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ON LEXICOGRAPHIC PRODUCTS OF TWO GO-SPACES WITH A GENERALIZED ORDERED TOPOLOGY

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Abstract. In this paper, we investigate lexicographic products of two GO-spaces with a generalized ordered topology that we call a generalized ordered topological product of the two GO-spaces. We concentrate on the relationship of the properties, such as Lindelöfness, paracompactness, and perfectness, of the two GO-spaces and their generalized ordered topological product.

1. Introduction

Starting with two generalized ordered (GO) spaces X and Y, we introduced, in [7], a new topology on the lexicographic product set $X \times Y$. This new topology contains the usual open-interval topology of the lexicographic order and also reflects in a natural way the fact that X and Y carry a GO-topology, rather than just the open interval topology of their linear orderings. (Precise definitions appear in section 2.) This new topology on the lexicographic product is called a *generalized ordered topological product* (GOTP) of the GO-spaces X and Y and is denoted by GOTP$(X * Y)$. In this paper, for GO-spaces X and Y, we show that the GOTP$(X * Y)$ is

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Lindelöf (or paracompact) if and only if X, Y are Lindelöf (or paracompact), provided that Y has two endpoints. We prove that the GOTP$(X * Y)$ is a paracompact GO-space if and only if X, Y are paracompact GO-spaces, provided that X does not have neighbor points. Moreover, we investigate when the GOTP is metrizable, (or perfectly normal, or a p-space, or an M-space).

Let $X = (X, \tau, \leq)$ be a GO-space. If p and q are points of X such that $p < q$ and $(p, q) = \emptyset$, then p and q are called *neighbor points* in X; p is the *left neighbor point* of q and q is the *right neighbor point* of p. Let

$$I_X = \{x \in X \mid x \text{ is an isolated point of } X\},$$

$$R_X = \{x \in X \mid [x, \to) \in \tau\},$$
$$L_X = \{x \in X \mid (\leftarrow, x] \in \tau\},$$
$$E_X = R_X \cup L_X,$$
$$N_X = \{x \in E_X - I_X \mid \exists\, y \in E_X - I_X \text{ such that } x, y \text{ are neighbor points in } X\}.$$

For example, suppose $X = (-1, 0] \cup \{1, 2\} \cup [3, 4)$ with the usual subspace topology from the real line. Then $\{0, 1, 2, 3\} \subset E_X$, but none of these points belongs to N_X. If C is a convex subset of X and $\xi = (A, B)$ is a (pseudo-)gap in X, then we say that C covers ξ if $C \cap A \neq \emptyset \neq C \cap B$. A subset A of X is said to be *left discrete* in X if for each $x \in X$, there exists a convex open neighborhood $O(x)$ such that $O(x) \cap (A - \{x\}) \cap (\leftarrow, x] = \emptyset$. A subset A of X is said to be *σ-l-discrete* if $A = \cup\{A_n \mid n \in \mathbb{N}\}$ where for each $n \in \mathbb{N}$, A_n is left discrete in X and $\mathbb{N}$ denotes the positive integers. *σ-r-discrete* is similarly defined. Define an equivalence relation $\sim$ on X by

$$x \sim y \iff x = y \text{ or } x, y \in N_X \ \& \ x, y \text{ are neighbor points in } X.$$

For a set V and a collection $\mathcal{U}$ of sets, we will write $V \prec \mathcal{U}$ to mean that V is a subset of some member of $\mathcal{U}$.

For undefined terminology refer to [3], [4], [5], [6].

2. Lindelöfness and the GOTP

In contrast to lexicographic products of two LOTS with the usual interval topology, we give a generalized ordered topology on lexicographic products of two GO-spaces.

Definition 2.1 ([4])**.** Let $(X, <_X)$, $(Y, <_Y)$ be linearly ordered sets. Then the lexicographic product $X * Y$ is defined as the Cartesian product $X \times Y$ supplied with the lexicographic ordering $\lessdot$; i.e., if $a = \langle x_1, y_1\rangle$ and $b = \langle x_2, y_2\rangle \in X \times Y$ then

$$a \lessdot b \text{ if and only if } x_1 <_X x_2 \text{ or } x_1 = x_2 \text{ and } y_1 <_Y y_2.$$

Definition 2.2 ([7])**.** Let $(X, \tau_X, <_X)$, $(Y, \tau_Y, <_Y)$ be GO-spaces. Let λ_X and λ_Y be the usual interval topology on X and Y, respectively, and let λ_{X*Y} be the usual interval topology on the linearly ordered set $X * Y$.

The *generalized ordered topology* (GOT) τ_{X*Y} is generated by a subbase $\lambda_{X*Y} \cup \tau_R \cup \tau_L \cup \{[\langle x, y\rangle, \rightarrow) \subseteq X * Y \mid x \in X,\ y \in Y \text{ and } [y, \rightarrow) \in \tau_Y - \lambda_Y\} \cup \{(\leftarrow, \langle x, y\rangle] \subseteq X * Y \mid x \in X,\ y \in Y \text{ and } (\leftarrow, y] \in \tau_Y - \lambda_Y\}$, where either

$\tau_R = \emptyset$ and $\tau_L = \emptyset$, if Y does not have endpoints,

or

$\tau_R = \{[\langle x, y_0\rangle, \rightarrow) \mid x \in X \text{ and } [x, \rightarrow) \in \tau_X - \lambda_X\}$ and $\tau_L = \emptyset$, if Y has a left endpoint y_0, but no right one,

or

$\tau_R = \emptyset$ and $\tau_L = \{(\leftarrow, \langle x, y_1\rangle] \mid x \in X \text{ and } (\leftarrow, x] \in \tau_X - \lambda_X\}$, if Y has a right endpoint y_1, but no left one,

or

$\tau_R = \{[\langle x, y_0\rangle, \rightarrow) \mid x \in X \text{ and} [x, \rightarrow) \in \tau_X - \lambda_X\}$ and $\tau_L = \{(\leftarrow, \langle x, y_1\rangle] \mid x \in X \text{ and } (\leftarrow, x] \in \tau_X - \lambda_X\}$, if Y has both a left endpoint y_0 and a right endpoint y_1.

