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THE SYMMETRIC SPAN OF A DISK AND ITS BOUNDARY

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Abstract. The following theorem (Theorem 1, p. 440) was given by M. T. Cuervo, E. Duda, and H. V. Fernandez in [*Upper semicontinuous continuum valued functions and spans of continua*, Houston J. Math. **27** (2001), no. 2, 439–444].

> Let J be a simple closed curve in the plane and D the closed topological disk which has J as its boundary. If the symmetric span of D is k, then the symmetric span of J is k.

We show in this paper that the proof given for this theorem is in error. While we have not been able to provide a correct proof, we have shown that the result holds when D is either restricted starlike or starlike about a point in the interior of D.

1. Introduction

The concept of the span of a compact metric space was introduced by A. Lelek in 1964 (see [4, p. 209]). Variations of the span have been defined since then (cf. [5], [6], [2]). Of particular interest in this paper is the symmetric span, which was defined by James Francis Davis in [2]. It is, in general, difficult to evaluate the various spans of geometric objects. The relationships of these spans is of interest. Also of interest is how related geometric objects compare to each other with respect to the various spans.

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The following theorem was given in [1, Theorem 1, p. 440].

Theorem 1. *Let J be a simple closed curve in the plane and D the closed topological disk which has J as its boundary. If the symmetric span of D is k, then the symmetric span of J is k.*

The proof of this theorem appears to be incorrect. The claim that the function, f^* , as given in the proof is upper semicontinuous (u.s.c.) is incorrect. The proof of Theorem 2 [1, p. 441] also depends on this theorem. An example will be given where the theorem holds, but the function $f^* : C \to C(J)$, as defined in the proof, fails to be u.s.c. While we have not been able to correct the proof of this theorem, we have shown that the result holds when the closed disk D is either restricted starlike or starlike.

2. Preliminaries

Let X be a continuum, that is a compact, connected metric space. The span of X, $\sigma(X)$, is the least upper bound of the set of real numbers r which satisfy the following conditions: there exist a continuum, C, and continuous functions f, g: $C \to X$, such that

$$d(f(c), g(c)) \geq r, \forall c \in C$$

and

(*span* σ) $\qquad f(C) = g(C).$

To obtain the various other spans, we replace the preceding equation with the following.

(*semispan* σ_0) $\qquad f(C) \subseteq g(C)$

(*surjective span* σ^*) $\qquad f(C) = g(C) = X$

(*surjective semispan* σ_0^*) $\qquad f(C) \subseteq g(C) = X$

(*symmetric span* s) $\qquad f(C) = g(C)$ and

$$\forall c \in C, \exists c' \text{ such that } f(c) = g(c') \text{ and } f(c') = g(c)$$

(*surjective symmetric span* s^*) $\qquad f(C) = g(C) = X$ and

$$\forall c \in C, \exists c' \text{ such that } f(c) = g(c') \text{ and } f(c') = g(c).$$

Alternatively, the span of a continuum X, $\sigma(X)$, can be defined as the least upper bound of the set of real numbers r which satisfy the following conditions:

p_1 and p_2 are the standard projection maps;

there exists a continuum C, contained in $X \times X$ such that

$$d(x, y) \geq r \quad \text{for all} \quad (x, y) \in C\,;$$

and

(*span* σ) $$p_1(C) = p_2(C).$$

To obtain the various other spans, we replace the preceding equation with the following.

(*semispan* σ_0) $p_1(C) \subseteq p_2(C)$;
(*surjective span* σ^*) $p_1(C) = p_2(C) = X$;
(*surjective semispan* σ_0^*) $p_1(C) \subseteq p_2(C) = X$;
(*symmetric span* s) $p_1(C) = p_2(C)$ and
$$\forall c \in C, \exists c' \text{ such that } p_1(c) = p_2(c') \text{ and } p_1(c') = p_2(c);$$
(*surjective symmetric span* s^*) $p_1(C) = p_2(C) = X$ and
$$\forall c \in C, \exists c' \text{ such that } p_1(c) = p_2(c') \text{ and } p_1(c') = p_2(c).$$

The following inequalities are immediate consequences of the definitions.

$$0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq diam X;$$
$$0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq diam X;$$
$$0 \leq s(X) \leq \sigma(X); \text{ and}$$
$$0 \leq s^*(X) \leq \sigma^*(X).$$

The result below follows from the definitions.

