuum will be symmetric.

Theorem 7. *For any symmetric pseudo-circle C separating 0 from ∞, there exists a symmetric decomposition map taking C and its bounded complementary domain to 0, and which is one-to-one elsewhere.*

Proof: Let C be a symmetric pseudo-circle in $\overline{\mathbb{C}}$ which separates 0 from ∞ and let B be the bounded complementary domain of C. Let $T_1\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ take $C \cup B$ to 0 and be one-to-one elsewhere. Let $\{x_1, -x_1\}$ be the leftmost and rightmost points on the real axis which are elements of C. Let A be the arc along the real axis containing ∞ whose endpoints are x_1 and $-x_1$. Since T_1 is one-to-one on $\overline{\mathbb{C}} \setminus (B \cup C)$, then any subcontinuum of $A \setminus (B \cup C)$ is homeomorphic via T_1 to its image and is therefore an arc. Furthermore, for any sequence $\{x_i\}_{i=1}^{\infty}$ in A converging to a point in $B \cup C$, we have $\{T_1(x_i)\}_{i=1}^{\infty}$ converges to 0. Then $T_1(A)$ is a simple closed curve containing 0 and ∞. Let α_1 and α_2 be the pair of arcs in $T_1(A)$ whose endpoints are 0 and ∞ such that $T_1(A) = \alpha_1 \cup \alpha_2$. Let C_1, C_2 denote the complementary domains of $T(A)$. The boundary of each of these spaces is $T(A)$. Let $T_2\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a homeomorphism satisfying $T_2(T_1(-x)) = -T_2(T_1(x))$ for all $x \in A$, and taking $T_1(A)$ to the real axis. Define $T\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as follows:
(1) If $z \in A$, let $T(z) = T_2(T_1(z))$.
(2) If $T_1(z) \in C_1$, let $T(z) = T_2(T_1(z))$.
(2) If $T_1(z) \in C_2$ (then $T_1(-z) \in C_1$), let $T(z) = -T_2(T_1(-z))$.

Then T is a decomposition map taking $B \cup C$ to 0, T is one-to-one elsewhere, and $T(-z) = -T(z)$ for all $z \in \overline{\mathbb{C}}$. □

Theorem 8. *There exists a continuum which properly contains a pseudo-circle whose preimage is hereditarily indecomposable.*

Proof: Let $f_2(z) = z^2$, and let X be the symmetric pseudo-arc constructed earlier. By Corollary 3, we know that 0 is hereditarily inaccessible in X. Let X_1 be a symmetric pseudo-circle separating 0 from ∞. This can be constructed by taking the preimage under $z \mapsto z^2$ of any pseudo-circle which separates 0 from ∞. By Bellamy [1], this is also a pseudo-circle. Let T be a symmetric decomposition map as constructed in Theorem 7. Then $X_2 = T^{-1}(X)$ is a symmetric hereditarily indecomposable continuum which properly contains a pseudo-circle. Then $Y = f_2(X_2)$ is hereditarily indecomposable and properly contains a pseudo-circle, and $f_2^{-1}(Y) = X_2$, so the preimage of Y under f_2 is hereditarily indecomposable. □

We have now answered both of Bellamy's questions in the affirmative. However, in doing so we have uncovered several new questions. In what ways can we generalize our existing results? The remaining section will provide extensions to our current findings.