We say that the space $(X * Y, \tau_{X*Y})$ is the *generalized ordered topological product* (GOTP) of GO-spaces $(X, \tau_X, <_X)$ and $(Y, \tau_Y, <_Y)$, and denote it by $\mathrm{GOTP}(X * Y)$. Similarly, we denote $(X * Y, \lambda_{X*Y})$ by $\mathrm{LOTP}(X * Y)$.

In Definition 2.2, if X, Y are LOTS, then $\tau_{X*Y} = \lambda_{X*Y}$. For each $x \in X$, the subspace $\{x\} * Y$ of the $\mathrm{GOTP}(X * Y)$ is homeomorphic to Y. Moreover, the topology on the $\mathrm{GOTP}(X * Y)$ is determined by the topologies on X and Y. So the $\mathrm{GOTP}(X * Y)$ is a natural generalization of the lexicographic product with the usual interval topology.

Convention. When the meanings are clear from the context, we do not distinguish notations for orderings on different ordered sets and use simply $<$ instead of $<_X$, $<_Y$, and $\lessdot$.

Definition 2.3. Let X be a GO-space and $S \subseteq X$ be convex in X. Define

$$I(S) = \{x \in S \mid \text{ there exist } a,\ b \in S \text{ with } a < x < b\}.$$

For any subset $G \subseteq X$, define

$$I(G) = \cup\{I(S) \mid S \text{ is a convex component of } G\}.$$

For any subset G of GO-space X, $I(G)$ is open in X. Next, we explore when the lexicographic product of two GO-spaces with a generalized ordered topology is Lindelöf. First, we need the following lemmas.

Lemma 2.1 ([7]). *Let X, Y be GO-spaces and y_0 (y_1) be a left (right) point of Y. Suppose $\mathcal{U}$ is an open cover of the* GOTP$(X*Y)$ *by convex sets and*

$$E = \{x \in X \mid \textit{no element of } \mathcal{U} \textit{ contains both } \langle u, y_0\rangle \textit{ and } \langle v, y_1\rangle \textit{ for some } u,\ v \in X \textit{ and } u < x < v\}.$$

Then E is a closed discrete subspace of X.

Lemma 2.2 ([7]). *Let X, Y be GO-spaces. Suppose π_1 is a mapping from the* GOTP$(X*Y)$ *onto X, which is defined by $\pi_1(\langle x, y\rangle) = x$ for each point $\langle x, y\rangle \in X * Y$. If Y has both a left and a right endpoint, then π_1 is continuous.*

Remark. In this paper, π_1 always denotes the map defined in Lemma 2.2.

Theorem 2.1. *Let X, Y be GO-spaces. If Y has both a left and a right endpoint, then the following are equivalent.*

(1) *X, Y are Lindelöf;*
(2) *the* GOTP$(X * Y)$ *is Lindelöf.*

Proof: (1) $\Longrightarrow$ (2) Let y_0 be a left endpoint of Y and y_1 be a right endpoint of Y. Suppose $\mathcal{U}$ is any open cover of the GOTP$(X * Y)$. Without loss of generality, suppose every member of $\mathcal{U}$ is convex.

Let E be defined as in Lemma 2.1. Since X is Lindelöf, E is countable by Lemma 2.1. Let

$$I(\pi_1(\mathcal{U})) = \{I(\pi_1(U)) \mid U \in \mathcal{U}\}$$

and

$$\mathcal{U}(I_X) = \{\{x\} \mid x \in E \cap I_X\}.$$

For each $x \in E \cap ((R_X \cup L_X) - I_X)$, choose $u_x,\ v_x \in X$ and $U_x \in \mathcal{U}$ satisfying the following properties:

(i) if $x \in E \cap (R_X - I_X)$, then $v_x > x$ and $[\langle x, y_1\rangle, \langle v_x, y_1\rangle] \subseteq U_x \in \mathcal{U}$;

(ii) if $x \in E \cap (L_X - I_X)$, then $u_x < x$ and $[\langle u_x, y_0\rangle, \langle x, y_0\rangle] \subseteq U_x \in \mathcal{U}$.

For each $x \in E - (R_X \cup L_X \cup I_X)$, choose $u_x,\ v_x \in X$ and U_{x0}, $U_{x1} \in \mathcal{U}$ satisfying the following property:

$$\begin{aligned} u_x < x < v_x,\ [\langle u_x, y_0\rangle, \langle x, y_0\rangle] \subseteq U_{x0} \in \mathcal{U} \text{ and}\\ [\langle x, y_1\rangle, \langle v_x, y_1\rangle] \subseteq U_{x1} \in \mathcal{U}. \end{aligned}$$

Then let

$$\begin{aligned} \mathcal{U}(R_X) &= \{[x, v_x) \mid x \in E \cap (R_X - I_X)\},\\ \mathcal{U}(L_X) &= \{(u_x, x] \mid x \in E \cap (L_X - I_X)\},\\ \mathcal{U}(T_X) &= \{(u_x, v_x) \mid x \in E - (R_X \cup L_X \cup I_X)\}. \end{aligned}$$

We claim that $\mathcal{U}' = I(\pi_1(\mathcal{U})) \cup \mathcal{U}(I_X) \cup \mathcal{U}(R_X) \cup \mathcal{U}(L_X) \cup \mathcal{U}(T_X)$ is an open cover of X. Obviously, every member of $\mathcal{U}'$ is open in X. It remains to show that $I(\pi_1(\mathcal{U})) \supseteq X - E$ since $\mathcal{U}(I_X) \cup \mathcal{U}(R_X) \cup \mathcal{U}(L_X) \cup \mathcal{U}(T_X) \supseteq E$. For every $x \in X$, if $x \notin E$, then there exist $u,\ v \in X$ with the property $u < x < v$ and $\langle x, y_0\rangle \in [\langle u, y_0\rangle, \langle v, y_1\rangle] \subseteq U \in \mathcal{U}$. Thus, $x \in (u, v) \prec I(\pi_1(U)) \in I(\pi_1(\mathcal{U}))$.