If $\lim_{i\to\infty} X_i = X_0$, then $\lim_{i\to\infty} \alpha(X_i) \leq \alpha(X_0)$,
and if $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots \supseteq X_0$,
then $\alpha(X_0) = \lim_{i\to\infty} \alpha(X_i)$ for $\alpha = \sigma, \sigma_0, s$.

We use the following definitions in our discussion.

$$C(X) = \{A \subseteq X \mid \ A \neq \emptyset, A \quad \text{is a continuum}\}.$$
$$N(A, \varepsilon) = \{x \in X \mid \ d(x, a) < \varepsilon \quad \text{for some} \quad a \in A\}.$$

The set $C(X)$ is a metric space with the Hausdorff metric, H, given by

$$H(A, B) = \text{glb}\{\varepsilon > 0 \mid \ A \ \subset N(B, \varepsilon) \quad and \quad B \subset N(A, \varepsilon)\}.$$

Let (C, T_1) and (X, T_2) be topological spaces. A continuum valued function $f : C \to C(X)$ is said to be u.s.c. at $p \in C$, if for $U \in T_2$ such that $f(p) \subset U$, there exists a $V \in T_1$ such that $p \in V$ and for all $c \in V, f(c) \subset U$. The function f is said to be u.s.c. if it is u.s.c. at every point $c \in C$.

The lemmas that follow can either be found in [7, 7.16, 7.15d] or be readily established from the definition of u.s.c.

Lemma 1. *The function $f : C \to C(X)$ is u.s.c. if and only if for any sequence $(c_i)_{i=1}^{\infty}$ converging to c, $\limsup f(c_i) \subseteq f(c)$.*

Lemma 2. *If $f : C \to C(X)$ is an u.s.c. function and $Y \subseteq C$ is a closed (connected) subset of C, then $\cup_{y \in Y} f(y)$ is a closed (connected) subset of X.*

Lemma 3. *If $f_1 : C_1 \to X_1$ and $f_2 : C_2 \to X_2$ are u.s.c. functions, then $f_1 \times f_2 : C_1 \times C_2 \to X_1 \times X_2$ is an u.s.c. function.*

A set $X \subseteq R^2$ is starlike if there is a point p in X such that for each other point x in X, $\overline{px} \subseteq X$ [3, p. 155]. In addition, we say that X is restricted starlike if for each x in the BdX, $p \neq x$, $\text{Bd}X \cap \overline{px} = \{x\}$.

Let D be a planar closed topological disk with symmetric span, $s(D) = k$. Let $f, g : C \to D$ be continuous functions from a continuum C into D such that for all $c \in C, d(f(c), g(c)) \geq k$, there is a $c \in C$ such that $d(f(c), g(c)) = k$, and for all $c \in C$ there is a $c' \in C$ such that $g(c') = f(c)$ and $f(c') = g(c)$.

For $c \in C$, let $\overline{f(c)g(c)}$ be the line segment in the plane with endpoints $f(c)$ and $g(c)$. Let L_c be the line perpendicular to $\overline{f(c)g(c)}$ through the point $f(c)$. Similarly, let K_c be the line perpendicular to $\overline{f(c)g(c)}$ through the point $g(c)$. Let H_c be the portion of the plane that is bound by L_c and K_c, not including L_c or K_c. Let $f^*(c) = D_c$ be the continuum in $D - H_c$ containing $f(c)$, and $g^*(c) = E_c$ be the continuum in $D - H_c$ containing $g(c)$. Let HP_c be the open half plane in $R^2 - L_c$ that does not contain $g(c)$. Also, let LP_c be the open half plane in $R^2 - K_c$ that does not contain $f(c)$. The functions f^* and g^* were shown in [1, p. 440] to be u.s.c. functions from C into $C(D)$.

The set $f^*(c) \cap L_c \cap J$, where J is the boundary of D, is a compact set. If $f^*(c) \cap L_c \cap J = \{f(c)\}$, let $J_c = \{f(c)\}$. Otherwise, let $\overline{a_c b_c}$ be the minimal interval on L_c containing $f^*(c) \cap L_c \cap J$. In this case, let J_c be the arc on J which joins a_c to b_c in D_c.

