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A NOTE ON SURLINDELÖF SPACES

O. OKUNEV AND E. REZNICHENKO

Abstract. We consider separable compact subspaces of the spaces of continuous functions with the topology of pointwise convergence on Lindelöf spaces and prove, in particular, that it is consistent that all such compact subspaces are metrizable.

Introduction

An important topic in the theory of function spaces with the topology of pointwise convergence is the description of subspaces, in particular, compact subspaces of the spaces of the form $C_p(X)$ where X belongs to some class of spaces. In this article we present a few results concerning *surlindelöf* spaces, that is, subspaces of $C_p(X)$ where X is a Lindelöf space [2].

About 1985 the second author proved (see [4])

Theorem 0.1. (MA(ω_1)) *If K is a separable compact space such that $C_p(K)$ is Lindelöf, then K is metrizable.*

A few years later the first author obtained the following result[10]:

Theorem 0.2. (MA(ω_1)) *If X is a space whose all finite powers are Lindelöf, and K is a separable compact subspace of $C_p(X)$, then K is metrizable.*

In both cases the theorems were originally formulated with the assumption of MA+¬CH, but in fact only MA(ω_1) was used

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in the proofs. Note that both theorems need some set-theoretic assumptions, because the equality $\mathfrak{b} = \omega_1$ implies the existence of a nonmetrizable compact space X whose all finite powers are hereditarily separable [13]; by the theorem of Zenor [14], all finite powers of $C_p(X)$ are hereditarily Lindelöf.

Theorems 0.1 and 0.2 have even more in common than it may appear; in fact, 0.1 would be a corollary of 0.2 if we had a positive answer to the following problem, stated many times by Arhangel'skii (see e.g. [3]):

Problem 0.3. *Is it true that if $C_p(X)$ is Lindelöf, then $C_p(X) \times C_p(X)$ is Lindelöf? Is this true if X is compact?*

On the other hand, if K is a compact subspace of $C_p(X)$ for some X whose *countable* power is Lindelöf, then $C_p(K)$ is Lindelöf [1], so a version of Theorem 0.2 for subspaces of $C_p(X)$ whose countable power is Lindelöf can be obtained from Theorem 0.1. The natural question that arises is

Question 0.4. *Does the following statement follow from* $\mathrm{MA}(\omega_1)$*: If all finite powers of X are Lindelöf and K is a compact subspace of $C_p(X)$, then $C_p(K)$ is Lindelöf? Is this statement consistent with ZFC?*

Note that there is an example, under CH, of a compact, separable, zero-dimensional space K such that all finite powers of $C_p(K, 2)$ are Lindelöf, and $C_p(K)$ is not Lindelöf [11]; the space K embeds in $C_p(C_p(K, 2))$, so the statement in Question 0.4 is false for K.

In this article we try to find a common generalization for Theorems 0.1 and 0.2; the most natural candidate would be

Question 0.5. *Assume* $\mathrm{MA}(\omega_1)$*; must every surlindelöf, compact, separable space be metrizable?*

(This question is essentially the same as Problem IV.1.8 in [4].)

We do not know the answer to this question; in this article we prove that the answer "yes" follows from the Proper Forcing Axiom and is consistent with ZFC.

All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We mostly use terminology and notation as in [7], with the exception that the tightness of a space X is denoted by $t(X)$. The *i-weight* of a space X, $iw(X)$ is the minimum weight of a Tychonoff space Y such that X has a continuous bijection to Y (or, in different words, the minimum cardinality of a family of continuous real-valued functions that separates points of X). A space X is called $\aleph_0$-*monolithic* if the closure of every countable set in X has a countable network.

We abbreviate as $\mathrm{MA}(\omega_1)$ the Martin's Axiom for families of dense sets of cardinality ω_1, and PFA is an abbreviation for the Proper Forcing Axiom. A subset C of a partially ordered set $\mathbb{P}$ is called *centered* if every two elements of C are compatible; it is well known that $\mathrm{MA}(\omega_1)$ implies that every uncountable partially ordered set with ccc has an uncountable centered subset.