Suppose that $\mathcal{V}$ is a countable subcover of $\mathcal{U}'$. Define

$$\begin{aligned} \mathcal{W}_1 &= \{U \in \mathcal{U} \mid I(\pi_1(U)) \in \mathcal{V}\};\\ \mathcal{W}_2 &= \{U_x \mid x \in E \cap (R_X - I_X) \text{ and } [x, v_x) \in \mathcal{V}\};\\ \mathcal{W}_3 &= \{U_x \mid x \in E \cap (L_X - I_X) \text{ and } (u_x, x] \in \mathcal{V}\};\\ \mathcal{W}_4 &= \{U_{x_0}, U_{x_1} \mid x \in E - (R_X \cup L_X \cup I_X) \text{ and } (u_x, v_x) \in \mathcal{V}\}. \end{aligned}$$

Since Y is Lindelöf and homeomorphic to $\{x\} * Y$, there is a countable open subfamily $\mathcal{V}_x$ of $\mathcal{U}$ covers $\{x\} * Y$ for every $x \in X$. Then let

$$\mathcal{W}_5 = \cup\{\mathcal{V}_x \mid x \in E\}.$$

Hence, for $i = 1, 2, 3, 4, 5$, $\mathcal{W}_i$ is a countable open collection of the GOTP$(X * Y)$. Define $\mathcal{W} = \cup\{\mathcal{W}_i \mid i = 1, 2, 3, 4, 5\}$. By the above construction, $\mathcal{W}$ is a subset of $\mathcal{U}$. To conclude the proof, it suffices to show that $\mathcal{W}$ is an open cover of the GOTP$(X * Y)$. Let $s = \langle z, y\rangle \in X * Y$. There are two cases:

(i) If $z \in E$, then $s = \langle z, y\rangle$ is covered by $\mathcal{W}_5$.

(ii) If $z \notin E$, then there exists $V \in \mathcal{V}$ with $z \in V$. Furthermore, V must be a member of $I(\pi_1(\mathcal{U})) \cup \mathcal{U}(R_X) \cup \mathcal{U}(L_X) \cup \mathcal{U}(T_X)$. If $V \in I(\pi_1(\mathcal{U}))$, then there exists a $U \in \mathcal{U}$ such that $V = I(\pi_1(U))$. Hence, $s \in U \in \mathcal{W}_1$. If $V \in \mathcal{U}(T_X)$, then there exists $x \in E - (R_X \cup L_X \cup I_X)$ with $V = (u_x, v_x)$ and $x \neq z$. Thus, $s = \langle z, y\rangle$ belongs to $(\langle u_x, y_0\rangle, \langle x, y_0\rangle)$ or $(\langle x, y_1\rangle, \langle v_x, y_1\rangle)$. Hence, $s = \langle z, y\rangle$ belongs to U_{x_0} or U_{x_1}, either of which belongs to $\mathcal{W}_4$. The other cases are similar.

(2) $\Longrightarrow$ (1) Clearly, Y is Lindelöf because $\{x\} * Y$ is a closed set of the GOTP$(X * Y)$ and homeomorphic to Y for every x of X. Next, we will prove that X is Lindelöf. For any open cover $\mathcal{V}$ of X by convex sets, $\pi_1^{-1}(\mathcal{V}) = \{\pi_1^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of the GOTP$(X * Y)$ by Lemma 2.2. So, there is a countable open subcover $\pi_1^{-1}(\mathcal{V}')$ of $\pi_1^{-1}(\mathcal{V})$ with $\mathcal{V}' \subseteq \mathcal{V}$. Then, $\mathcal{V}'$ is a countable open subcover of $\mathcal{V}$. In fact, for each $x \in X$, there exists $V \in \mathcal{V}'$ such that $\langle x, y_0\rangle \in \pi_1^{-1}(V)$. Hence, $x \in V$. □

In Theorem 2.1, if Y has only one endpoint, then the GOTP$(X * Y)$ may not be Lindelöf.

Example 2.1. Let $Y = [0, 1)$ with the usual topology and let X denote $[0, 1]$ having a base consisting of all intervals $(z, 1]$, $[x, y)$, where x, y, $z \in [0, 1]$; $x < y$; and $x \neq 1$. Then the GOTP$(X * Y)$ is not Lindelöf.

In addition, we have the following theorem.

Theorem 2.2. *Let X, Y be GO-spaces. If Y has neither a left nor a right endpoint, then the* GOTP$(X * Y)$ *is Lindelöf if and only if Y is Lindelöf and $|X| < \omega_1$.*

Proof: Obvious, since $X * Y$ is the disjoint union of the open subset of the GOTP$(X * Y)$. □

3. Paracompactness and the GOTP

Lemma 3.1 ([4]). *Let X be a GO-space. Then X is hereditarily paracompact iff $X - \{x\}$ is paracompact for each point $x \in X$.*

Lemma 3.2. *Let X, Y be GO-spaces. If Y has both a left and a right endpoint, and*

(1) *if A is discrete in X and B is discrete in Y, then $A * B$ is discrete in the* GOTP$(X * Y)$*;*
(2) *if D is discrete in the GOTP $X * Y$, then $\pi_1(D)$ is discrete in X.*

Proof: Let y_0 denote the left endpoint of Y and y_1 denote the right endpoint of Y.