5. Composants and accessibility

5.1. Accessibility of 0

Let p be a point in a continuum X. The *composant of* p *in* X, denoted $C_p(X)$, is the union of all proper subcontinua of X containing p. Every indecomposable continuum contains uncountably many composants, any two of which are disjoint. We say p is *accessible* if there exists an arc in $\overline{\mathbb{C}}$ which has p as an endpoint and every other point is contained in the complement of X. We say that p is *weakly accessible* if there exists an arc in $\overline{\mathbb{C}}$ which has p as an endpoint and every other point is contained in the complement of $C_p(X)$. A composant C is *accessible* if it contains an accessible point. A composant C is *weakly accessible* if it contains a weakly accessible point. C is a K*-composant*, denoted $C \in K$, if there is a continuum D such that $D \setminus X \neq \emptyset$, $D \cap C \neq \emptyset$, and $D \cap X$ is a proper subcontinuum of X. C is a Z*-composant*, denoted $C \in Z$, if there is a continuum D such that $D \setminus X \neq \emptyset$, $D \cap C \neq \emptyset$, and $D \cap C$ is a compact set. A composant C is called *external* if there is a continuum D intersecting C but not all other composants of X. Otherwise, C is called *internal*. More information on these types of composants can be found in [4].

Lemma 7. *If* X *is a nonseparating continuum such that* $0 \in X$ *is accessible, then* $f_n^{-1}(X)$ *is decomposable.*

Proof: If 0 is accessible, let α be an arc from 0 to ∞ that only intersects X at 0. Then $f_n^{-1}(\alpha)$ shows that 0 is a separation point in $f_n^{-1}(X)$. Then $f_n^{-1}(X)$ is decomposable. □

Corollary 4. *If* X *is a nonseparating continuum such that* $0 \in X$ *is not hereditarily inaccessible, then* $f_n^{-1}(X)$ *is not hereditarily indecomposable.*

5.2. The Role of Nonseparation

In the above and in future theorems, we require that X be nonseparating. When considering separating continua containing 0, certain problems may arise. An example is given below.

Let X be a symmetric pseudo-circle such that for any $\epsilon > 0$, X can be covered by an ϵ-circular-chain with convex links. Let p_1 be a point in X such that there exists a straight line segment from 0 to p_1 which only intersects X at p_1. Let T be a symmetric decomposition taking the straight arc whose endpoints are p_1 and $-p_1$ to 0, with T being one-to-one elsewhere. Then for any $\epsilon > 0$, we have that $T(X)$ can be covered by ϵ-graph-chains with connected links such that the nerve of each graph chain is a figure eight. Also, 0 is accessible in $T(X)$. To see this, let α be an arc in the bounded complement of X which has 0 as an endpoint and does not intersect the line from p_1 to $-p_1$ anywhere else. Then $T(\alpha)$ is an arc with 0 as an endpoint, which does not intersect $T(X)$ anywhere else. Then 0 is also accessible in $Y = f_2(T(X))$, but from its bounded complement.

Suppose $T(X)$ is decomposable. Let $T(X) = A \cup B$, both proper subcontinua of $T(X)$. If $0 \in A$ and $T^{-1}(A)$ is a continuum, then either X is a decomposable continuum or $T^{-1}(A) \cup \phi_2(T^{-1}(A))$ is a proper subcontinuum of X which separates the plane, both contradictions. So if $0 \in A$, then $T^{-1}(A) = A_1 \cup A_2$, the disjoint union of two continua. If $0 \notin A$, then $T^{-1}(A) = A_1$. Similarly, $T^{-1}(B)$ is either a continuum or the disjoint union of two continua. Furthermore, at least one of A and B must contain 0. Then X is the union of either three or four continua, contradicting our assumption that X is indecomposable. So $T(X)$ is indecomposable. Then by Theorem 1, we have that Y is indecomposable. Then we have a continuum Y in which 0 is accessible, but $f_2^{-1}(Y)$ is also indecomposable.