We denote by $C_p(X, Z)$ the space of all continuous functions from X to Z equipped with the topology of pointwise convergence (that is, the topology of the subspace of the set of all functions from X to Z, Z^X, with the Tychonoff product topology; see [4] for a thorough presentation of the theory of spaces of functions equipped with this topology). The space $C_p(X, \mathbb{R})$ is denoted as $C_p(X)$.

For every $\bar{x} = (x_1, \ldots, x_n) \in X^n$ and an open set $U \subset \mathbb{R}^n$, we denote

$$O(\bar{x}, U) = \{\, f \in C_p(X) : (f(x_1), \ldots, f(x_n)) \in U \,\}.$$

Note that if for every $n \in \omega$, $\mathcal{B}_n$ is an open base for $\mathbb{R}^n$, then the family of all sets of the form $O(\bar{x}, U)$, $\bar{x} \in X^n$, $U \in \mathcal{B}_n$, $n \in \omega$, is an open base for the topology of $C_p(X)$.

The *dual mapping* $p^*\colon C_p(Y) \to C_p(X)$ of a continuous mapping $p\colon X \to Y$ is defined by the rule: $p^*(g) = g \circ p$ for all $g \in C_p(Y)$. It is well known (see e.g., [4]) that p^* is always continuous, that it is an embedding of $C_p(Y)$ into $C_p(X)$ if $p(X) = Y$, and that $p^*(C_p(Y))$ is closed in $C_p(X)$ if p is quotient.

For a subset A of $C_p(X)$, *the reflection mapping* $\Psi_{XA}\colon X \to C_p(A)$ is defined by the rule:

$$\Psi_{XA}(x)(a) = a(x) \text{ for all } x \in X \text{ and } a \in A.$$

It is easy to verify that the mapping Ψ_{XA} coincides with the diagonal product $\Delta A\colon X \to \mathbb{R}^A$. Thus, Ψ_{XA} is always continuous, and is one-to-one if and only if A separates points of X. In particular, if X is compact and A separates points of X, then Ψ_{XA} is an embedding of X into $C_p(A)$.

For every point $x \in X$, $\hat{x}$ is the *evaluation function* on $C_p(X)$ defined by the rule: $\hat{x}(f) = f(x)$ for all $f \in C_p(X)$. Obviously, for every $x \in X$ and $A \subset C_p(X)$, $\Psi_{XA}(x) = \hat{x}|A$.

A space Z is called an *Eberlein-Grothendieck* space (or, shortly, an *EG-space*) if Z is homeomorphic to a subspace of $C_p(K)$ for some compact space K. As we already mentioned, Z is a *surlindelöf* space if it is homeomorphic to a subspace of $C_p(X)$ for some Lindelöf space X.

1. Surlindelöf, separable compact spaces

Theorem 1.1. (MA(ω_1)) *Let K be a separable, compact space. Then every Lindelöf subspace of $C_p(K)$ is hereditarily Lindelöf.*

Proof. Let X be a Lindelöf subspace of $C_p(K)$; assume for contradiction that X is not hereditarily Lindelöf. By the Baturov Theorem [5], then X contains a discrete subspace S of cardinality ω_1; we may assume without loss of generality that S is dense in X.

For every $n \in \omega$ let $\mathcal{B}_n$ be a countable open base for $\mathbb{R}^n$. Then for every $s \in S$ there are an $n_s \in \omega$, $\bar{y}_s \in K^{n_s}$ and $U_s, V_s \in \mathcal{B}_{n_s}$ such that $\overline{U_s} \subset V_s$, $s \in O(\bar{y}_s, U_s)$ and $O(\bar{y}_s, V_s) \cap X = \{s\}$.

By the standard uncountability argument, we may assume without loss of generality that there are $n \in \omega$ and $U, V \in \mathcal{B}_n$ such that $n_s = n$, $U_s = U$ and $V_s = V$ for all $s \in S$.