(1) For every $x \in X$ and $y \in Y$, there exist convex open subsets $O(x, A)$ (in X) and $U(y, B)$ (in Y) such that $O(x, A) \cap (A - \{x\}) = \emptyset$ and $U(y, B) \cap (B - \{y\}) = \emptyset$, respectively. It suffices to show that there exist open subsets $V(\langle x, y\rangle)$ of the GOTP$(X * Y)$ such that $V(\langle x, y\rangle) \cap (A * B - \{\langle x, y\rangle\}) = \emptyset$ for each $\langle x, y\rangle \in X * Y$. If $y \neq y_0,\ y_1$, then let $V(\langle x, y\rangle) = \{x\} * (U(y, B) - \{y_0,\ y_1\})$. If $y = y_0$, then let $V(\langle x, y\rangle) = \{x\} * U(y_0, B) \cup (\pi_1^{-1}(O(x, A)) \cap (\leftarrow, \langle x, y_0\rangle))$. If $y = y_1$, then let $V(\langle x, y\rangle) = \{x\} * U(y_1, B) \cup (\pi_1^{-1}(O(x, A)) \cap (\langle x, y_1\rangle, \rightarrow))$.

(2) For every $\langle x, y\rangle \in X * Y$, there exists a convex open neighborhood $O(\langle x, y\rangle)$ of $\langle x, y\rangle$ in the GOTP $X * Y$ such that $O(\langle x, y\rangle) \cap (D - \{\langle x, y\rangle\}) = \emptyset$. We shall prove that there exists an open neighborhood $U(x)$ of x in X such that $U(x) \cap (\pi_1(D) - \{x\}) = \emptyset$. There are four cases to consider: (i) $x \in I_X$ is clear. (ii) $x \in X - (R_X \cup L_X \cup I_X)$. Then there exists $u_x,\ v_x \in X$, $O(\langle x, y_0\rangle)$ and $O(\langle x, y_1\rangle)$ with $u_x < x < v_x$, $\langle v_x, y_1\rangle \in O(\langle x, y_1\rangle)$ and $\langle u_x, y_0\rangle \in O(\langle x, y_0\rangle)$. Then, let $U(x) = (u_x, v_x)$. (iii) $x \in R_X - I_X$. For $\langle x, y_1\rangle \in X * Y$, there exists $v_x > x$ with $\langle v_x, y_1\rangle \in O(\langle x, y_1\rangle)$. Then let $U(x) = [x, v_x)$. (iv) $x \in L_X - I_X$. Similarly, there exists $u_x < x$ with $\langle u_x, y_0\rangle \in O(\langle x, y_0\rangle)$, then let $U(x) = (u_x, x]$. □

Corollary 3.1. *Let X, Y be GO-spaces. If Y has both a left and a right endpoint, then the following are equivalent.*

(1) *The* GOTP$(X * Y)$ *is σ-discrete;*
(2) *X, Y are σ-discrete.*

Proof: (1) $\Longrightarrow$ (2) Let the GOTP$(X * Y)$ be σ-discrete. Obviously, Y is σ-discrete. Moreover, let

$$X * Y = \cup\{D_n \mid n \in \mathbb{N} \text{ and } D_n \text{ is discrete in the GOTP}(X * Y) \text{ for each } n \in \mathbb{N}\}.$$

Then, $X = \pi_1(X * Y) = \cup\{\pi_1(D_n) \mid n \in \mathbb{N}\}$ is σ-discrete in X by Lemma 3.2(2).

(2) $\Longrightarrow$ (1) Since X, Y are σ-discrete, we have

$X = \cup\{A_n \mid n \in \mathbb{N}$ and A_n is discrete in $X\}$ and
$Y = \cup\{B_m \mid m \in \mathbb{N}$ and B_m is discrete in $Y\}$.

For each $n, m \in \mathbb{N}$, let $C_{n,m} = \cup\{\{x\} * B_m \mid x \in A_n\}$. Then $X * Y = \cup\{C_{n,m} \mid n, m \in \mathbb{N}\}$. By Lemma 3.2(1), $C_{n,m}$ is discrete in the GOTP$(X * Y)$. Thus, the GOTP$(X * Y)$ is σ-discrete. $\square$

Lemma 3.3 ([4], Theorem 2.4.6)**.** *Let X be a GO-space. Then X is paracompact if and only if for each gap and each pseudo-gap (A, B) in X, there exist discrete subsets $C \subset A$ and $D \subset B$ which are cofinal in A and coinitial in B, respectively.*

Theorem 3.1. *If X, Y are (hereditarily) paracompact GO-spaces, then the* GOTP$(X * Y)$ *is a (hereditarily) paracompact GO-space.*

Proof: Let X, Y be paracompact. Suppose (A, B) is a left-pseudo-gap in the GOTP$(X * Y)$. (The other cases can be proved similarly.) Then, the singleton set $\{\langle b_{0X}, b_{0Y}\rangle\}$ of the left endpoint of B is a discrete subset of the GOTP$(X * Y)$ which is coinitial in B. So, it suffices to prove that there exists a cofinal subset D of A which is discrete in the GOTP$(X * Y)$. Let

$A_X = \{a_X \in X \mid$ there exists $a_Y \in Y$ such that $a = \langle a_X, a_Y\rangle \in A\}$,
$B_X = \{b_X \in X \mid$ there exists $b_Y \in Y$ such that $b = \langle b_X, b_Y\rangle \in B\}$.

Then $X = A_X \cup B_X$, $a_X \leq b_X$ for all $a_X \in A_X$, and $b_X \in B_X$. Further, $|A_X \cap B_X| \leq 1$ and B_X has a left endpoint b_{0X}.