We may also note that Y is a pseudo-circle. To see this, first notice that Y is circularly chainable, as we can take the image under f_2 of any symmetric figure eight chain covering $T(X)$ to give a circular chain covering Y. Next, suppose $Y' \subset Y$ is a proper subcontinuum. If $0 \notin Y'$, then $f_2^{-1}(Y') = Y_1 \cup Y_2$, where $Y_1 \cong Y'$ and $0 \notin Y_1$. But then $T^{-1}(Y_1) \cong Y'$, and since X is a pseudo-circle, we have that $T^{-1}(Y_1)$ is a pseudo-arc. So Y' is a pseudo-arc. If

$0 \in Y'$, then $f_2^{-1}(Y')$ is a proper subcontinuum of $T(X)$ containing 0, so $T^{-1}(f_2^{-1}(Y'))$ is the disjoint union of two pseudo-arcs in X, each one being the rotation of the other by π radians. Then $f_2^{-1}(Y')$ is the one-point union of two pseudo-arcs in $T(X)$, each of which maps homeomorphically via f_2 onto Y'. So Y' is a pseudo-arc. Then Y is hereditarily indecomposable. Finally, Y is not chainable since it separates the plane. Thus, Y is a pseudo-circle.

In this example, although $f_2^{-1}(Y)$ is indecomposable, it is not hereditarily indecomposable. If 0 is accessible in a continuum X, this will always be the case. Let α be an arc with 0 as an endpoint such that $\alpha \cap X = \{0\}$. Then we may take a neighborhood about 0 of diameter less than the diameter of $f_2^{-1}(\alpha)$. Then the component of $f_2^{-1}(X)$ containing 0 contained in the closure of this neighborhood would be decomposable, since $f_2^{-1}(\alpha)$ separates it and only intersects it at one point.

5.3. Accessibility of $C_0(X)$

Theorem 7 and Corollary 4 demonstrate that it is necessary for 0 to be (hereditarily) inaccessible in X for the preimage of X to be (hereditarily) indecomposable. However, this is not sufficient, as the following theorems will show.

Theorem 9. *Let X be a nonseparating indecomposable continuum such that $0 \in X$. Let $C_0(X)$ be the composant of X containing 0. If $C_0(X)$ is a K-composant, then $f_n^{-1}(X)$ is decomposable.*

Proof: Suppose $C_0(X)$ is a K-composant. Then there exists a continuum D such that $D \setminus X \neq \emptyset$, $D \cap C_0(X) \neq \emptyset$, and $D \cap X$ is a proper subcontinuum of X and thus is contained in $C_0(X)$.

Let $X'' = D \cap X$ and let X' be a subcontinuum of X which contains 0 and X''. Suppose $X \setminus X''$ is not contained in just one complementary domain of D. Then since $(X \setminus X'') \cap D = \emptyset$, we have X is decomposable, a contradiction to our assumption. Therefore, $X \setminus X''$ is contained in one complementary domain of D. Let α be an arc in $\overline{\mathbb{C}} \setminus X$ from some point in $D \setminus X$ to ∞. Then $f_n^{-1}(D \cup X' \cup \alpha)$ is a symmetric continuum which separates the plane into some number of components, exactly n of which contain points of $f_n^{-1}(X)$. Call these components C_i for $i = 1, \ldots, n$, where $C_i = \phi_{i-1}(C_1)$. Then let $X_i = (f^{-1}(X) \cap C_i) \cup f_n^{-1}(X')$. Then $f_n^{-1}(X) = \bigcup_{i=1}^{n}(X_i)$ and is therefore decomposable. □

Theorem 10. *Let X be a nonseparating indecomposable continuum such that $0 \in X$. Let $C_0(X)$ be the composant of X containing 0. If $C_0(X)$ is a weakly accessible composant, then $f_n^{-1}(X)$ is not hereditarily indecomposable.*

Proof: Let p be a point in $C_0(X)$ that is weakly accessible in X, and let α be an arc with p as an endpoint which only intersects $C_0(X)$ at p. Let X' be a continuum in $C_0(X)$ containing 0 and p, and let X'' be a continuum in $C_0(X)$ properly containing X'. Then X' is an accessible subcontinuum of X'', so $C_0(X'')$ is a K-composant. Then by Theorem 9, we have that $f_n^{-1}(X'')$ is decomposable. So $f_n^{-1}(X)$ is not hereditarily indecomposable. □

5.4. Relation between composants of X and composants of $f_n^{-1}(X)$

Since composants seem to play an important part in our results for the preimages of continua, we will now establish some results regarding properties of composants.

