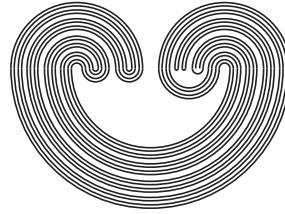

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by

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SELECTIVE SEPARABILITY: GENERAL FACTS AND BEHAVIOR IN COUNTABLE SPACES

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ABSTRACT. A space X is called selectively separable if, for any sequence $\{D_n : n \in \omega\}$ of dense subsets of X , we can choose a finite set $F_n \subset D_n$ for each $n \in \omega$ in such a way that $\bigcup_{n \in \omega} F_n$ is dense in X ; this notion was introduced by Marion Scheepers [*Combinatorics of open covers. VI. Selectors for sequences of dense sets*, Quaest. Math. **22** (1999), no. 1, 109–130]. Every space of countable π -weight is selectively separable and if X is selectively separable, then all dense subsets of X are separable. We study the general properties of selective separability together with the behavior of this notion in some special classes, such as function spaces or countable spaces. We prove, in particular, that some dense countable subsets of $\{0, 1\}^{\mathfrak{c}}$ are selectively separable and some are not. We also show that $C_p(X)$ is selectively separable if and only if it is separable and has countable fan tightness and we give a consistent example of a countable regular maximal space which is not selectively separable.

INTRODUCTION

In any context of general topology, separable spaces play a notable role. The existence of a dense countable subset brings a crucial information on the global properties of a space so there was a

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lot of results obtained recently on separable spaces and their generalizations. For example, hereditary separability turned out to be important also in set theory and logic; the quest for the spaces whose dense subspaces are all separable resulted in a beautiful theorem of I. Juhász and S. Shelah which implies that a compact space X has countable π -weight whenever every dense subspace of X is separable [7].

In [8], Marion Scheepers introduced a general notation for selection principles. For any notions (or classes) \mathcal{A} and \mathcal{B} , he denotes by $S_{fin}(\mathcal{A}, \mathcal{B})$ the following statement:

- (*) if $\{A_n : n \in \omega\} \subset \mathcal{A}$ then, for every $n \in \omega$, we can choose a finite $A'_n \subset A_n$ in such a way that $\bigcup\{A'_n : n \in \omega\}$ belongs to the class \mathcal{B} .

If \mathcal{A} and \mathcal{B} stand for the family of all dense subsets of a space X , then (*) gives us a property of X which we prefer to call *selective separability* of X . Selective separability of X follows from countable π -weight of X and implies that all dense subspaces of X are separable. Therefore, the above-mentioned theorem of Juhász and Shelah implies that, in compact spaces, selective separability coincides with countable π -weight. Scheepers characterized in [8] selective separability of $C_p(X)$ in terms of X for second countable spaces. He also gave other equivalencies for selective separability of $C_p(X)$ which do not hold in general spaces.

We show that selective separability has reasonably good categorical properties: it is preserved by open maps and closed irreducible maps, as well as by dense subspaces and open subspaces. We observe that separability, together with countable fan tightness, implies selective separability, while in spaces $C_p(X)$, selective separability implies countable fan tightness thus giving a characterization of selective separability of $C_p(X)$ for an arbitrary Tychonoff space X ; this generalizes the Scheepers result we mentioned above (see [8, Theorem 35]). We give sufficient conditions for a countable space to be selectively separable and show that there are ZFC examples of countable spaces which are not selectively separable. We also provide a consistent example of a regular maximal countable space which is not selectively separable.

1. NOTATION AND TERMINOLOGY

All spaces under consideration are assumed to be Tychonoff, i.e., $T_{3\frac{1}{2}}$; if X is a space, then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$. If $A \subset X$, then $\tau(A, X)$ is the family of all open subsets of X which contain A ; we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. A family $\mathcal{B} \subset \tau^*(X)$ is a π -base of X (at a point $x \in X$) if, for every $U \in \tau^*(X)$ ($U \in \tau(x, X)$, respectively), there is $B \in \mathcal{B}$ such that $B \subset U$. The cardinal $\pi w(X)$ (called the π -weight of X) is the minimal cardinality of a π -base of the space X . The minimal cardinality of a π -base at a point $x \in X$ is denoted by $\pi\chi(x, X)$ and $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$. Let $iw(X)$ be the minimal cardinal κ such that X has a weaker Tychonoff topology of weight κ ; evidently, the statement $iw(X) = \omega$ is equivalent to saying that X has a weaker separable metrizable topology.

A space X has *countable fan tightness* if, for any $x \in X$ and any sequence $\{A_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap_{n \in \omega} \overline{A_n}$, we can choose a finite set $B_n \subset A_n$ for each $n \in \omega$ in such a way that $x \in \overline{\bigcup\{B_n : n \in \omega\}}$. A space X has *countable tightness* (which is denoted by $t(X) \leq \omega$) if for any $x \in X$ and $A \subset X$ if $x \in \overline{A}$, then there is a countable set $B \subset A$ such that $x \in \overline{B}$. A space is *scattered* if every non-empty subspace of X has an isolated point.