Case 1. $A_X \cap B_X = \emptyset$. Since (A, B) is a left-pseudo-gap, Y has a left endpoint $y_0 = b_{0Y}$. First we claim that (A_X, B_X) must be a left-pseudo-gap in X. In fact, if (A_X, B_X) were not a left-pseudo-gap in X, then it would be a jump. So b_{0X} has an immediate predecessor b_{0X}^- in X that is the maximal point of A_X. Then Y

does not have a right endpoint since (A, B) is a left-pseudo-gap of the GOTP$(X * Y)$. But by the definition of GOTP, $[\langle b_{0X}, b_{0Y}\rangle, \rightarrow)$ is not open in the GOTP$(X * Y)$. This contradicts that (A, B) is a left-pseudo-gap of the GOTP$(X * Y)$.

Now we know that (A_X, B_X) is a left-pseudo-gap in X. Then there exists $D_X \subseteq A_X$ such that D_X is a discrete subset of X and cofinal in A_X. Pick arbitrary $y \in Y$. Let $D = \{\langle x, y\rangle \mid x \in D_X\}$. Obviously, D is a cofinal subset of A. By Lemma 3.2, D is discrete in the GOTP$(X * Y)$.

Case 2. $A_X \cap B_X \neq \emptyset$. Then $A_X \cap B_X = \{b_{0X}\}$. Let

$$A_Y = \{a_Y \in Y \mid \langle b_X, a_Y\rangle \in A\}$$

and

$$B_Y = \{b_Y \in Y \mid \langle b_X, b_Y\rangle \in B\}.$$

Then (A_Y, B_Y) is a left-pseudo-gap in Y since $\{b_X\} * Y$ is homeomorphic to Y. Hence, there exists a subset $D_Y \subseteq A_Y$ which is discrete in Y and cofinal in A_Y. Now, define $D = \{\langle b_{0X}, y\rangle \mid y \in D_Y\}$. Then D is a discrete subset of the GOTP$(X * Y)$ and a cofinal subset of A.

Next, let X, Y be hereditarily paracompact. By Lemma 3.1, it suffices to prove that there are discrete subsets D and R of $X * Y - \{p\}$, such that D is cofinal in $\{s \in X * Y \mid s < p\}$ and R is coinitial in $\{s \in X * Y \mid p < s\}$ for every $p \in X * Y$. Let $p = \langle p_X, p_Y\rangle \in X * Y$. If p_Y is not an endpoint of Y, then the proof is clear since Y is hereditarily paracompact. Let p_Y be the left endpoint of Y. Then there are two possibilities to consider:

(i) p_X has an immediate predecessor p_X^- in X. If Y has the right endpoint y_1, then let $D = \{\langle p_X^-, y_1\rangle\}$. If Y does not have a right endpoint, then there exists $D_Y \subseteq Y$ such that D_Y is discrete and cofinal in Y. Thus, let $D = \{\langle p_X^-, y\rangle \mid y \in D_Y\}$.

(ii) p_X does not have an immediate predecessor in X. Then there exists $D_X \subseteq X - \{p_X\}$ such that D_X is discrete in $X - \{p_X\}$ and cofinal in $\{x \mid x < p_X\}$. Thus, let $D = \{\langle x, p_Y\rangle \mid x \in D_X\}$.

In either case, D is discrete in $X * Y - \{p\}$ and cofinal in $\{s \in X * Y \mid s < p\}$. Further, there exists $R_Y \subseteq Y$ such that R_Y is discrete and coinitial in $Y - \{p_Y\}$. Hence, let $R = \{\langle p_X, y\rangle \mid y \in R_Y\}$. Then R

is discrete in $X * Y - \{p\}$ and cofinal in $\{s \in X * Y \mid s > p\}$. If p_Y is the right endpoint of Y, the proof is similar. $\square$

The converse of the above theorem is not true. (See Example 3.1.)

Example 3.1. Let $X = \omega_0$ and $Y = \omega_1$ with the usual order topology. Then Y is not paracompact. However, the GOTP$(X * Y)$ is paracompact. (In fact, the GOTP$(X * Y)$ is Lindelöf.)

Theorem 3.2. *Let X, Y be GO-spaces. If Y has neither a left nor a right endpoint, then the* GOTP$(X * Y)$ *is paracompact if and only if Y is paracompact.*

The proof of Theorem 3.2 is obvious because the GOTP$(X * Y)$ is the disjoint union of open subsets of $X * Y$, each homeomorphic to Y. Moreover, when "paracompact" is replaced by σ-discrete, or perfectly normal, or p-space, or M-space, or metrizable, the conclusion of Theorem 3.2 is also true.

Theorem 3.3. *Let X, Y be GO-spaces. If Y has both a left and a right endpoint, then the following are equivalent.*

(1) *The* GOTP$(X * Y)$ *is paracompact;*
(2) *X, Y are paracompact.*

Proof: (2) $\Longrightarrow$ (1) is clear by Theorem 3.1.

(1) $\Longrightarrow$ (2) Obviously, Y is paracompact since it is homeomorphic to the closed $\{x\} * Y$ of the GOTP$(X * Y)$ for every $x \in X$. Next, we prove that X is paracompact. Let (A, B) be a left-pseudo-gap in X. (The other cases are proved similarly.) Then $(\pi_1^{-1}(A), \pi_1^{-1}(B))$ is a left-pseudo-gap in the GOTP$(X * Y)$. Hence, there exists a discrete subset D of the GOTP$(X * Y)$ such that D is a cofinal subset of $\pi_1^{-1}(A)$. Thus, $\pi_1(D)$ is a cofinal subset of A and a discrete subset of X by Lemma 3.2(2). $\square$

Theorem 3.4. *Let X, Y be GO-spaces. If X does not have neighbor points, then the following are equivalent.*

(1) *the* GOTP$(X * Y)$ *is paracompact;*
(2) *X, Y are paracompact.*

Proof: (2) $\Longrightarrow$ (1) is clear by Theorem 3.1.