A function is defined in the proof of Theorem 1 in [1, p. 441] that is named f^* and $f^* : C \to C(J)$. We call it $\hat{f}^*$ here in order to easily distinguish the functions

$$f^* : C \to C(D) \text{ and } \hat{f}^* : C \to C(J).$$

This function is defined as $\hat{f}^*(c) = J_c$. The function $\hat{g}^*$ is defined similarly. It is claimed in this proof that $\hat{f}^*$ is u.s.c. In the example that follows, it is shown that $\hat{f}^*$ is not u.s.c. However, it is the case that $s(D) = s(J)$ in this example.

3. Example

Let $l = (0,0)$, $m = (0,18)$, $g = (2,2)$, $f = (2,16)$, $b = (4,4)$, $c = (4,14)$, $d = (32,14)$, $e = (32,16)$, $a = (34,4)$, $o = (34,16)$, $n = (34,18)$, $h = (36,2)$, $i = (36,18)$, $k = (38,0)$, and $j = (38,18)$. Let

$$J = \overline{ab} \cup \overline{bc} \cup \overline{cd} \cup \overline{de} \cup \overline{ef} \cup \overline{fg} \cup \overline{gh} \cup \overline{hi} \cup \overline{ij} \cup \overline{jk} \cup \overline{kl} \cup \overline{lm} \cup \overline{mn} \cup \overline{na}.$$

Let D be the disc in the plane which is bound by the simple closed curve J.

Let π be the continuous projection from D onto the arc K

$$\pi : D \to K$$

where $K = \overline{ba} \cup \overline{an} \cup \overline{nm} \cup \overline{ml} \cup \overline{lk} \cup \overline{kj}$ and π is defined as given below.

Definition of π: For a point p which is in the portion of D bound by a simple closed curve S, define $\pi(p)$ considering the following cases.

Case 1: $S = \overline{ab} \cup \overline{bc} \cup \overline{cd} \cup \overline{da}$. Project p on the line through it of slope -5 onto $\overline{ab}$.

Case 2: $S = \overline{(3,16)(31,16)} \cup \overline{(31,16)(31,18)} \cup \overline{(31,18)(3,18)} \cup \overline{(3,18)(3,16)}$. Project p vertically onto $\overline{(31,18)(3,18)}$.

Case 3: $S = \overline{(3,0)(35,0)} \cup \overline{(35,0)(35,2)} \cup \overline{(35,2)(3,2)} \cup \overline{(3,2)(3,0)}$. Project p vertically onto $\overline{(3,0)(35,0)}$.

Case 4: $S = \overline{(2,3)(2,15)} \cup \overline{(2,15)(0,15)} \cup \overline{(0,15)(0,3)} \cup \overline{(0,3)(2,3)}$. Project p horizontally onto $\overline{(0,15)(0,3)}$.

Case 5: $S = \overline{da} \cup \overline{a(34,15)} \cup \overline{(34,15)(32,15)} \cup \overline{(32,15)d}$. For $t \in [14,15]$, let L_t be the line segment with endpoints $(32,t)$ and $(34, 11t - 150)$. Project each point p on L_t to the point $(34, 11t - 150)$.

Case 6: $S = \overline{(32,16)(32,15)} \cup \overline{(32,15)(34,15)} \cup \overline{(34,15)(34,18)} \cup \overline{(34,18)(32,16)}$. For $t \in [15,16]$, let L_t be the line segment with endpoints $(32, t)$ and $(34, 3t-30)$. Project each point p on L_t to the point $(34, 3t-30)$.

Case 7: $S = \overline{(32,16)(31,16)} \cup \overline{(31,16)(31,18)} \cup \overline{(31,18)(34,18)} \cup \overline{(34,18)(32,16)}$. For $t \in [31, 32]$, let L_t be the line segment with endpoints $(t, 16)$ and $(3t-62, 18)$. Project each point p on L_t to the point $(3t-62, 18)$.

Case 8: $S = \overline{(2,16)(3,16)} \cup \overline{(3,16)(3,18)} \cup \overline{(3,18)(0,18)} \cup \overline{(0,18)(2,16)}$. Similar to case 7.

Case 9: $S = \overline{(2,16)(2,15)} \cup \overline{(2,15)(0,15)} \cup \overline{(0,15)(0,18)} \cup \overline{(0,18)(2,16)}$. Similar to case 6.