We denote by \mathbb{R} the real line with its natural topology, $\mathbb{N} = \omega \setminus \{0\}$, and $\mathbb{Q} \subset \mathbb{R}$ is the set of rational numbers. Let $\mathbb{I} = [0, 1] \subset \mathbb{R}$; the symbol \mathbb{D} stands for the doubleton $\{0, 1\}$ with the discrete topology. If X is a space, then $C_p(X)$ is the set of continuous real-valued functions on X endowed with the topology of pointwise convergence. A space X is *Hurewicz* if, for any sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , we can find a finite $\mathcal{U}'_n \subset \mathcal{U}_n$ for each $n \in \omega$ in such a way that $\bigcup_{n \in \omega} \mathcal{U}'_n$ is a cover of X . In the terminology of Scheepers [8], this is the same as $S_{fin}(\mathcal{O}, \mathcal{O})$ where \mathcal{O} stands for the family of all open covers of the space X . The specialists on the selection principles call $S_{fin}(\mathcal{O}, \mathcal{O})$ *the Menger property*; to avoid confusion, we will use the expression $S_{fin}(\mathcal{O}, \mathcal{O})$ instead of using either of the two mentioned names.

If \mathcal{U} is a family of subsets of X and $A \subset X$, then $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$. Given $f, g \in \omega^\omega$, we say that $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \omega$. The cardinal \mathfrak{d} is the minimal cardinality of a cofinal subset

of (ω^ω, \leq) (see [3]). The rest of our terminology is standard and follows [5].

2. GENERAL PROPERTIES OF SELECTIVELY SEPARABLE SPACES

In a systematic study of a generalization of separability, it is natural to look at the properties which are analogous and which make a difference with separability. This section's results show the positioning of selective separability with respect to countable π -weight and separability.

Definition 2.1. ([8]). A space X is called selectively separable if, for any sequence $\{D_n : n \in \omega\}$ of dense subsets of X , there exists a family $\{F_n : n \in \omega\}$ of finite subsets of X such that $F_n \subset D_n$ for every $n \in \omega$ and $\bigcup_{n \in \omega} F_n$ is dense in X .

Proposition 2.2. *Assume that X is selectively separable. Then*

- (1) *every dense subspace of X is selectively separable and hence separable;*
- (2) *every open subspace of X is selectively separable;*
- (3) *every open continuous image of X is selectively separable;*
- (4) *every closed irreducible continuous image of X is selectively separable.*

Proof: Properties (1) and (2) are straightforward from the definition. Now, if a continuous onto map $f : X \rightarrow Y$ is either open or closed irreducible, then, for any sequence $\{E_n : n \in \omega\}$ of dense subsets of Y , every set $D_n = f^{-1}(E_n)$ is dense in X , so we can choose $F_n \subset D_n$ in such a way that $\bigcup_{n \in \omega} F_n$ is dense in X . Then $G_n = f(F_n)$ is a finite subset of E_n for every $n \in \omega$ and $\bigcup_{n \in \omega} G_n$ is dense in Y ; i.e., we checked properties (3) and (4). \square

Proposition 2.3. *Given a space X ,*

- (1) *if $\pi w(X) = \omega$, then X is selectively separable;*
- (2) *if X is separable and has countable fan tightness, then it is selectively separable;*
- (3) *if X has a dense open selectively separable subspace, then it is selectively separable.*

Proof: Suppose that $\pi w(X) = \omega$, and fix a π -base $\{B_n : n \in \omega\}$ in the space X . If $\{D_n : n \in \omega\}$ is a sequence of dense subspaces of X , then we can choose a point $x_n \in D_n \cap B_n$ for each $n \in \omega$; it is

clear that $\{x_n : n \in \omega\}$ is dense in X , so X is selectively separable and hence, we proved (1).

If X has countable fan tightness, then fix a set $\{a_n : n \in \omega\} \subset X$ which is dense in X and take any sequence $\{D_n : n \in \omega\}$ of dense subspaces of X . Choose a disjoint family $\mathcal{L} = \{L_n : n \in \omega\}$ of infinite subsets of ω such that $\bigcup \mathcal{L} = \omega$. We have $a_n \in \bigcap \{\overline{D}_k : k \in L_n\}$ for every $n \in \omega$, so it follows from countable fan tightness of X that we can find, for every $k \in L_n$, a finite $F_k \subset D_k$ such that $a_n \in \overline{\bigcup \{F_k : k \in L_n\}}$. It is evident that $\{F_k : k \in \omega\}$ is a sequence of finite sets which witnesses that the space X is selectively separable; i.e., we settled (2).

Finally, it is easy to deduce (3) from the fact that any dense subset of X intersected with a dense open subset of X is still dense in X . \square

Proposition 2.4. *A compact space is selectively separable if and only if it has countable π -weight.*

Proof: Proposition 2.3 provides sufficiency; if a compact space K is selectively separable, then every dense subset of K is separable by Proposition 2.2, so we can apply the main result of [7] to see that $\pi w(K) = \omega$. \square

Since Proposition 2.4 characterizes selective separability in compact spaces, a natural question is, “When are all continuous images of a compact space selectively separable (or, equivalently, have countable π -weight)?”