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $h(\overline{U}) = \{1\}$ and $h(\mathbb{R}^n \setminus V) = \{0\}$. The induced mapping $h_*\colon C_p(X)^n \to C_p(X)$ defined by the rule $h_*(g) = h \circ g$ for every $g \in C_p(X)^n = C_p(X, \mathbb{R}^n)$ is continuous, because it is the restriction to $C_p(X, \mathbb{R}^n)$ of the product of $|X|$ copies of the continuous mapping h.

Let $\hat{K} = \Psi_{KX}(K) \subset C_p(X)$ and $K_0 = h_*(\hat{K}^n)$; for every $s \in S$ put $z_s = h_*(\bar{y}_s)$. Then K_0 is a separable compact subspace of $C_p(X)$, and for every $s \in S$, $z_s \in K_0$, $z_s(s) = 1$ and $z_s(X \setminus \{s\}) = \{0\}$. Let M be a dense, countable subset of K_0, $Z = \{\, z_s : s \in S \,\} \setminus M$, $S_0 = \{\, s \in S : z_s \in Z \,\}$, and F the closure of Z in K_0. Note that Z is an uncountable discrete subset of $C_p(X)$ (because $\hat{s}(z_s) = 1$ and $\hat{s}(z_t) = 0$ if $t \neq s$), and F is a nowhere dense, compact subspace of K_0.

Since X has the discrete dense subset S of cardinality ω_1, we have $iw(C_p(X)) = d(X) = \omega_1$ (see [9]) or Theorem I.1.5 in [4]), and therefore, the weight of K_0 is equal to ω_1.

By Reznichenko's Lemma (see Lemma IV.8.9 in [4]) we can find a countable discrete subset $D = \{\, d_i : i \in \omega \,\}$ of M so that $F = \overline{D} \setminus D$; note that every neighborhood of F contains all but finitely many elements of D.

Put $T = \overline{S}_0 \setminus S_0$, and for every $i \in \omega$,

$$P_i = \{\, x \in X : |d_i(x)| \geq 1/2 \,\}$$

and

$$P = \bigcap_{k \in \omega} \bigcup_{i \geq k} P_i.$$

Obviously, P is a $F_{\sigma\delta}$-set in X.

For every $t \in T$, the evaluation function $\hat{t}$ is continuous, and $\hat{t}(z) = z(t) = 0$ for all $z \in Z$; it follows that there is a $k \in \omega$ such that $t \notin P_i$ for all $i \geq k$, so $t \notin P$.

On the other hand, if s is in S_0, then $z_s(s) = 1$, so $d_i(s) \geq 1/2$ for infinitely many elements of D. Thus,

$$S_0 \subset P \quad \text{and} \quad T \cap P = \emptyset.$$

It follows that S_0 is an $F_{\sigma\delta}$-set in its closure, and hence in X. The contradiction now follows from the next observation:

Lemma 1.2. (MA(ω_1)) *If X is a Lindelöf EG-space, then every $F_{\sigma\delta}$-subspace of X is Lindelöf.*

For a proof, see, e.g., Lemmas 2 and 3 in [10]. □

Corollary 1.3. (MA(ω_1)) *If K is a separable, compact surlindelöf space, then there is a hereditary Lindelöf subspace X of $C_p(K)$ such that K is homeomorphic to a subspace of $C_p(X)$.*

Proof. Let X_0 be a Lindelöf space such that $K \subset C_p(X_0)$; put $X = \Psi_{X_0 K}(X_0)$. Then X is a Lindelöf subspace of $C_p(K)$, and by Theorem 1.1, X_0 is hereditarily Lindelöf. Since $K \subset C_p(X_0)$, the set X separates points of K, so the reflection mapping Ψ_{KX} embeds K in $C_p(X)$. □

Recall that a space X is called an *L-space* if X is hereditarily Lindelöf and not separable. A space S is called *left-separated* if there is a well-ordering $\leq$ on S such that for every $s \in S$, the set $\{\, t \in S : s \leq t \,\}$ is open in S. It is well-known that every L-space contains a left-separated subspace of cardinality ω_1.