Case 10: $S = \overline{(0,3)(2,3)} \cup \overline{(2,3)(2,2)} \cup \overline{(2,2)(0,0)} \cup \overline{(0,0)(0,3)}$. Similar to case 6.

Case 11: $S = \overline{(0,0)(2,2)} \cup \overline{(2,2)(3,2)} \cup \overline{(3,2)(3,0)} \cup \overline{(3,0)(0,0)}$. Similar to case 7.

Case 12: $S = \overline{(38,0)(36,2)} \cup \overline{(36,2)(35,2)} \cup \overline{(35,2)(35,0)} \cup \overline{(35,0)(38,0)}$. Similar to case 7.

Case 13: $S = \overline{(38,0)(36,2)} \cup \overline{(36,2)(36,3)} \cup \overline{(36,3)(38,3)} \cup \overline{(38,3)(38,0)}$. Similar to case 6.

Case 14: $S = \overline{(36,3)(38,3)} \cup \overline{(38,3)(38,18)} \cup \overline{(38,18)(36,18)} \cup \overline{(36,18)(36,3)}$. Project p horizontally onto $\overline{(38,3)(38,18)}$.

Let C be any continuum and let $\hat{f}$ and $\hat{g}$ be continuous functions

$$\hat{f}, \hat{g} : C \to D$$

such that either

(σ) $\hat{f}[C] = \hat{g}[C]$;

(σ_0) $\hat{f}[C] \subseteq \hat{g}[C]$;

(σ_0^*) $\hat{f}[C] \subseteq \hat{g}[C] = D$;

(s) $\hat{f}[C] = \hat{g}[C]$ and for all $c \in C$ there exists c' such that $\hat{f}(c) = \hat{g}(c')$ and $\hat{f}(c') = \hat{g}(c)$; or

(s^*) $\hat{f}[C] = \hat{g}[C] = D$ and for all $c \in C$ there exists c' such that $\hat{f}(c) = \hat{g}(c')$ and $\hat{f}(c') = \hat{g}(c)$.

Consider $\pi \circ \hat{f}, \pi \circ \hat{g} \to K$. Since K is an arc and $\pi \circ \hat{f}[C] \subseteq \pi \circ \hat{g}[C]$, there exists $c \in C$ such that $\pi \circ \hat{f}(c) = \pi \circ \hat{g}(c) = t \in K$ and $\operatorname{diam}(\pi^{-1}(t)) \leq \sqrt{104}$. Consequently, $\alpha(D) \leq \sqrt{104}$ where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$.

Next we show that $s(J) = \sqrt{104}$. Let f and g be piecewise linear functions $f, g : [0, 36] \to J$ such that

t	$f(t)$	$g(t)$	t	$f(t)$	$g(t)$
0	b	a	18	a	b
1	c	a	19	a	c
2	d	a	20	a	d
3	e	a	21	a	e
4	f	a	22	a	f
5	g	a	23	a	g
6	g	n	24	n	g
7	g	m	25	m	g
8	h	m	26	m	h
9	i	m	27	m	i
10	j	m	28	m	j
11	k	m	29	m	k
12	l	m	30	m	l
13	l	n	31	n	l
14	l	a	32	a	l
15	m	a	33	a	m
16	n	a	34	a	n
17	n	b	35	b	n
18	a	b	36	b	a .

Also, for $t \in [0, 18]$, $f(t) = g(t + 18)$ and $g(t) = f(t + 18)$.

Note that for all $t \in [0, 36]$, $d(f(t), g(t)) \geq \sqrt{104}$, and for all $t \in [0, 36]$, there exists $t' \in [0, 36]$ such that $f(t) = g(t')$ and $f(t') = g(t)$. So, $s(J) \geq \sqrt{104}$. Hence, $s(J) = s(D) = \sqrt{104}$ since $J \subseteq D$ and $\sqrt{104} \leq s(J) = s(D) \leq \sqrt{104}$. If fact, $\sqrt{104} = s(J) = s^*(J) = s(D) = s(D) = \sigma(D) = \sigma_0(D)$.

Let B be the closure of the region of D which is bound by $\overline{bc} \cup \overline{cd} \cup \overline{bd}$. Let A be the closure of the region of D which is bound by $\overline{ad} \cup \overline{d(34, 14)} \cup \overline{(34, 14)a}$.