Corollary 2.5. *Suppose that X is a separable compact space. If X is either scattered or has countable tightness, then every continuous image of X is selectively separable.*

Proof: If X is scattered, then every continuous image Y of the space X is scattered and separable, so it has a countable dense set of isolated points; this implies the equality $\pi w(Y) = \omega$. If $t(X) \leq \omega$ and Y is a continuous image of X , then $t(Y) \leq \omega$ (see [1, 1.1.1]) and hence, $\pi \chi(Y) = \omega$ by [9], so it follows from separability of Y that Y has countable π -weight and hence, it is selectively separable. \square

Corollary 2.6. *If a compact space K is either scattered or has countable tightness, then every separable subspace of K is selectively separable. Therefore, if X is a countable non-selectively separable*

space, then X cannot be embedded in a compact space of countable tightness.

Remark 2.7. A continuous image of a selectively separable compact space can fail to be selectively separable; to see this, observe that $\beta\omega$ is selectively separable because it has a dense set of isolated points and hence, $\pi w(\beta\omega) = \omega$. However, $\beta\omega$ can be continuously mapped onto the Tychonoff cube \mathbb{I}^c which is not selectively separable because it has non-separable dense subspaces. The next example shows that selective separability of all continuous images of a compact space does not imply its countable tightness.

Example 2.8. There exists a compact scattered separable space X of uncountable tightness; therefore, all continuous images of X are selectively separable.

Proof: Let $K = \omega_1 + 1$ and apply Remark 7.2 of [3] to find a compact space X such that $X = A \cup K$ where A is dense in X and all points of A are isolated in X . It is evident that X is separable and scattered, so every continuous image of X is selectively separable by Corollary 2.5. However, $t(X) \geq t(K) = \omega_1$. \square

Let V be the countable Fréchet–Urysohn fan (see [1, 2.3.1]). The space V has a dense set of isolated points, so $\pi w(V) = \omega$ and hence, V is selectively separable. However, V does not have countable fan tightness. The following theorem shows that in selectively separable spaces $C_p(X)$, countable fan tightness is obligatory.

Theorem 2.9. *A space $C_p(X)$ is selectively separable if and only if it is separable and has countable fan tightness.*

Proof: By Proposition 2.3, it suffices to prove necessity, so assume that $C_p(X)$ is selectively separable. We adapt for our purposes the respective part of the proof of Theorem II.2.2 from [2]. Fix $n \in \mathbb{N}$ and an arbitrary sequence $\{\mathcal{U}_k : k \in \omega\}$ of open covers of the space X^n . Given a finite family \mathcal{V} of open subsets of X say that \mathcal{V} is *k-small* if, for any $U_1, \dots, U_n \in \mathcal{V}$, there exists $G \in \mathcal{U}_k$ such that $U_1 \times \dots \times U_n \subset G$. For each $k \in \omega$, let \mathcal{E}_k be the family of all *k-small* finite collections of open subsets of X . For any $\mathcal{V} \in \mathcal{E}_k$, consider the set $F_{\mathcal{V}} = \{f \in C_p(X) : f^{-1}(\mathbb{R} \setminus \{0\}) \subset \bigcup \mathcal{V}\}$; we claim that the set $P_k = \bigcup \{F_{\mathcal{V}} : \mathcal{V} \in \mathcal{E}_k\}$ is dense in $C_p(X)$.

Indeed, fix a finite set $K \subset X$ and a function $q : K \rightarrow \mathbb{R}$. It is easy to find a finite family $\mathcal{W} \subset \tau(X)$ such that for any n -tuple

$(y_1, \dots, y_n) \in K^n$ there are $V_1, \dots, V_n \in \mathcal{W}$ such that $V_i \in \tau(x_i, X)$ for each $i \leq n$ and $V_1 \times \dots \times V_n \subset G$ for some $G \in \mathcal{U}_k$. Clearly, $K \subset \bigcup \mathcal{W}$; let $W_x = \bigcap \{U : x \in U \in \mathcal{W}\}$ for every $x \in K$. It is easy to check that the family $\mathcal{V} = \{W_x : x \in K\}$ is k -small and $K \subset \bigcup \mathcal{V}$. Take a function $f \in C_p(X)$ such that $f|K = g$ and $f(X \setminus \bigcup \mathcal{V}) = \{0\}$; then $f \in F_{\mathcal{V}} \subset P_k$ and $f|K = g$. An immediate consequence is that P_k is dense in $C_p(X)$.

The space $C_p(X)$ being selectively separable, we can choose a finite set $Q_k \subset P_k$ in such a way that $Q = \bigcup \{Q_k : k \in \omega\}$ is dense in $C_p(X)$. Fix $k \in \omega$ and a finite family $\mathcal{P}_k \subset \mathcal{E}_k$ such that $Q_k \subset \bigcup \{F_{\mathcal{V}} : \mathcal{V} \in \mathcal{P}_k\}$. There is a finite family $\mathcal{U}'_k \subset \mathcal{U}_k$ such that, for any $\mathcal{V} \in \mathcal{P}_k$ and $V_1, \dots, V_n \in \mathcal{V}$, there exists $G \in \mathcal{U}'_k$ with $V_1 \times \dots \times V_n \subset G$. We claim that $\mathcal{U}' = \bigcup \{\bigcup \mathcal{U}'_k : k \in \omega\}$ is a cover of X^n .

