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COMPOSANT STRUCTURE OF AN INTERESTING INVERSE LIMIT SPACE

RANJITH MUNASINGHE

Abstract. We present an interesting inverse limit space on the closed interval with one bonding map. Even though the bonding map satisfies the property that $f(x) \geq x$ for all x, we find that the composant structure of the inverse limit is very complex and interesting. In addition to the composant structure, we describe unstable sets of the inverse limit.

1. Introduction

Let I be the closed interval $[0,1]$ and $f : I \to I$ be a continuous function. Let $(I,f) = \{(x_0, x_1, x_2, \ldots) : f(x_{i+1}) = x_i\}$. We denote elements of (I,f) by subbarred lowercase letters (i.e., $\underline{x} = (x_0, x_1, x_2, \ldots)$) and utilize the metric $d(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$.

(I,f), which is called the *inverse limit space of* f, is an example of what R. H. Bing [1] has called a snake-like continuum. In [2], Marcy Barge and Joe Martin show that (I,f) can be realized as a global attractor in the plane.

A *continuum* is a compact, connected metric space. A compact connected subspace of a continuum is called a *subcontinuum*. A continuum X is said to be *snake-like* if and only if, for each $\epsilon > 0$, there is a finite open cover $\{g_1, g_2, \ldots, g_n\}$ of the continuum X so that (i) $\operatorname{diam}(g_i) < \epsilon$ for each $i = 1, 2, \ldots, n$ and (ii) $g_i \cap g_j \neq \phi$ if and only if $| i - j | \leq 1$. Snake-like continua have also been called

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chainable. A proof that (I, f) is a continuum can be found in [3]. It is easy to show that (I, f) is snake-like. For each $n = 0, 1, 2, \ldots$, let $\prod_n : (I, f) \to I$ be the *nth projection map* defined by $\prod_n(\underline{x}) = x_n$. For each n, $\prod_n$ is continuous. For $\underline{x} \in (I, f)$, $\underline{x} = (x_0, x_1, x_2, \ldots)$, let the *induced homeormorphism* $\hat{f} : (I, f) \to (I, f)$ be defined by $\hat{f}(\underline{x}) = (f(x_0), x_0, x_1, x_2, \ldots)$. It is straightforward to verify that $\hat{f}$ is a homeomorphism of (I, f) onto (I, f). The statement that X is an *arc* means that X is a homeomorphic image of the closed interval $[0, 1]$.

Let $[a, b]$ denote the smallest closed interval which contains both a and b and let (a, b) denote the corresponding open interval.

The following theorem describes how to construct subcontinua of (I, f). The proof of the theorem is easy and therefore, we omit the proof.

Theorem 1.1. *Let $f : I \to I$ be a continuous function and K be a subcontinuum of (I, f).*

(1) *For each $n = 0, 1, 2, \ldots$, $\prod_n(K)$ is a closed interval.*
(2) *For each $n = 0, 1, 2, \ldots$, $f(\prod_{n+1}(K)) = \prod_n(K)$.*
(3) *If $\{J_n\}_{n=0}^{\infty}$ is a sequence of closed intervals so that $f(J_{n+1}) = J_n$ for each $n = 0, 1, 2, \ldots$, then there is a unique subcontinuum H of (I, f) with $\prod_n(H) = J_n$ for each $n = 0, 1, 2, \ldots$.*
(4) *If there is a positive integer N so that f maps $\prod_{n+1}(K)$ homeomorphically onto $\prod_n(K)$ whenever $n \geq N$, then K is an arc.*
(5) *Let $\underline{x}$, $\underline{y} \in (I, f)$ and let K be the smallest subcontinuum of (I, f) which contains both $\underline{x}$ and $\underline{y}$. (That is, if $\underline{x}$, $\underline{y} \in H$, a subcontinuum of (I, f), then $K \subseteq H$.) Then, for each $k = 0, 1, 2, \ldots$, $\prod_k(K) = \mathrm{cl}\{\cup_{n=k}^{\infty} f^{n-k}([x_n, y_n])\}$.*
(6) *If f is onto, then $K = (I, f)$ if and only if $\prod_n(K) = I$ for each $n = 0, 1, 2, \ldots$.*

2. Composants and unstable sets

Definition 2.1. Let X be a continuum and $\underline{x} \in X$. Then the *composant of x*, denoted $C(\underline{x})$, is the union of all proper subcontinua of X containing x.