Let

$$f(t) = \begin{cases} (4, 4 + 10t) & 0 \leq t \leq 1 \\ (4 + 28(t - 1), 14) & 1 \leq t \leq 2 \,. \end{cases}$$

The slope of $\overline{f(t)g(t)}$ for $t \in [0, 2]$ is given by the function m where

$$m(t) = \begin{cases} -\frac{1}{3}t & 0 \leq t \leq 1 \\ \frac{10}{28t - 58} & 1 \leq t \leq 2 \,. \end{cases}$$

For $t = 0$, L_0 is the vertical line through b, $D_0 = \overline{bc} = J_0$.

For $0 < t < 1$, the slope of L_t is $\frac{3}{t} \in (3, +\infty)$, but the slope between d and $f(t)$ is $\frac{10(1-t)}{28} \in (0, \frac{10}{28})$. Consequently, $D_t \subset B - \{d\}$ and $J_t \subset \overline{bc} \cup \overline{cd} - \{d\}$.

For t=1, the slope of L_1 is 3, $D_1 = \{c\} = J_1$.

For $1 < t < 2$, the slope of L_t is $\frac{58-28t}{10} \in (\frac{1}{5}, 3)$. Consequently, A is beneath $L_t, D_t \subset B$, and $J_t \subset \overline{bc} \cup \overline{cd} - \{d\}$.

When $t = 2$, the equation for L_2 is $y = \frac{1}{5}x + \frac{38}{5}$, D_2 is the closure of the portion of D which is bound by $\overline{(0, \frac{38}{5})m} \cup \overline{mn} \cup \overline{n(34, \frac{72}{5})} \cup \overline{(34, \frac{72}{5})(4, \frac{42}{5})} \cup \overline{(4, \frac{42}{5})c} \cup \overline{cd} \cup \overline{de} \cup \overline{ef} \cup \overline{f(2,8)} \cup \overline{(2,8)(0, \frac{38}{5})}$, and $J_2 = \overline{(0, \frac{38}{5})m} \cup \overline{mn} \cup \overline{n(34, \frac{72}{5})}$. Consequently, $\hat{f}^*$ is not u.s.c. at 2 since for $0 \le t < 2$, $J_t \subset (\overline{bc} \cup \overline{cd} - \{d\})$ but $(\overline{bc} \cup \overline{cd}) \cap J_2 = \emptyset$.

4. Main Results

Theorem 2. *Let D be a closed disk in the plane which is restricted starlike about a point $P \in D^\circ$. Let J be the simple closed curve which bounds D. If $s(D) = k$ then $s(J) = k$.*

Proof: Assume that the point about which D is restricted starlike is the origin, O. Let $r : R \to (0, +\infty)$ be a continuous function from the reals into the positive reals, such that $r(\theta + 2n\pi) = r(\theta)$ for $n \in Z$ where $\{r(\theta)e^{i\theta}\} = \overrightarrow{Oe^{i\theta}} \cap J$. Also, let $h : S^1 \to J$ be the homeomorphism from the unit circle onto J given by $h(e^{i\theta}) = r(\theta)e^{i\theta}$. Let $a, b \in J, a \ne b$. There are angles θ_a and θ_b such that $0 \le \theta_a \le 2\pi$, $0 \le \theta_b \le 2\pi$, $h(e^{i\theta_a}) = a$, and $h(e^{i\theta_b}) = b$.

If $0 \le \theta_a < \theta_b \le 2\pi$, then let

$$J_{ab} = \{h(e^{i\theta}) \mid \theta_a \le \theta \le \theta_b\} \quad \text{and}$$

$$J_{ba} = \{h(e^{i\theta}) \mid \theta_b \le \theta \le 2\pi \quad \text{or} \quad 0 \le \theta \le \theta_a\}.$$

If $0 \le \theta_b < \theta_a \le 2\pi$, then let

$$J_{ab} = \{h(e^{i\theta}) \mid \theta_a \le \theta \le 2\pi \quad \text{or} \quad 0 \le \theta \le \theta_b\} \quad \text{and}$$

$$J_{ba} = \{h(e^{i\theta}) \mid \theta_b \le \theta \le \theta_a\}.$$

Suppose f and g are continuous functions from a continuum C into D which give the symmetric span of D; that is, for all $c \in C$, $d(f(c)g(c)) \ge k$, there is a $c^* \in C$ such that $d(f(c^*), g(c^*)) = k$, and for all $c \in C$ there is a $c' \in C$ such that $f(c) = g(c')$ and $g(c) = f(c')$.