Theorem 11. *Let X be a nonseparating continuum containing 0, such that $f_n^{-1}(X)$ is an indecomposable continuum containing p. If $p \in f_n^{-1}(X)$, then $f_n(C_p(f_n^{-1}(X))) = C_{f_n(p)}(X)$.*

Proof: Let $y \in f_n(C_p(f_n^{-1}(X)))$, and let $x \in C_p(f_n^{-1}(X))$ such that $f_n(x) = y$. Then there is a proper subcontinuum $X'$of $f_n^{-1}(X)$ containing p and x. If $f_n(X') = X$, then $f_n^{-1}(X)$ would be decomposable, contradicting our assumption. Therefore, we know that $f_n(X')$ is a proper subcontinuum of X containing $f_n(p)$ and y. So $y \in C_{f_n(p)}(X)$. Then $f_n(C_p(f_n^{-1}(X))) \subset C_{f_n(p)}(X)$. Now let $y \in C_{f_n(p)}(X)$. Then there is a proper subcontinuum Y' of $f_n(X)$ containing $f_n(p)$ and y. Then $f_n^{-1}(Y')$ is a continuum if $0 \in Y'$ or is the disjoint union of n continua if $0 \notin Y'$. In either case, $f_n^{-1}(Y')$ contains a component X' such that $p \in X'$. Furthermore, there exists $x \in X'$ such that $f_n(x) = y$. So $y \in f_n(C_p(f_n^{-1}(X)))$. Then $C_{f_n(p)}(X) \subset f_n(C_p(f_n^{-1}(X)))$. So $f_n(C_p(f_n^{-1}(X))) = C_{f_n(p)}(X)$. □

In fact, if $C = C_p(f_n^{-1}(X)) \neq C_0(f_n^{-1}(X))$, then $f_n|_C$ is a bijection onto $C_{f_n(p)}(X)$. To see this, we need only show that $f_n|C$ is one-to-one, since we have shown above that it is onto. Let $p_1, p_2 \in C$ and suppose $f_n(p_1) = f_n(p_2)$. Then p_2 is a rotation of p_1 by $2\pi/n$ radians, and there exists $X' \subset f_n^{-1}(X)$ containing these points. Then $\bigcup_{i=1}^{n} \phi_n^i(X')$ is a proper subcontinuum of X which separates 0 from ∞, a contradiction. So $f_n|_C$ is a bijection.

If a composant C is a K-composant, denote this by $C \in K$, and if a composant is a Z-composant, denote this by $C \in Z$.

Theorem 12. *Let X be a nonseparating continuum containing* 0*, such that $f_n^{-1}(X)$ is an indecomposable continuum containing p.*

(1) $C_p(f_n^{-1}(X)) \in K \iff C_{f_n(p)}(X) \in K$;
(2) $C_p(f_n^{-1}(X)) \in Z \Rightarrow C_{f_n(p)}(X) \in Z$;
(3) $C_0(f_n^{-1}(X)) \in Z \iff C_0(X) \in Z$;
(4) $C_p(f_n^{-1}(X))$ *weakly accessible* $\iff C_{f_n(p)}(X)$ *weakly accessible;*
(5) $C_p(f_n^{-1}(X))$ *internal* $\Rightarrow C_{f_n(p)}(X)$ *internal;*
(6) $C_p(f_n^{-1}(X))$ *accessible* $\iff C_{f_n(p)}(X)$ *accessible.*

Proof: (1) Suppose $C_p(f_n^{-1}(X))$ is a K-composant; let D be a continuum such that $D \cap f_n^{-1}(X)$ is a subcontinuum of $C_p(f_n^{-1}(X))$ and D is not contained in $f_n^{-1}(X)$. Then $f_n(D)$ is a continuum such that $f_n(D) \cap X$ is a subcontinuum of $C_{f_n(p)}(X)$ and $f_n(D)$ is not contained in X. So if $C_p(f_n^{-1}(X))$ is a K-composant, then $C_{f_n(p)}(X)$ is a K-composant.