Indeed, if $x = (x_1, \dots, x_n) \in X^n$, then the set $O = \{f \in C_p(X) : f(x_i) > 0 \text{ for all } i \leq n\}$ is open in $C_p(X)$ and non-empty. The set Q is dense in $C_p(X)$, so there exists $k \in \omega$ such that $Q_k \cap O \neq \emptyset$ and hence, we can find $\mathcal{V} \in \mathcal{P}_k$ and $f \in F_{\mathcal{V}}$ for which $f(x_i) > 0$ for all $i \leq n$. It follows from $f^{-1}(\mathbb{R} \setminus \{0\}) \subset \bigcup \mathcal{V}$ that $\{x_1, \dots, x_n\} \subset \bigcup \mathcal{V}$, so there are $V_1, \dots, V_n \in \mathcal{V}$ such that $x_i \in V_i$ for all $i \leq n$. By our choice of \mathcal{U}'_k , there is $G \in \mathcal{U}'_k$ such that $V_1 \times \dots \times V_n \subset G$; therefore, $x \in G$ and hence, \mathcal{U}' is a cover of the space X^n .

We proved that selective separability of $C_p(X)$ implies that every X^n satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$; applying Theorem II.2.2 of [2], we conclude that $C_p(X)$ has countable fan tightness. \square

Theorem II.2.2 of [2] states that X^n has the property $S_{fin}(\mathcal{O}, \mathcal{O})$ for every $n \in \mathbb{N}$ if and only if $C_p(X)$ has countable fan tightness. It is folklore (the respective result can be established mimicking the proof of Proposition on p. 156 in [6]) that this happens if and only if X belongs to the class obtained by substituting “covers” with “ ω -covers” in the definition of Hurewicz spaces; in [8], this property is denoted by $S_{fin}(\Omega, \Omega)$.

Recall that a family \mathcal{U} of subsets of X is called an ω -cover of X if, for every finite set $K \subset X$, there exists $U \in \mathcal{U}$ with $K \subset U$. Using the fact that $d(C_p(X)) = iw(X)$ for every space X , we obtain the following corollary.

Corollary 2.10. *For any space X which has a weaker second countable topology, the following conditions are equivalent:*

- (i) the space $C_p(X)$ is selectively separable;
- (ii) the space $C_p(X)$ has countable fan tightness;
- (iii) for any sequence $\{\mathcal{U}_n : n \in \omega\}$ of open ω -covers of X , we can choose a finite $\mathcal{U}'_n \subset \mathcal{U}_n$ for each $n \in \omega$ in such a way that $\bigcup_{n \in \omega} \mathcal{U}'_n$ is an ω -cover of X .

Corollary 2.11. *If $C_p(X)$ is selectively separable, then every separable subspace of $C_p(X)$ is selectively separable. In particular, if X is second countable and $C_p(X)$ is selectively separable, then $C_p(X)$ is hereditarily selectively separable.*

Proof: Observe that countable fan tightness is hereditary and apply Proposition 2.3 together with Theorem 2.9. \square

Corollary 2.12. *If $C_p(X)$ is selectively separable, then $C_p(X^n)$ is selectively separable for all $n \in \mathbb{N}$, and the space $(C_p(X))^\omega$ is selectively separable as well.*

Corollary 2.13. *The space $C_p(C_p(X))$ is selectively separable if and only if X is finite.*

Proof: If X is finite, then $C_p(X)$ is σ -compact and metrizable so $w(C_p(X)) = \omega$, and every finite power of $C_p(X)$ has the property $S_{fin}(\mathcal{O}, \mathcal{O})$ which makes it possible to apply Theorem II.2.2 of [2] and Corollary 2.10 to conclude that $C_p(C_p(X))$ is selectively separable. If, on the other hand, the space $C_p(C_p(X))$ is selectively separable, then $S_{fin}(\mathcal{O}, \mathcal{O})$ holds for $C_p(X)$ and hence, X is finite by Theorem II.2.10 of [2]. \square

Example 2.14. It follows from Corollary 2.10 that $C_p(\mathbb{I})$ is selectively separable. However, $\pi w(C_p(\mathbb{I})) = w(C_p(\mathbb{I})) = |\mathbb{I}| > \omega$ which shows that, in general, selective separability does not imply countable π -weight; i.e., compactness is essential in Proposition 2.4.

It is well known (and easy to prove) that the space $X = \omega^\omega$ of the irrationals does not have $S_{fin}(\mathcal{O}, \mathcal{O})$. Corollary 2.10 shows that $C_p(X)$ is a hereditarily separable, non-selectively separable space. Therefore, we can choose a countable dense set $A_n \subset C_p(X)$ for every $n \in \omega$ such that the family $\{A_n : n \in \omega\}$ witnesses that $C_p(X)$ is not selectively separable. If $A = \bigcup_{n \in \omega} A_n$, then A is a countable dense subspace of $C_p(X)$ which is not selectively separable. Since $C_p(X)$ is densely embeddable in \mathbb{I}^c , we conclude that \mathbb{I}^c has a dense countable subspace D which fails to be selectively separable.

Another interesting observation is that $X = \omega^\omega$ condenses onto a compact space K and hence, $C_p(K)$ is densely embeddable in $C_p(X)$. It follows from Corollary 2.10 that $C_p(K)$ is selectively separable. Taking a dense countable $B \subset C_p(K)$, we obtain a dense countable subset of $C_p(X)$ which is selectively separable. Thus, a countable dense subspace of $C_p(X)$ (or a countable dense subspace of \mathbb{I}^c , for that matter) can be selectively separable.