We will define functions $\hat{f}^*, \hat{g}^* : C \to C(J)$ such that $\hat{f}^*$ and $\hat{g}^*$ are u.s.c. We define $\hat{f}^* : C \to C(J)$ according to the following cases.

Case 1. $O \in L_c$.
Consequently, $O \in \overline{a_c b_c} - \{a_c, b_c\}$ and $D_c \cap J = HP_c \cap J$. Let $\hat{f}^*(c) = D_c \cap J = J_c$. Note that J_c is either $J_{a_c b_c}$ or $J_{b_c a_c}$.

Case 2. $O \in R^2 - (HP_c \cup L_c)$.
There are two subcases to consider.

(2i) $f^*(c) \cap L_c \cap J = \{f(c)\}$.
For all $\varepsilon > 0$, there are points P_{l_ε} and P_{r_ε} on L_c, such that $P_{l_\varepsilon} < f(c) < P_{r_\varepsilon}$, $d(P_{l_\varepsilon}, f(c)) < \frac{\varepsilon}{2}$, $d(P_{r_\varepsilon}, f(c)) < \frac{\varepsilon}{2}$, and P_{l_ε} and P_{r_ε} are not in D. Otherwise, there would be an interval I_ε of diameter ε centered about $f(c)$ contained in D, and consequently, also contained in D_c. So, $(f^*(c) \cap L_c \cap J) \cap (L_c - \{f(c)\}) \neq \emptyset$, contrary to our assumption. So, D_c is contained in the region of $HP_c \cup L_c$ that is bound by $\overrightarrow{OP_{l_\varepsilon}} \cup \overline{P_{l_\varepsilon} P_{r_\varepsilon}} \cup \overrightarrow{OP_{r_\varepsilon}}$ for all $\varepsilon > 0$ and $D_c \cap (\overrightarrow{Of(c)} - \overline{Of(c)}) = \emptyset$. Hence $D_c = \{f(c)\}$ and $\hat{f}^*(c) = f^*(c) = \{f(c)\} = D_c \cap J$.

(2ii) $f^* \cap L_c \cap J \neq \{f(c)\}$.
In this case, $f^*(c) \cap L_c \cap J \subset \overline{a_c b_c}$ and $a_c \neq b_c$. We define $\hat{f}^*(c) = D_c \cap J = J_c$. Note that $J_c = J_{a_c b_c}$ or $J_c = J_{b_c a_c}$.

Case 3. $O \in HP_c$.
In this case, $f^*(c) \cap J \cap L_c \neq \{f(c)\}$.

There are two subcases to consider.

(3i) $\overline{a_c b_c} - \{a_c, b_c\} \subset D^\circ$.
It must be the case that $(L_c - \overline{a_c b_c}) \cap D = \emptyset$. So, either $J_{a_c b_c} \subseteq HP_c \cup L_c$, $J_{a_c b_c} \cap L_c = \{a_c, b_c\}$, $J_{b_c a_c} \subseteq R^2 - HP_c$, and we define $\hat{f}^*(c) = D_c \cap J = J_c = J_{a_c b_c}$, or $J_{b_c a_c} \subseteq HP_c \cup L_c$, $J_{b_c a_c} \cap L_c = \{a_c, b_c\}$, $J_{a_c b_c} \subseteq R^2 - HP_c$, and we define $\hat{f}^*(c) = D_c \cap J = J_c = J_{b_c a_c}$.

(3ii) $\overline{a_c b_c} - \{a_c, b_c\} \nsubseteq D^\circ$.
Either $J_{a_c b_c}$ or $J_{b_c a_c}$ is contained in $(HP_c \cup \{a_c, b_c\})$. If $J_{a_c b_c} \subset HP_c \cup \{a_c, b_c\}$, then let $J_{c'} = J_{a_c b_c}$. Otherwise, let $J_{c'} = J_{b_c a_c}$. Note that $L_c \cap D \subset \overline{a_c b_c}$. Let $\overline{a_{c'} b_{c'}} \subset \overline{a_c b_c}$ be the smallest interval such that if $x \in \overline{a_c a_{c'}}$, then $\overline{Ox} \cap J \neq \emptyset$ and if $x \in \overline{b_{c'} b_c}$, then $\overline{Ox} \cap J \neq \emptyset$.