Now suppose $C_{f_n(p)}(X)$ is a K-composant; let D be a continuum such that $D \cap X$ is a subcontinuum of $C_{f_n(p)}(X)$ and D is not contained in X. Then $f_n^{-1}(D)$ is either a continuum or the disjoint union of n continua. In either case, D contains a continuum D' such that $D' \cap f_n^{-1}(X)$ is a subcontinuum of $C_p(f_n^{-1}(X))$ and $f_n^{-1}(D)$ is not contained in $f_n^{-1}(X)$. So if $C_{f_n(p)}(X)$ is a K-composant, then $C_p(f_n^{-1}(X))$ is a K-composant.

(2) Suppose $C_p(f_n^{-1}(X))$ is a Z-composant; let D be a continuum such that $D \cap f_n^{-1}(X)$ is a compact subset of $C_p(f_n^{-1}(X))$ and D is not contained in $f_n^{-1}(X)$. Then $f_n(D)$ is a continuum such that $f_n(D) \cap X$ is a compact subset of $C_{f_n(p)}(X)$ and $f_n(D)$ is not contained in X. So if $C_p(f_n^{-1}(X))$ is a Z-composant, then $C_{f_n(p)}(X)$ is a Z-composant.

(3) We already have one direction from (2). Now suppose $C_0(X)$ is a Z-composant; let D be a continuum such that $D \cap X$ is a compact subset of $C_0(X)$ and D is not contained in X. Then

$f_n^{-1}(D)$ is either a continuum or the disjoint union of n continua. In either case, D contains a continuum D' such that $D' \cap f_n^{-1}(X)$ is a compact subset of $C_0(f_n^{-1}(X))$ and $f_n^{-1}(D)$ is not contained in $f_n^{-1}(X)$. So if $C_0(X)$ is a Z-composant, then $C_0(f_n^{-1}(X))$ is a Z-composant.

(4) Suppose $C_p(f_n^{-1}(X))$ is weakly accessible; let $x \in C_p(f_n^{-1}(X))$ be weakly accessible point, and let α be an arc containing x such that $(\alpha \setminus \{x\}) \cap C_p(f_n^{-1}(X)) = \emptyset$. Then $f_n(\alpha)$ contains an arc which contains $f_n(x)$, with $f_n(x) \in C_{f_n(p)}(X)$, and $(f_n(\alpha) \setminus \{f_n(x)\}) \cap C_{f_n(p)}(X) = \emptyset$. So if $C_p(f_n^{-1}(X))$ is weakly accessible, then $C_{f_n(p)}(X)$ is weakly accessible.

Now suppose $C_{f_n(p)}(X)$ is weakly accessible; let $x \in C_{f_n(p)}(X)$ be a weakly accessible point, and let α be an arc containing x such that $(\alpha \setminus \{x\}) \cap C_{f_n(p)}(X) = \emptyset$. Then $f_n^{-1}(\alpha)$ contains an arc which intersects $C_p(f_n^{-1}(X))$ only at one point x_1 of $f_n^{-1}(x)$, with x_1 in the same composant as p. Then if $C_{f_n(p)}(X)$ is weakly accessible, then $C_p(f_n^{-1}(X))$ is weakly accessible.

(5) Suppose $C_p(f_n^{-1}(X))$ is internal, and let D be a continuum which is not contained in X which intersects $C_{f_n(p)}(X)$. Then $f_n^{-1}(D)$ contains a continuum D_2 which is not contained in $f_n^{-1}(X)$ which intersects $C_p(f_n^{-1}(X))$. Since $C_p(f_n^{-1}(X))$ is internal, D_2 intersects every other composant of $f_n^{-1}(X)$. Then $f_n(D_2)$ intersects every other composant of X, so $C_{f_n(p)}(X)$ is internal.