The following statement is a particular case of [8, Theorem 40].

Proposition 2.15. *If every dense subspace of a space X is separable and we have the inequality $\pi w(X) < \mathfrak{d}$, then X is selectively separable.*

Corollary 2.16. *Under Martin's Axiom and the negation of CH, every countable subspace of \mathbb{D}^{ω_1} is selectively separable.*

Corollary 2.17. *If X is a space such that $iw(X) = \omega$, $|X| < \mathfrak{d}$, and $l(X^n) = \omega$ for any $n \in \mathbb{N}$, then $C_p(X)$ is selectively separable. In particular, if X is a second countable space with $|X| < \mathfrak{d}$, then $C_p(X)$ is selectively separable.*

Proof: Indeed, for such a space X , we have $w(C_p(X)) = |X| < \mathfrak{d}$ and $t(C_p(X)) = \omega$ by the Arhangel'skii–Pytkeev theorem (see [2, Theorem II.1.1]) so every dense subspace of $C_p(X)$ is separable and hence, Theorem 2.15 is applicable to $C_p(X)$. \square

Theorem 2.18. *The space $\mathbb{D}^{\mathfrak{d}}$ contains a countable dense subspace which is not selectively separable.*

Proof: Fix a set $F \subset \omega^\omega$ such that $|F| = \mathfrak{d}$ and F is cofinal in (ω^ω, \leq) . Since $|F| \leq \mathfrak{c}$, we can choose a second countable topology τ on the set F ; let $\mathcal{B} \subset \tau \setminus \{\emptyset\}$ be a countable base of τ . The set $P = \{f \in \mathbb{D}^F : \text{there exists a finite disjoint family } \mathcal{V} \subset \mathcal{B} \text{ such that } f \text{ is constant on every element of } \mathcal{V} \text{ and } f(F \setminus (\bigcup \mathcal{V})) = \{0\}\}$ is countable; take an enumeration $\{p_i : i \in \omega\}$ of the set P in which every $f \in P$ occurs infinitely many times.

For any $n, i \in \omega$ and $f \in F$, let $d_i^n(f) = p_i(f)$ if $f(n) \leq i$ and $d_i^n(f) = 0$, otherwise. The set $D_n = \{d_i^n : i \in \omega\}$ is dense in \mathbb{D}^F for any $n \in \omega$. To see this, take any finite set $A \subset F$ and a function $w : A \rightarrow \mathbb{D}$. Choose a finite disjoint family $\mathcal{V} = \{U_a : a \in A\} \subset \mathcal{B}$ such that $a \in U_a$ for every $a \in A$. If $a \in A$ and $f \in U_a$, then let

$p(f) = w(a)$; if $f \in F \setminus (\bigcup\{U_a : a \in A\})$, then let $p(f) = 0$. Clearly, $p \in P$ so there exists $i \in \omega$ such that $p = p_i$ and $f(n) \leq i$ for all $f \in A$. Then $d_i^n(f) = p_i(f) = p(f) = w(f)$ for each $f \in A$, i.e., $d_i^n|A = w$; this proves that every set D_n is, indeed, dense in \mathbb{D}^F . Therefore, $D = \bigcup_{n \in \omega} D_n$ is a countable dense subset of \mathbb{D}^F ; evidently, \mathbb{D}^F is homeomorphic to $\mathbb{D}^{\mathfrak{d}}$.

Assume that we are given a finite set $K_n \subset D_n$ and take a function $f \in F$ such that $K_n \subset \{d_i^n : i < f(n)\}$ for every $n \in \omega$. Then $d_i^n(f) = 0$ whenever $n \in \omega$ and $i < f(n)$, which shows that $q(f) = 0$ for any $q \in K = \bigcup_{n \in \omega} K_n$ and hence, K is not dense in \mathbb{D}^F . Therefore, the family $\{D_n : n \in \omega\}$ witnesses that the space D is not selectively separable. \square

Corollary 2.19. *It is independent of ZFC whether every dense countable subspace of \mathbb{D}^{ω_1} is selectively separable.*

Theorem 2.18 shows that the countable union of selectively separable spaces can fail to be selectively separable. It turns out that this can happen even if all summands are dense.

Example 2.20. If $X = \omega^\omega$, then we saw in Example 2.14 that there exists a dense selectively separable subspace $Y \subset C_p(X)$. This set Y is actually a copy of a space $C_p(K)$ for some compact K so it is a linear subspace of $C_p(X)$. The space $C_p(X)$ is not selectively separable by Corollary 2.10, so we can find a countable dense $D \subset C_p(X)$ which is not selectively separable. Then $Z = \bigcup\{Y + f : f \in D\}$ is a countable union of translates of Y (and hence, a countable union of selectively separable dense subspaces of Z). Since $D \subset Z$, the space Z is not selectively separable.

Recall that a space X is called *maximal* if it is dense-in-itself but any strictly stronger topology on X has an isolated point. Since countable discrete spaces are selectively separable and maximal countable spaces are, in a certain way, the closest ones to the discrete spaces, it is interesting to find out how selective separability behaves in this class. This turned out to be non-trivial and we only have an answer when $\mathfrak{d} = \omega_1$.