Let $\widehat{a_c a_{c'}}$ be the arc on J that connects a_c and $a_{c'}$ and is bound by $\overline{Oa_c} \cup \overline{a_c b_c} \cup \overline{Ob_c}$. Let $\widehat{b_{c'} b_c}$ be the arc on J that connects $b_{c'}$ and b_c and is bound by $\overline{Oa_c} \cup \overline{a_c b_c} \cup \overline{Ob_c}$. Let $J_c = J_{c'} \cup \widehat{a_c a_{c'}} \cup \widehat{b_{c'} b_c}$. In this case, $J_c \subset D_c \cap J$, but it may not be the case that $J_c = D_c \cap J$.

Now we need to show that $\hat{f}^*$ is u.s.c. at c for all $c \in C$. Let $c \in C$ such that $\hat{f}^*(c) = f^*(c) \cap J$ and let (c_i) be a sequence in C that converges to c. For each i, $\hat{f}^*(c_i) \subseteq f^*(c_i) \cap J$. So, lim sup $\hat{f}^*(c_i) \subseteq$ lim sup $(f^*(c_i) \cap J) \subseteq$ (lim sup$f^*(c_i)) \cap J = f^*(c) \cap J = \hat{f}^*(c)$. Consequently, $\hat{f}^*$ is u.s.c. at all $c \in C$ such that $\hat{f}^*(c) = f^*(c) \cap J$, as in case 1, subcase (2i), subcase (2ii), subcase (3i), and may be the situation in subcase (3ii).

Let $c \in C$ such that $\hat{f}^*(c) \subset f^*(c) \cap J$, but $\hat{f}^*(c) \neq f^*(c) \cap J$, as may be the situation in subcase (3ii). Let $j \in (f^*(c) \cap J) - \hat{f}^*(c)$. To simplify the following argument we assume that there are four angles $\theta_{a_c}, \theta_{a_{c'}}, \theta_{b_{c'}}$ and θ_{b_c} such that $\pi < \theta_{a_c} < \theta_{a_{c'}} < \theta_{b_{c'}} < \theta_{b_c} < 2\pi$; $h(\theta_{a_c}) = a_c, h(\theta_{a_{c'}}) = a_{c'}, h(\theta_{b_{c'}}) = b_{c'}$; and $h(\theta_{b_c}) = b_c$. There is an angle θ_j such that $\theta_{a_{c'}} < \theta_j < \theta_{b_{c'}}$ and $h(\theta_j) = j$. Let $\theta_{f(c)}$ be the angle such that $h(\theta_{f(c)}) = f(c)$ and $\theta_{a_{c'}} < \theta_{f(c)} < \theta_{b_{c'}}$.

We will consider the case where $\theta_{a_{c'}} < \theta_j < \theta_{f(c)} < \theta_{b_{c'}}$. The case where $\theta_{a_{c'}} < \theta_{f(c)} < \theta_j < \theta_{b_{c'}}$ is similar. Based on the definition of $a_{c'}$ and our choice for j, there must be a point $j' \in J$ and an angle $\theta_{j'}$ such that $h(\theta_{j'}) = j'$, $\theta_{a_{c'}} < \theta_{j'} < \theta_j$, and $j' \in R^2 - (HP_c \cup L_c)$.

For $c \in C$, let $l_c, r_c \in L_c$ such that $d(l_c, f(c)) = d(r_c, f(c)) =$ diam$X$. If (c_i) is a sequence in C such that (c_i) converges to c, then $(f(c_i))$ converges to $f(c)$ and $(g(c_i))$ converges to $g(c)$. Consequently, $(\overline{f(c_i)g(c_i)})$ converges to $\overline{f(c)g(c)}$, and $(\overline{l_{c_i} r_{c_i}})$ converges to $\overline{l_c r_c}$ with respect to the Hausdorff metric.