(6) Suppose $C_p(f_n^{-1}(X))$ is accessible; let $x \in C_p(f_n^{-1}(X))$ be an accessible point, and let α be an arc containing x such that $(\alpha \setminus \{x\}) \cap f_n^{-1}(X) = \emptyset$. Then $f_n(\alpha)$ contains an arc which contains $f_n(x)$, with $f_n(x) \in C_{f_n(p)}(X)$, and $(f_n(\alpha) \setminus \{f_n(x)\}) \cap X = \emptyset$. So if $C_p(f_n^{-1}(X))$ is accessible, then $C_{f_n(p)}(X)$ is accessible.

Now suppose $C_{f_n(p)}(X)$ is accessible, let $x \in C_{f_n(p)}(X)$ be an accessible point, and let α be an arc containing x such that $(\alpha \setminus \{x\} \cap X = \emptyset$. Then $f_n^{-1}(\alpha)$ contains an arc which intersects $f_n^{-1}(X)$ only at one point x_1 of $f_n^{-1}(x)$, with x_1 in the same composant as p. Then if $C_{f_n(p)}(X)$ is accessible, then $C_p(f_n^{-1}(X))$ is accessible. □

Corollary 5. *Let X be a continuum containing* 0*.*

(1) $C_0(f_n^{-1}(X)) \in K \Rightarrow f_n^{-1}(X)$ *is not hereditarily indecomposable.*

(2) *$C_0(f_n^{-1}(X))$ weakly accessible $\Rightarrow f_n^{-1}(X)$ is not hereditarily indecomposable.*

Proof: These results follow directly from theorems 9, 10, and 12. □

5.5. The preimage of the symmetric pseudo-arc

We now apply some of our results to the preimage under $z \mapsto z^2$ of our symmetric pseudo-arc.

Let N be a simple $2n$-od, with branches $I_1, \dots, I_{2n}$. For each i, let t_i be the endpoint of I_i which is not an endpoint of any other branch.

Lemma 8. *If $N = A \cup B$, then one of these sets contains a complete branch of N.*

Proof: Suppose $N = A \cup B$. If $0 \notin A$, then A is contained in one branch of N; then B contains all other branches of N, so B contains a full branch. If $0 \in A \cap B$, then WLOG $t_1 \in A$, so the first branch of N is contained in A. □

If X is a symmetric pseudo-arc constructed using our method discussed earlier, then for each chain C_i covering X, we have a corresponding tree-chain T_i covering $f_n^{-1}(X)$, where each link of T_i is the preimage under f_n of a link of C_i. Then $f_n^{-1}(X)$ can be expressed as an inverse limit of N, where the bonding maps correspond to the nesting of the T_i. We can represent C_{i+1} as the union of two subchains $C_{i+1,1}$ and $C_{i+1,2}$, with their intersection being the link of C_{i+1} containing 0. Then by crookedness, each of these subchains contains links in all but possibly one of the links of C_i. Since each link of C_i is a round ball, then there exists a sequence $\{\epsilon_1, \epsilon_2, \dots\}$ converging to 0 such that each bonding map f_i satisfies the following properties, where addition is taken to be over $\mathbb{Z}_n$:

(1) $f_i(I_k) \supset (I_k \cup I_{k+1}) \setminus B_{\epsilon_i}(t_k)$, and
(2) $f_i(I_k \setminus B_{\epsilon_i}(t_k)) \supset I_{k+1}$.