Definition 2.21. Given a set S and an ordinal λ , call a family $\{\tau_\alpha : \alpha < \lambda\}$ of topologies on S continuous if $\alpha < \beta$ implies $\tau_\alpha \subset \tau_\beta$ and for any limit ordinal $\alpha < \lambda$, the family $\bigcup\{\tau_\beta : \beta < \alpha\}$ is a base of τ_α .

The proof of the following statement is standard and can be left to the reader.

Proposition 2.22. *Suppose that S is a set and $\{\tau_\alpha : \alpha < \lambda\}$ is a continuous family of topologies on S and denote by τ the topology generated by $\bigcup\{\tau_\alpha : \alpha < \lambda\}$. If $N \subset S$ is nowhere dense in (S, τ_α) for any $\alpha < \lambda$, then N is nowhere dense in (S, τ) .*

We are going to construct a maximal space which is not selectively separable. Our first step is to inductively produce a continuous family of topologies on \mathbb{Q} whose least upper bound makes certain sets nowhere dense. The possibility to begin the induction is guaranteed by the following lemma.

Lemma 2.23. *Given any $q \in \mathbb{Q}$, let $O(q, \varepsilon) = \{p \in \mathbb{Q} : |p - q| < \varepsilon\}$ for every $\varepsilon > 0$. Choose a faithful enumeration $\{p_i : i \in \omega\}$ of the set \mathbb{Q} . Using rational sequences that converge to the relevant irrationals, it is easy to construct, for any $j \in \omega$, a closed and discrete set $Q_j \subset \mathbb{Q}$ such that $\{p_i : i \leq j\} \subset Q_0 \cup \dots \cup Q_j$ and the set $O(p_i, 2^{-j}) \cap Q_j$ is infinite for any $i \leq j$, while the family $\{Q_j : j \in \omega\}$ is disjoint. Take a faithful enumeration $\{q_i^j : i \in \omega\}$ of every set Q_j . Then $\{q_i^j : i, j \in \omega\}$ is a faithful indexation of \mathbb{Q} and, for each $s \in \omega^\omega$, if $K_s = \{q_i^j : 0 \leq i \leq s(j), j \in \omega\}$, then, for any infinite $N \subset \omega$, the set $(\bigcup\{Q_j : j \in N\}) \setminus K_s$ is dense in \mathbb{Q} .*

Proof: Given $p \in \mathbb{Q}$ and $\varepsilon > 0$, there exist $i, j \in \omega$ such that $p = p_i$, $j \in N$, and $2^{-j} < \varepsilon$. The set $Q_j \cap O(p, 2^{-j})$ is infinite while $K_s \cap Q_j$ is finite, so there is a point $q \in Q_j \setminus K_s$ for which $q \in O(p, 2^{-j})$ and hence, $|p - q| < \varepsilon$. \square

Lemma 2.24. *If $\mathfrak{d} = \omega_1$, then there exists a regular topology $\tau \supset \tau(\mathbb{Q})$ on the set \mathbb{Q} such that the set K_s is nowhere dense in (\mathbb{Q}, τ) for any $s \in \omega^\omega$.*

Proof: Call a set $A \subset \mathbb{Q}$ adequate if there exists $m \in \omega$ such that $A \cap Q_n$ is infinite for all $n \geq m$. Choose an enumeration $\{s_\alpha : \alpha < \omega_1\}$ of a dominating set in ω^ω and let $\tau_0 = \tau(\mathbb{Q})$. Given any non-empty set $U \in \tau_0$, fix a point $p_i \in U$ and $j \in \omega$ for which $O(p_i, 2^{-j}) \subset U$. If $n \geq i + j$, then the set $Q_n \cap O(p_i, 2^{-n}) \subset Q_n \cap O(p_i, 2^{-j})$ is infinite so $Q_n \cap U$ is infinite as well. This shows that every non-empty element of τ_0 is adequate.

Assume that, for some $\alpha < \omega_1$, we constructed a continuous family $\{\tau_\beta : \beta < \alpha\}$ of second countable topologies on \mathbb{Q} in such a way that

- (1) if $\gamma < \beta < \alpha$, then the set K_{s_γ} is closed in (\mathbb{Q}, τ_β) ;
- (2) if $\beta < \alpha$ and $U \in \tau_\beta \setminus \{\emptyset\}$, then U is an adequate set.

If α is a limit ordinal, then let τ_α be the topology generated by the family $\bigcup\{\tau_\beta : \beta < \alpha\}$ as a base. It is straightforward that properties (1) and (2) now hold for all $\beta \leq \alpha$.

If $\alpha = \gamma + 1$, then fix a base $\{B_n : n \in \omega\} \subset \tau_\gamma \setminus \{\emptyset\}$ in the space $(\mathbb{Q}, \tau_\gamma)$. For every $j \in \omega$, it is easy to find a disjoint partition $\{Y_j^i : i \in \omega\}$ of the set Q_j such that every Y_j^i is infinite and, for every $n \in \omega$, if the set $B_n \cap Q_j$ is infinite, then $Y_j^i \cap B_n$ is infinite for any $i \in \omega$. Letting $Z_i = (\bigcup\{Y_j^i : j \in \omega\}) \setminus K_{s_\gamma}$ for all $i \in \omega$, we obtain a disjoint partition $\{Z_i : i \in \omega\}$ of the set $\mathbb{Q} \setminus K_{s_\gamma}$, and it is straightforward that every Z_i is dense in $(\mathbb{Q}, \tau_\gamma)$.