Definition 2.2. Suppose that $f : I \to I$ is continuous and that $\underline{x} \in (I, f)$. Then the *unstable set of* $\underline{x}$, denoted $U(\underline{x})$, is the set

$$\{\underline{y} \in (I, f) : \lim_{n \to \infty} d(\hat{f}^{-n}(\underline{x}), \hat{f}^{-n}(\underline{y})) = 0\}.$$

Notice that (1) $\underline{x} \in U(\underline{x})$, (2) $\underline{y} \in U(\underline{x})$ if and only if $\underline{x} \in U(\underline{y})$, and (3) if $\underline{x} \in U(\underline{y})$ and $\underline{y} \in U(\underline{z})$, then $\underline{x} \in U(\underline{z})$. Therefore, the relation defined by the statement that $\underline{x} \sim \underline{y}$ if and only if $\underline{x} \in U(\underline{y})$ is an equivalence relation. That is, the unstable sets of (I, f) partition (I, f).

We will use the following characterization of unstable sets of (I, f) which is easy to establish.

Lemma 2.3. *Let* $\underline{x} = (x_0, x_1, x_2, \ldots)$ *and* $\underline{y} = (y_0, y_1, y_2, \ldots)$ *be points of* (I, f). $\underline{x}$ *and* $\underline{y}$ *are in the same unstable set if and only if* $\lim_{n \to \infty} \mid x_n - y_n \mid = 0$.

The proofs of theorems 2.4 and 2.5 are found in [4].

Theorem 2.4. *If* $f : I \to I$ *is continuous and* U *is an unstable set in* (I, f)*, then* U *is connected.*

Unstable sets of (I, f) partition (I, f) into connected subsets, while, in general, composants tend to intersect with each other. When the bonding map has interesting dynamics, the composants of (I, f) tend to align with unstable sets of (I, f) to some extent.

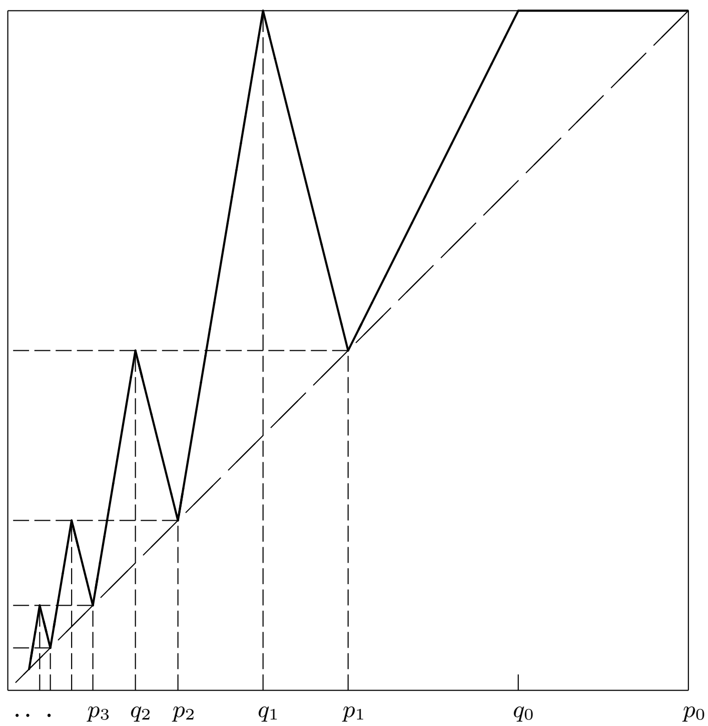
Theorem 2.5. *Suppose that* $f : I \to I$ *is continuous and* f^2 *has a dense orbit. Then for each* $\underline{x} \in (I, f)$, $C(\underline{x}) \subseteq U(\underline{x})$.