Let

$$\varepsilon_1 = \frac{1}{4} d(f(c), g(c));$$

$$\varepsilon_2 = \frac{1}{2} d(j, L_c);$$

$$\varepsilon_3 = \frac{1}{2} d(j', L_c);$$

$$\varepsilon_4 = \frac{1}{2} d(l_c, \overleftrightarrow{Oa_c}) \quad \text{if} \quad l_c \neq a_c;$$

$$\varepsilon_5 = \frac{1}{2} d(f(c), \overleftrightarrow{Oj});$$
$$\varepsilon_6 = \frac{1}{2} d(a_c, \overleftrightarrow{Oa_{c'}}) \quad \text{if} \quad a_c \neq a_{c'};$$
$$\varepsilon_7 = \frac{1}{2} d(a_{c'}, \overleftrightarrow{Oa_c}) \quad \text{if} \quad a_c \neq a_{c'};$$
$$\varepsilon_8 = \frac{1}{2} d(a_{c'}, \overleftrightarrow{Oj});$$
$$\varepsilon_9 = \frac{1}{2} d(f(c), \overleftrightarrow{Oj'}); \text{ and}$$
$$\varepsilon = \min\{\varepsilon_i \mid \; i = 1, 2, \cdots, 9 \quad \text{and} \quad \varepsilon_i \quad \text{has been defined } \}.$$

There is an $N \in Z^+$ such that for $i \geq N$, $H(\overline{l_{c_i} r_{c_i}}, \overline{l_c r_c}) < \varepsilon$. Let $B_\varepsilon = \cup_{x \in \overline{l_c r_c}} B(x, \varepsilon)$. The set $(B_\varepsilon - \overline{Oj'})$ has two components. Let C'_{l_c} be the component that contains l_c, and C'_{r_c} be the component that contains r_c. Also, the set $B_\varepsilon - \overrightarrow{Oj}$ has two components. Let C_{l_c} be the component that contains l_c, and C_{r_c} be the component that contains r_c. Clearly, for $i \geq N$,

$$\overline{l_{c_i} r_{c_i}} \cap C'_{l_c} \neq \emptyset;$$
$$\overline{l_{c_i} r_{c_i}} \cap C'_{r_c} \neq \emptyset;$$
$$\overline{l_{c_i} r_{c_i}} \cap C_{l_c} \neq \emptyset; \text{ and}$$
$$\overline{l_{c_i} r_{c_i}} \cap C_{r_c} \neq \emptyset.$$

So,

$$\overline{l_{c_i} r_{c_i}} \cap \overline{Oj'} \neq \emptyset;$$
$$\overline{l_{c_i} r_{c_i}} \cap \overline{Oj} = \emptyset; \text{ and}$$
$$J \cap L_{c_i} \cap C'_{l_c} \neq \emptyset.$$

We can see that $j \in D_{c_i} = f^*(c_i)$, but that $j \notin J_{c_i} = \hat{f}^*(c_i)$. Consequently,

$$\limsup \hat{f}^*(c_i) \subset \hat{f}^*(C).$$

We conclude that for all $c \in C$ and for any sequences (c_i) converging to c, lim sup $\hat{f}^*(c_i) \subset \hat{f}^*(c)$ and that $\hat{f}^*$ is upper semicontinuous.

Define g^* and $\hat{g}^*$ similarly, using K_c instead of L_c, E_c instead of D_c, and LP_c instead of HP_c. Also, $\hat{g}^*$ is u.s.c. By lemmas 2 and 3, the set $Z = \cup_{c \in C} \hat{f}^*(c) \times \hat{g}^*(c)$ is a continuum in $J \times J$ with the properties that if $(x, y) \in Z$, then $d(x, y) \geq k$ and $(y, x) \in Z$. Consequently, $s(J) = k$. □

Corollary 1. *Let D be a closed disk in the plane which is starlike about a point $p \in D^\circ$. Let J be the simple closed curve which bounds D. If $s(D) = k$, then $s(J) = k$.*

Proof: There is a sequence (D_i) of closed disks in the plane such that each D_i is restricted starlike about $p \in D_i^\circ$ and $D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots \supseteq D$. Consequently, lim $s(D_i) = s(D)$. For each i, let J_i be the simple closed curve that bounds D_i. Hence, lim $J_i = J$, lim $s(J_i) \leq s(J) \leq s(D) = k$, for each i $s(J_i) = s(D_i)$, $k = s(D) =$ lim $s(D_i) =$ lim $s(J_i) \leq s(J) \leq s(D) = k$. So, $s(J) = s(D) = k$. □

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