Corollary 1.4. (MA(ω_1)) *If K is a separable, non-metrizable, compact surlindelöf space, then there is an L-subspace X of $C_p(K)$ such that K is homeomorphic to a subspace of $C_p(X)$.*

Indeed, let X be as in Corollary 1.3. Then X is hereditarily Lindelöf and non-separable; otherwise we would have $w(K) = iw(K) \leq iw(C_p(X)) = d(X) = \omega$, and K would be metrizable.

Theorem 1.5. (MA(ω_1)) *Let K be a compact space of countable tightness. Then $C_p(K)$ contains no L-subspaces.*

Proof. Suppose X is an L-subspace of $C_p(K)$. Then X has a left-separated hereditarily Lindelöf subspace $S = \{\, s_\alpha : \alpha < \omega_1 \,\}$ where for every $\alpha < \omega_1$, the set $\{\, s_\beta : \alpha \leq \beta \,\}$ is open in S. For every $n \in \omega$ fix a countable open base $\mathcal{B}_n$ for $\mathbb{R}^n$. Then for every $\alpha < \omega_1$ there are $n_\alpha \in \omega$, $\bar{y}_\alpha \in K^{n_\alpha}$ and $U_\alpha, V_\alpha \in \mathcal{B}_{n_\alpha}$ such that $\overline{U_\alpha} \subset V_\alpha$, $s_\alpha \in O(\bar{y}_\alpha, U_\alpha)$, and the set $W_\alpha = O(\bar{y}_\alpha, V_\alpha) \cap S$ is contained in $\{\, s_\beta : \alpha \leq \beta \,\}$. By a standard uncountability argument, we may assume (taking an uncountable subspace of S instead of S) that $n_\alpha = n$, $U_\alpha = U$ and $V_\alpha = V$ for all $\alpha \in \omega_1$. Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $h(\overline{U}) = \{1\}$ and $h(\mathbb{R}^n \setminus V) = \{0\}$, and let $h_*\colon C_p(S, \mathbb{R}^n) = C_p(S)^n \to C_p(S)$ be the induced mapping. Put $\hat{K} = \Psi_{KS}(K)$ and $K_1 = h_*(\hat{K}^n)$; for every $\alpha < \omega_1$ put $z_\alpha = h_*(\bar{x}_\alpha)$. Then K_1 is a compact subspace of $C_p(S)$; we have $t(K_1) \leq \omega$, because the tightness of compact spaces is not raised by finite products and continuous mappings [8], [4]. Obviously, $z_\alpha(s_\beta) = 0$ whenever $\beta < \alpha$, and $z_\alpha(s_\beta) = 1$ if $s_\beta \in W_\alpha$.

For every $m \in \omega$ let $\phi_m\colon C_p(S)^m \to C_p(S)$ be the mapping defined by the rule $\phi_m(f_1, \ldots, f_m) = \max(f_1, \ldots, f_m)$ and let $K_m = \phi_m(K_1^m)$. Since ϕ_m is continuous, K_m is a compact subspace of $C_p(S)$ and $t(K_m) \leq \omega$. For every finite subset p of ω_1 denote $W_p = \bigcup\{\, W_\alpha : \alpha \in p \,\}$ and $z_p = \max\{\, z_\alpha : \alpha \in p \,\}$. Then $z_p \in K_{|p|}$, $z_p(s_\beta) = 0$ for all $\beta < \min p$, and $z_p(s_\alpha) = 1$ if $s_\alpha \in W_p$.

We now use the argument as in [12]. Let $\mathbb{P}$ be the set of all finite subsets p of ω_1 such that whenever $\alpha, \beta \in p$ and $\alpha < \beta$, we have $s_\beta \notin W_\alpha$, ordered by the inverse inclusion. Then $\mathbb{P}$ is a partially ordered set of cardinality ω_1.

If $\mathbb{P}$ satisfied the countable chain condition, then by MA(ω_1), it would have an uncountable centered subset C; then $\bigcup C$ would be an uncountable discrete subspace of S, in contradiction with S being hereditarily Lindelöf.