Then

$$f_i \circ f_{i+1} \circ \cdots \circ f_{i+n-1}(I_k) \supset N \setminus (\bigcup_{j=1}^{n} B_{\epsilon_i}(t_j)).$$

Theorem 13. *If X is a symmetric pseudo-arc constructed using our method, then $f_n^{-1}(X)$ is indecomposable.*

Proof: First note that $f_n^{-1}(X) \cong \varprojlim(N, f_i) = X_\infty$. Suppose $X_\infty = A \cup B$. Then for each $i \in \mathbb{N}$, we have $A_i = \pi_i(A)$ and $B_i = \pi_i(B)$. By Lemma 8, one of these contains a branch. Suppose A_i contains a branch. Then by the above equations, $X \setminus A_{i-n+1}$ can be contained in n balls of radius less than ϵ_{i-n+1}. This would be true for B_i if B_i contained a branch. Since one of these must contain a branch for each i, there exists WLOG an infinite subsequence j such that $X \setminus A_j$ can be contained in n balls each of radius less than ϵ_j. Then $A = \varprojlim(A_j, f_j^{j+1})$, where f_{j-1}^j is formed by composition of the various f_i maps. Then A is dense in X since ϵ_i goes to 0, so $A = X_\infty$. □

We now extend the above claim to be true for any subcontinuum of $f_n^{-1}(X)$.

Theorem 14. *The preimage of the symmetric pseudo-arc under $z \mapsto z^n$ is hereditarily indecomposable.*

Proof: Let X denote the symmetric pseudo-arc, and suppose that $X' \subset X$ and $0 \in X'$. If X' were not symmetric, then $X' \cup \phi_2(X')$ would be a decomposable subcontinuum of X, so X' must be symmetric. Furthermore, since any nondegenerate subcontinuum of a pseudo-arc is also a pseudo-arc, X' is a symmetric pseudo-arc. Finally, X' can be formed using subchains of the chains forming X, and since X' is symmetric, we can use symmetric subchains. Then by Theorem 13, $f_n^{-1}(X')$ is indecomposable.

Now let $Y \subset f_n^{-1}(X)$ such that $0 \in Y$. Then $f_n(Y) \subset X$ and $0 \in f_n(Y)$. So $f_n^{-1}(f_n(Y))$ is indecomposable. But $f_n^{-1}(f_n(Y)) = \bigcup_{i=0}^{n-1} \phi_n^i(Y)$, so $Y = f_n^{-1}(f_n(Y))$. So Y is indecomposable.

Finally, if $Y \subset f_n^{-1}(X)$ such that $0 \notin Y$, then $f_n^{-1}(f_n(Y))$ is the disjoint union of n rotations of Y, each of which is homeomorphic under f_n to $f_n(Y)$. Since $f_n(Y)$ is indecomposable, we have Y is indecomposable.

So $f_n^{-1}(X)$ is hereditarily indecomposable. □

In the above example, $f_n^{-1}(X)$ is $2n$-od like. However, letting $Y = f_2(X)$ and noting that $f_2^{-1}(Y) = X$, we have that $f_n^{-1}(X) = f_{2n}^{-1}(Y) = f_2^{-1}(f_n^{-1}(Y))$. Then by Corollary 1 we have that $f_n^{-1}(Y)$

is hereditarily indecomposable, so we have constructed for $n = 3, 4, 5, \ldots$ a hereditarily indecomposable n-od like continuum.

References

[1] David P. Bellamy, *Certain analytic preimages of pseudocircles are pseudocircles*, Topology Proc. **29** (2005), no. 1, 19–25.

[2] R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. Math. **1** (1951), 43–51.

[3] Jo Heath, 2-*to*-1 *maps with hereditarily indecomposable images*, Proc. Amer. Math. Soc. **113** (1991), no. 3, 839–846.

[4] J. Krasinkiewicz, *On internal composants of indecomposable plane continua*, Fund. Math. **84** (1974), no. 3, 255–263.

[5] A. Lelek, *On confluent mappings*, Colloq. Math. **15** (1966), 223–233.

Department of Mathematics; Tulane University; New Orleans, LA 70118

E-mail address: `rvernon@math.tulane.edu`