For the set $S = \{2^{-n} : n \in \omega\} \cup \{0\} \subset \mathbb{Q}$, let $f(x) = 2^{-n}$ whenever $x \in Z_n$ and $f(x) = 0$ for every $x \in K_{s_\gamma}$. This gives us a function $f : \mathbb{Q} \rightarrow S$ so the natural projection π condenses its graph $\Gamma = \{(x, f(x)) : x \in \mathbb{Q}\} \subset (\mathbb{Q}, \tau_\gamma) \times S$ onto the space $(\mathbb{Q}, \tau_\gamma)$. Therefore, $\tau_\alpha = \{\pi(U) : U \in \tau(\Gamma)\}$ is a second countable topology on \mathbb{Q} with $\tau_\gamma \subset \tau_\alpha$.

The space $(\mathbb{Q}, \tau_\alpha)$ is homeomorphic to Γ ; besides, $f : (\mathbb{Q}, \tau_\alpha) \rightarrow S$ is continuous, so the set $K_{s_\gamma} = f^{-1}(0)$ is closed in $(\mathbb{Q}, \tau_\alpha)$. It is straightforward that the set $Z = \bigcup\{Z_n \times \{2^{-n}\} : n \in \omega\}$ is dense in Γ , so $\pi(Z) = \mathbb{Q} \setminus K_{s_\gamma}$ is dense in $(\mathbb{Q}, \tau_\alpha)$. Fix a non-empty set $U \in \tau_\alpha$; the set Z being dense in $(\mathbb{Q}, \tau_\alpha)$, there exists $k \in \omega$ such that $U \cap Z_k \neq \emptyset$. The topologies $\tau_\alpha|_{Z_k}$ and $\tau_\gamma|_{Z_k}$ coincide so there is $n \in \omega$ such that $B_n \cap Z_k \subset U \cap Z_k$. The set B_n being adequate, there is $m \in \omega$ for which $B_n \cap Q_j$ is infinite for all $j \geq m$. Our choice of Z_k guarantees that $B_n \cap Y_j^k \subset U \cap Y_j^k$ is infinite for all $j \geq m$. Therefore, $U \cap Y_j^k \subset U \cap Q_j$ is infinite for any $j \geq m$; this proves that U is an adequate set and hence, properties (1) and (2) hold for all $\beta \leq \alpha$.

Thus, we can construct a continuous family $\{\tau_\beta : \beta < \omega_1\}$ of topologies on \mathbb{Q} such that (1) and (2) are satisfied for all $\alpha < \omega_1$. If τ is generated by the family $\bigcup\{\tau_\beta : \beta < \omega_1\}$ as a base, then every K_{s_β} is closed in (\mathbb{Q}, τ) by (1). It is immediate that every $U \in \tau \setminus \{\emptyset\}$ is an adequate set; since K_{s_β} is not adequate, it cannot contain a

non-empty open subset of (\mathbb{Q}, τ) . Therefore K_{s_β} is nowhere dense for all $\beta < \omega_1$. Finally, if we take an arbitrary $s \in \omega^\omega$, then there exists $\alpha < \omega_1$ such that $s \leq s_\alpha$, so the set $K_s \subset K_{s_\alpha}$ is nowhere dense in (\mathbb{Q}, τ) . \square

Theorem 2.25. *If $\mathfrak{d} = \omega_1$, then there exists a regular maximal countable space X which is not selectively separable.*

Proof: Take the space (\mathbb{Q}, τ) from Lemma 2.24 and choose a maximal regular topology $\nu \supset \tau$ on the set \mathbb{Q} . The space (\mathbb{Q}, ν) is hereditarily irresolvable (see [4, Fact 1.6 and Fact 1.11]). Apply Lemma 3.2 and Theorem 2.2 of [4] to see that there is a nowhere dense closed set F in the space $Q = (\mathbb{Q}, \nu)$ such that $V = Q \setminus F$ is a maximal space. If $K_s \cap V$ is dense in V , then K_s is dense in (\mathbb{Q}, ν) and hence in (\mathbb{Q}, τ) , which is a contradiction. Therefore,

(3) the set $K_s \cap V$ is not dense in V for all $s \in \omega^\omega$.

Finally, observe that the set $Q_j \cap V$ is closed and discrete in V for each $j \in \omega$, so the set $H_n = V \setminus (\bigcup\{Q_j : j \leq n\})$ is dense in V for every $n \in \omega$. If $G_n \subset H_n$ is finite for each $n \in \omega$, then let $G = \bigcup_{n \in \omega} G_n$. It is straightforward that $G \cap Q_j$ is finite for every $j \in \omega$, so there exists $s \in \omega^\omega$ such that $G \subset K_s$ and hence, G is not dense in V by property (3). Therefore, V is a maximal countable space which is not selectively separable. \square

We will finish this paper with two observations about the behavior of selective separability in products. Our suspicion is that a product of two selectively separable spaces need not be selectively separable. However, if one of the factors has countable π -weight, then we have a positive result.