(1) $\Longrightarrow$ (2) Since X does not have neighbor points, $\{x\} * Y$ is a closed subset of the GOTP$(X * Y)$ for every $x \in X$. Then Y

is paracompact because Y is homeomorphic to $\{x\} * Y$ for every $x \in X$. Next, we prove that X is paracompact. Let (A, B) be a left-pseudo-gap in X. (The other cases proved similarly.) There are two cases.

Case 1. If Y has a left endpoint, then $(\pi_1^{-1}(A), \pi_1^{-1}(B))$ is a left-pseudo-gap in the $\mathrm{GOTP}(X * Y)$.

Case 2. If Y does not have a left endpoint, then $(\pi_1^{-1}(A), \pi_1^{-1}(B))$ is a gap in the $\mathrm{GOTP}(X * Y)$.

In either case, there is a discrete subset D in the $\mathrm{GOTP}(X * Y)$ which is cofinal in $\pi_1^{-1}(A)$. Thus, $\pi_1(D)$ must be cofinal in A. Further, $\pi_1(D)$ is a discrete subset of X by Lemma 3.2 (2). The proof is finished. □

In Theorem 3.4, the condition "X does not have neighbor points" cannot be removed. Otherwise, Y may not be paracompact (see Example 3.1).

Lemma 3.4 ([4]). *Let X be a GO-space. Then the following are equivalent.*

(1) *X is metrizable;*
(2) *there exists a subset $D \subseteq X$ such that*
  (i) $\overline{D} = X$; (ii) $E_X \subseteq D$; (iii) D is σ-discrete (in X).

Theorem 3.5. *Let X, Y be GO-spaces. If Y has both a left and a right endpoint, then the following are equivalent.*

(1) *The $\mathrm{GOTP}(X * Y)$ is metrizable;*
(2) *X is σ-discrete and Y is metrizable.*

By Lemma 3.2 and Lemma 3.4, the proof of Theorem 3.5 is easy.

4. Other results on the GOTP

Theorem 4.1. *Let X, Y be GO-spaces.*

(1) *If $|Y| > 2$ and Y has both a left and a right endpoint, then the $\mathrm{GOTP}(X * Y)$ is perfectly normal $\Longleftrightarrow$ X is σ-discrete and Y is perfectly normal.*
(2) *If $|Y| = 2$, then the $\mathrm{GOTP}(X * Y)$ is perfectly normal $\Longleftrightarrow$ X is perfectly normal and E_X is σ-discrete in X.*

Proof: (1) Necessity. Clearly, Y is perfectly normal. Since $|Y| > 2$, there exists $y \in Y$ such that y is not an endpoint of Y. Then

$\{\langle x,y\rangle \mid x\in X\}$ is a relatively discrete subset in the GOTP$(X*Y)$. Hence, $\{\langle x,y\rangle \mid x\in X\}$ is a $\sigma-$discrete subset in the GOTP$(X*Y)$ by Theorem 2.4.5 in [4]. Therefore, X is $\sigma-$discrete by the Lemma 3.2(2).

Sufficiency. Let P be a relatively discrete subset in the GOTP$(X*Y)$. Let $P_x = (\{x\}*Y)\cap P$. Then P_x is a relatively discrete subset in $\{x\}*Y$ for each $x\in X$. Since $\{x\}*Y$ is homeomorphic to Y, P_x is σ-discrete in Y for each $x\in X$. In addition, $P\subseteq \{\{x\}*P_x \mid x\in \pi_1(P)\}$. Thus, P is σ-discrete in the GOTP$(X*Y)$ by Lemma 3.2(1). By Theorem 2.4.5 in [4], the GOTP$(X*Y)$ is perfectly normal.

(2) Necessity. Suppose $Y=\{y_0,y_1\}$ with $y_0<y_1$. Let P be relatively discrete in X. Then $\pi_1^{-1}(P)$ is relatively discrete in the GOTP$(X*Y)$. Hence, $\pi_1^{-1}(P)$ is σ-discrete in the GOTP$(X*Y)$. By Lemma 3.2(2), P is σ-discrete in X. Then X is perfectly normal.

For each $x\in L_X$, $\langle x,y_1\rangle$ is an isolated point in GOTP$(X*Y)$ since $(\leftarrow,\langle x,y_1\rangle]$ is open and $\langle x,y_1\rangle$ has an immediate predecessor $\langle x,y_0\rangle$. Similarly, for each $x\in R_X$, $\langle x,y_0\rangle$ is isolated. Thus, $A=\{\langle x,y_1\rangle | x\in L_X\}\cup\{\langle x,y_0\rangle | x\in R_X\}$ is a relatively discrete subset of GOTP$(X*Y)$. By perfectness of GOTP$(X*Y)$, A is σ-discrete. Again by Lemma 3.2, $E_X=\pi_1(A)$ is σ-discrete in X.