Thus, $\mathbb{P}$ must have an uncountable antichain. By the standard Δ-lemma argument, we can find an uncountable antichain $T = \{ p_i : i \in \omega_1 \}$ in $\mathbb{P}$ so that all elements of T have the same cardinality m, and $\alpha < \beta$ whenever $\alpha \in p_i$, $\beta \in p_j$ and $i < j$. By the incompatibility, we have $p_i \cap W_{p_j} \neq \emptyset$ whenever $j < i$.

Put

$$A = \{ z_{p_i} : i < \omega_1 \}.$$

Then A is an uncountable set in K_m, and for every $\alpha \in \omega_1$ all but countably many elements of A are equal to 0 at s_α. It follows that every neighborhood of the zero function 0 in $C_p(S)$ contains all but countably many points of A, so 0 is a limit point for A; in particular, $0 \in K_m$. On the other hand, if B is a countable subset of A, then there is an ordinal $l < \omega_1$ such that $B \subset \{ z_{p_i} : i < l \}$, so $p_l \cap W_{p_i} \neq \emptyset$, and hence $\max\{ z_{p_l}(s_\alpha) : \alpha \in p_i \} = 1$ for all i with $z_{p_i} \in B$. Let $\bar{x}$ be a point of S^m whose set of coordinates is equal to p_l; then we have

$$O\big(\bar{x}, (-1/2, 1/2)^m\big) \cap B = \emptyset,$$

so 0 is not a limit point of B. Thus, K_m has uncountable tightness at the point 0, a contradiction that completes the proof. $\square$

Corollary 1.6. (MA(ω_1)) *Every separable, surlindelöf compact space of countable tightness is metrizable.*

Remark 1.7. Corollary 1.4 can be viewed as a common generalization of Theorems 0.1 and 0.2. Indeed, each of the conditions "$C_p(K)$ is Lindelöf" and "K is a subspace of $C_p(X)$ where all finite powers of X are Lindelöf" implies the countability of the tightness of K (see Arhangel'skii-Pytkeev Theorem II.1.1 and Asanov Theorem I.4.1 in [4]).

It is proved in [2] that PFA implies that every surlindelöf compact space has countable tightness. Thus,

Theorem 1.8. (PFA) *Every surlindelöf separable compact space is metrizable.*

Theorem 1.9. (PFA) *Every surlindelöf compact space is $\aleph_0$-monolithic.*

The consistency of PFA with ZFC depends on the existence of large cardinals; it was noted in [2] that in fact the countability of tightness of every surlindelöf compact space follows from the following statement:

(*) *Every compact space of weight* ω_1 *and uncountable tightness contains a homeomorphic copy of* $\omega_1 + 1$.

It is easy to see that (*) follows from the next statement:

(**) *If* Z *is a closed preimage of* ω_1, *and the character of* Z *is at most* ω_1, *then* Z *contains a homeomorphic copy of* ω_1.

Indeed, let X be a compact space with $w(X) = t(X) = \omega_1$. Then X contains a free sequence $S = \{ s_\alpha : \alpha < \omega_1 \}$; it is easy to see that the set $Z = \overline{\{ s_\alpha : \alpha < \omega_1 \}}$ has a closed continuous mapping onto $\omega_1 + 1$. By (**), Z must contain a homeomorphic copy of ω_1. Since $\omega_1 + 1$ is the only compactification of the space ω_1, X must contain a homeomorphic copy of $\omega_1 + 1$.

It is shown in [6] that there is a model of ZFC where both (**) and $\mathrm{MA}(\omega_1)$ hold. Thus,

Theorem 1.10. *It is consistent with ZFC that every surlindelöf compact space is* $\aleph_0$*-monolithic.*

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Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Apdo postal 1152, Puebla, Puebla CP 72000, Mexico

E-mail address: oleg@servidor.unam.mx

Department of Mathematics and Mechanics, Moscow State University

E-mail address: erezn@inbox.ru