Theorem 2.26. *Suppose X is selectively separable and $\pi w(Y) = \omega$. Then $X \times Y$ is selectively separable.*

Proof: Fix a π -base $\{B_n : n \in \omega\}$ in the space Y and assume that $\{D_k : k \in \omega\}$ is a family of dense subspaces of $X \times Y$. Let $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ be the natural projections and choose a disjoint family $\{A_n : n \in \omega\}$ of infinite subsets of ω such that $\omega = \bigcup_{n \in \omega} A_n$. Fix any $n \in \omega$ and observe that the set $G_k = \{\pi_X(z) : z \in D_k \cap \pi_Y^{-1}(B_n)\}$ is dense in X for every $k \in A_n$; the space X being selectively separable, we can find a finite set $F_k \subset D_k \cap \pi_Y^{-1}(B_n)$ in such a way that the set $\bigcup\{\pi(F_k) : k \in A_n\}$

is dense in X . The family $\{F_k : k \in \omega\}$ is the required selection of finite sets; i.e., $F_k \subset D_k$ for each $k \in \omega$ and the set $\bigcup_{k \in \omega} F_k$ is dense in $X \times Y$. \square

Theorem 2.27. *Given a sequence $\{X_n : n \in \omega\}$ of topological spaces, suppose that the space $Y_n = X_0 \times \dots \times X_n$ is selectively separable for every $n \in \omega$. Then the space $X = \prod_{n \in \omega} X_n$ is selectively separable.*

Proof: Let $\pi_n : X \rightarrow Y_n$ be the natural projection for every $n \in \omega$. Take an arbitrary sequence $\{D_n : n \in \omega\}$ of dense subspaces of X and choose a disjoint family $\{L_n : n \in \omega\}$ of infinite subsets of ω such that $\bigcup_{n \in \omega} L_n = \omega$. By selective separability of Y_n , we can find a finite set $F_k \subset D_k$ for each $k \in L_n$ in such a way that $\bigcup\{\pi_n(F_k) : k \in L_n\}$ is dense in Y_n . If $F = \bigcup_{k \in \omega} F_k$, then $\pi_n(F)$ is dense in Y_n for all $n \in \omega$; an easy consequence is that F is dense in X , so F witnesses selective separability of the space X . \square

3. OPEN PROBLEMS

Studying selective separability is motivated by a desire to obtain new information about separable (and even countable) spaces. The following list shows that our knowledge on the topic is far from exhaustive.

If $K = \mathbb{D}^\omega$ is the Cantor set, then the space $C_p(K, \mathbb{D})$ is countable, selectively separable, and dense in \mathbb{D}^K . Therefore, $\mathbb{D}^\mathfrak{c}$ has a dense, countable, selectively separable subspace in ZFC. Corollary 2.16 shows that *every countable dense subspace* of \mathbb{D}^{ω_1} is selectively separable under $\text{MA} + \neg\text{CH}$. However, the following problem remains open.

Problem 3.1. Is it true in ZFC that the Cantor cube \mathbb{D}^{ω_1} has a countable dense selectively separable subset?

Theorem 2.25 shows that the following question might have a positive answer.

Problem 3.2. Does there exist in ZFC a regular maximal countable space which is not selectively separable?

It is also interesting to find classes of countable spaces which cannot be selectively separable; therefore, maximal spaces are natural candidates.

Problem 3.3. Is it true (at least, consistently) that every countable regular maximal space fails to be selectively separable?

Since Proposition 2.4 completely characterizes selective separability in compact spaces (and it is easy to see that its statement is also true for locally compact ones), it would be interesting to check the limits of this characterization.

Problem 3.4. Suppose that X is a selectively separable Čech-complete space. Must X have countable π -weight?

Problem 3.5. Suppose that X is a separable Čech-complete space of countable tightness. Must X be selectively separable?

Problem 3.6. Is it true (at least, consistently) that, for each second countable X , the space $C_p(X)$ has a dense selectively separable subspace?

At the moment, we know very little about the behavior of selective separability in products. Theorem 40 of [8] (see also Proposition 2.15) shows that the most interesting case of the following problem occurs when $\pi w(X) \geq \mathfrak{d}$ or $\pi w(Y) \geq \mathfrak{d}$.

Problem 3.7. Suppose that X and Y are selectively separable spaces. Must $X \times Y$ be selectively separable?

Strangely enough, it is not even clear what happens with selective separability under finite unions.

Problem 3.8. Suppose that $X = A \cup B$ and the spaces A and B are selectively separable. Must X be selectively separable?

Problem 3.9. Suppose that $C_p(X)$ and $C_p(Y)$ are selectively separable spaces. Must $C_p(X) \times C_p(Y)$ be selectively separable?

Problem 3.10. Let K be an infinite metrizable compact space. Is it true that $L_p(K)$ is selectively separable? This question is non-trivial even if K is a convergent sequence. Observe that we know from Corollary 2.13 that $C_p(C_p(K))$ is not selectively separable.

Problem 3.11. Suppose that A is a countable selectively separable subspace of \mathbb{D}^κ for some uncountable cardinal κ . Must the group generated by A in \mathbb{D}^κ be selectively separable? What happens if A is dense in \mathbb{D}^κ ?

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