Sufficiency. Let $F=(X*Y)/\sim$ be the quotient space of GOTP(X,Y) (See section 1 for the definition of the equivalence relation $\sim$). Then F is a GO-space with respect to the ordering inherited from $X*Y$. Observe that

$$N_{\mathrm{GOTP}(X*Y)}=\{\langle x,y_0\rangle,\langle x,y_1\rangle | x\in X-(L_X\cup R_X)\}$$

and

$$\begin{aligned}F=\{\{\langle x,y_0\rangle,\langle x,y_1\rangle\} | x\in X-(L_X\cup R_X)\}&\\ \cup\{\{\langle x,y\rangle\} | \langle x,y\rangle\in (X*Y)-N_{\mathrm{GOTP}(X*Y)}\}&.\end{aligned}$$

Define $g: X\to F$ as follows

$$g(x)=\begin{cases}\{\langle x,y_0\rangle,\langle x,y_1\rangle\} & \text{if } x\in X-(L_X\cup R_X)\\ \{\langle x,y_0\rangle\} & \text{if } x\in L_X\\ \{\langle x,y_1\rangle\} & \text{if } x\in R_X-I_X.\end{cases}$$

Then g is an embedding map from X to F. We regard x and $g(x)$ as the same thing. Suppose $I=F-X$. Then I is a set consisting of isolated points of the GOTP$(X*Y)$. Hence, any family consisting of disjoint convex (in F) subsets of I is σ-discrete in the GOTP$(X*Y)$ and also in F. Hence, F may be regarded as the union of X and I, and I is an open σ-discrete subset of F. Suppose $\mathcal{O}$ is a family of disjoint open convex subsets of F. Then $\mathcal{O}|X = \{O \cap X | O \in \mathcal{O}\}$ is a family of disjoint open convex subsets of X. It follows that $\mathcal{O}|X$ is σ-discrete in X since X is perfectly normal. So we may put $\mathcal{O} = \cup\{\mathcal{O}_n | n \in \mathbb{N}\}$ such that $\mathcal{O}_n|X$ is discrete in X. Moreover, $\{O \in \mathcal{O}_n | O \cap X \neq \emptyset\}$ is discrete in F as well since each member of $\mathcal{O}_n$ is convex. Therefore, $\{O \in \mathcal{O} | O \cap X \neq \emptyset\}$ is σ-discrete in F. Next, if $O \in \mathcal{O}$ does not meet X, then $O \subset I$. So $\{O \in \mathcal{O} | O \cap X = \emptyset\}$ is σ-discrete in F. Hence, $(X*Y)/\sim$ is perfectly normal. Thus, the GOTP$(X*Y)$ is perfectly normal by [4, Lemma 3 (p. 26)]. $\square$

Let X be a GO-space and suppose $\xi = (A, B)$ is a (pseudo-)gap, possibly an endgap. Then ξ is said to be *countable* from the left if some strictly increasing countably infinite sequence is cofinal in A, and ξ is said to be *countable* from the right if there is a strictly decreasing countably infinite sequence coinitial in B. The (pseudo-)gap ξ is said to be countable if it is countable from the left or from the right.

Lemma 4.1 ([6]). (1) *Let X be a GO-space. Then X is a p-space if and only if there exists a sequence $(\mathcal{V}(n))_{n\in\mathbb{N}}$ of convex open covers of X with the property that for each $x \in X$ and each (pseudo-)gap $\xi = (A, B)$ in X there exists an $n = n(x, \xi) \in \mathbb{N}$ such that $St(x, \mathcal{V}(n))$ does not cover the (pseudo-)gap ξ.*

(2) *Let X be a GO-space. Therefore, X is an M-space if and only if there exists a sequence $(\mathcal{V}(n))_{n\in\mathbb{N}}$ of convex open covers of X with the property that for each $x \in X$ and each countable (pseudo-)gap $\xi = (A, B)$ in X, there exists an $n = n(x, \xi) \in \mathbb{N}$ such that $St(x, \mathcal{V}(n))$ does not cover the (pseudo-)gap ξ.*

The following theorem can be proved in a way analogous to the proof of Theorem 3.1.2 in [6]. But we give here a different (direct) proof.

Theorem 4.2. *Let X and Y be GO-spaces. If Y has both a left and a right endpoint and no (interior) gaps, then the following are equivalent.*

(1) *X is a p-space;*

(2) *the* GOTP$(X * Y)$ *is a p-space.*

Proof: Let y_0 denote a left endpoint of Y and y_1 denote a right endpoint of Y.

(1) $\Longrightarrow$ (2) Suppose (A, B) is a (pseudo-)gap of the GOTP$(X * Y)$. Then $(\pi_1(A), \pi_1(B))$ is a (pseudo-)gap of X since Y is a compact LOTS. Let $(\mathcal{V}(n))_{n\in\mathbb{N}}$ be open covers of X with the properties of Lemma 4.1. Thus, $(\pi_1^{-1}(\mathcal{V}(n)))_{n\in\mathbb{N}}$ are open covers of the GOTP$(X * Y)$ with the properties of Lemma 4.1. In fact, for every $\langle x, y\rangle \in X * Y$ and (pseudo-)gap (A, B), there exists $n \in \mathbb{N}$ such that $St(x, \mathcal{V}(n))$ does not cover the (pseudo-)gap $(\pi_1(A), \pi_1(B))$. Hence, $St(\langle x, y\rangle, \pi_1^{-1}(\mathcal{V}(n)))$ does not cover the (pseudo-)gap (A, B).

(2) $\Longrightarrow$ (1) Let $(\mathcal{U}(n))_{n\in\mathbb{N}}$ be open covers of the GOTP$(X * Y)$ with the properties of Lemma 4.1. Without loss of generality, suppose that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$. For each $n \in \mathbb{N}$, let $I(\mathcal{U}(n)) = \{I(\pi_1((U)) \mid U \in \mathcal{U}(n))\}$ and $E(\mathcal{U}(n)) = X - \cup I(\mathcal{U}(n))$. For $x \in E(\mathcal{U}(n)) \cap I_X$, let $V(x, n) = \{x\}$. For $x \in E(\mathcal{U}(n)) \cap (R_X - I_X)$, there exists $v_x > x$ such that $\langle v_x, y_1\rangle \in St(\langle x, y_1\rangle, \mathcal{U}(n))$. Then let $V(x, n) = [x, v_x)$. Similarly, for $x \in E(\mathcal{U}(n)) \cap (L_X - I_X)$, there exists $u_x < x$ such that $\langle u_x, y_0\rangle \in St(\langle x, y_0\rangle, \mathcal{U}(n))$. Then let $V(x, n) = (u_x, x]$. For $x \in E(\mathcal{U}(n)) - (R_X \cup L_X \cup I_X)$, there exist u_x, v_x such that $u_x < x < v_x$, $\langle u_x, y_0\rangle \in St(\langle x, y_0\rangle, \mathcal{U}(n))$ and $\langle v_x, y_1\rangle \in St(\langle x, y_1\rangle, \mathcal{U}(n))$. Then let $V(x, n) = (u_x, v_x)$ and $\mathcal{V}(n) = I(\mathcal{U}(n)) \cup \{V(x, n) \mid x \in E(\mathcal{U}(n))\}$. Thus, the sequence $(\mathcal{V}(n))_{n\in\mathbb{N}}$ of open covers of X satisfies the properties of Lemma 4.1. In fact, let (A, B) be a (pseudo-)gap of X. Then $(\pi_1^{-1}(A), \pi_1^{-1}(B))$ is a (pseudo-)gap of the GOTP$(X * Y)$. For $x \in X$, there are m, $n \in \mathbb{N}$ such that $St(\langle x, y_0\rangle, \mathcal{U}(m))$ and $St(\langle x, y_1\rangle, \mathcal{U}(n))$ do not cover the (pseudo-)gap $(\pi_1^{-1}(A), \pi_1^{-1}(B))$ of the GOTP$(X * Y)$. Let $l = \max\{m, n\}$. Then $St(\langle x, y_0\rangle, \mathcal{U}(l))$ and $St(\langle x, y_1\rangle, \mathcal{U}(l))$ do not cover the (pseudo-)gap $(\pi_1^{-1}(A), \pi_1^{-1}(B))$. Therefore, $St(x, \mathcal{V}(l))$ does not cover the (pseudo-)gap (A, B) because $\pi_1^{-1}(St(x, \mathcal{V}(l))) \subseteq St(\langle x, y_0\rangle, \mathcal{U}(l)) \cup St(\langle x, y_1\rangle, \mathcal{U}(l))$. □

By an argument similar to the proof of Theorem 4.2, we have the following theorem.

Theorem 4.3. *Let X and Y be GO-spaces. If Y has both a left and a right endpoint and no countable gaps, then the following are equivalent.*

(1) *X is an M-space;*
(2) *the* GOTP$(X * Y)$ *is an M-space.*

The following theorems are improvements of theorems 3.1.3, 3.2.3, 3.1.5, 3.1.6, 3.2.5, 3.2.6 in [6]; Lemma 3 (page 81) and theorems 4.4.3, 4.4.7 in [4] can be proved by some modifications, respectively.

Theorem 4.4. *Let X, Y be GO-spaces. If Y has both a left and a right endpoint and*

(1) *if Y has at least one interior gap, then the* GOTP$(X * Y)$ *is a p-space $\Longleftrightarrow$ X is σ-discrete and Y is a p-space.*
(2) *if Y has at least one countable gap, then the* GOTP$(X * Y)$ *is an M-space $\Longleftrightarrow$ X is σ-discrete and Y is an M-space.*

Theorem 4.5. *Let X, Y be GO-spaces. If Y has a left* (*right*) *endpoint, but no right* (*left*) *one, and*

(1) *if Y has no interior gaps, then the* GOTP$(X * Y)$ *is a p-space $\Longleftrightarrow$ X is a left-(right) p-space, and $D = \{x \in X \mid x$ has no right* (*left*) *neighbor point$\}$ is σ-l-(σ-r-)discrete;*
(2) *if Y has at least one interior gap, then the* GOTP$(X * Y)$ *is a p-space $\Longleftrightarrow$ X is a σ-l-(σ-r-)discrete, Y is a p-space and if X contains neighbor points and the interior gaps are cofinal (coinitial) in Y, then Y has cofinality ω_0 (coinitiality ω_0^*);*
(3) *if Y has a countable right (left) endgap and no countable interior gaps, then the* GOTP$(X*Y)$ *is an M-space $\Longleftrightarrow$ X is a left-(right) M-space, and $D = \{x \in X \mid x$ has no right (left) neighbor point$\}$ is σ-l-(σ-r-)discrete;*
(4) *if Y has at least one countable interior gap, then the* GOTP $(X * Y)$ *is an M-space $\Longleftrightarrow$ X is a σ-l-(σ-r-)discrete, Y is an M-space and if X contains neighbor points and the countable interior gaps are cofinal (coinitial) in Y, then Y has cofinality ω_0 (coinitiality ω_0^*);*
(5) *if X has neighbor points, then*
   (a) *X is σ-l-(σ-r-)discrete, Y is σ-discrete, and ω_0* (ω_0^*) *is cofinal (coinitial) in Y $\Longleftrightarrow$ the* GOTP$(X * Y)$ *is σ-discrete;*
   (b) *X is σ-l-(σ-r-)discrete, Y is metrizable, and ω_0* (ω_0^*) *is cofinal (coinitial) in Y $\Longleftrightarrow$ the* GOTP$(X * Y)$ *is metrizable;*

(c) *X is σ-l-(σ-r-)discrete, Y is perfectly normal, and ω_0 (ω_0^*) is cofinal (coinitial) in $Y \iff$ the* GOTP$(X * Y)$ *is perfectly normal;*

(6) *if X does not have neighbor points, then*

(a) *X is σ-l-(σ-r-)discrete and Y is σ-discrete $\iff$ the* GOTP$(X * Y)$ *is σ-discrete;*

(b) *X is σ-l-(σ-r-)discrete and Y is metrizable $\iff$ the* GOTP$(X * Y)$ *is metrizable;*

(c) *X is σ-l-(σ-r-)discrete and Y is perfectly normal $\iff$ the* GOTP$(X * Y)$ *is perfectly normal.*

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