The reverse inclusion, $U(\underline{x}) \subseteq C(\underline{x})$, need not hold even if f^2 has a dense orbit. For example, it is shown in [4] that the inverse limit of the full tent map has two composants that lie in the same unstable set.

In the next section, we define a map f that has the following properties: f^2 does not have a dense orbit, (in fact, $f(x) \geq x$ for each $x \in I$); (I, f) has an unstable set that contains all but one of its uncountably many composants; and (I, f) has a composant that contains all but one of its countably many unstable sets.

3. An interesting inverse limit space

For each $n = 0, 1, 2, \ldots$, let $p_n = \frac{1}{2^n}$ and $q_n = \frac{3}{2^{n+2}}$. Let $f : I \to I$ be the continuous function which is linear on the intervals $[p_0, q_0]$, $[q_0, p_1]$, $[p_1, q_1]$, $[q_1, p_2]$, $\ldots$, $[p_n, q_n]$, $[q_n, p_{n+1}]$, $\ldots$, with the values $f(0) = 0$, $f(q_0) = p_0$, and for each $n = 1, 2, 3, \ldots$, $f(p_{n-1}) = p_{n-1}$ and $f(q_n) = p_{n-1}$ at the end points of the intervals.



Graph of the function f

Let $\underline{p_\infty} = (0, 0, 0, \ldots)$ and for each $n = 0, 1, 2, \ldots$, let $\underline{p_n} = (p_n, p_n, p_n, \ldots)$. Let $I_n = [p_n, p_{n-1})$ for $n = 1, 2, 3, \ldots$, and let $I_0 = \{1\}$, $I_\infty = \{0\}$. For each $\underline{x} \in (I, f)$, $\underline{x} = (x_0, x_1, x_2, \ldots)$, and for each nonnegative integer i, let $s_i(\underline{x}) = n_i$ where $x_i \in I_{n_i}$. Notice that $n_i = \infty$ for some nonnegative integer i if and only if $\underline{x} = \underline{p_\infty}$.

We notice that $\underline{p_n}$ and $\underline{p_\infty}$ are points of (I, f) and that for each $n \geq 0$, $\{s_i(\underline{p_n})\}_{i=0}^{\infty}$ is the constant sequence with every term equal to n.

The following theorem describes the unstable sets of (I, f) for the function f.

Theorem 3.1. *Let $f : I \to I$ be the function defined above. Then, for each $\underline{x} \in (I, f)$, $\{s_i(\underline{x})\}_{i=0}^{\infty}$ is a nondecreasing sequence. Moreover,*

(1) *if $\lim_{i\to\infty} s_i(\underline{x}) = k$, then $\underline{x} \in U(\underline{p_k})$, and*
(2) *if $\{s_i(\underline{x})\}_{i=0}^{\infty}$ does not converge (i.e., increases without a limit), then $\underline{x} \in U(\underline{p_\infty})$.*

Proof: Suppose that $\underline{x} = (x_0, x_1, x_2, \ldots) \in (I, f)$. Since for each $x \in I$, $f(x) \geq x$ for each $i = 0, 1, 2, \ldots$, we have $f(x_{i+1}) = x_i \geq x_{i+1}$. Therefore, for each $i = 0, 1, 2, \ldots$, $s_{i+1}(\underline{x}) \geq s_i(\underline{x})$; that is, $\{s_i(\underline{x})\}_{i=0}^{\infty}$ is a nondecreasing sequence.

To see (1), suppose that $\lim_{i\to\infty} s_i(\underline{x}) = k$. If $k = 0$, then $\underline{x} = \underline{p_0}$, and therefore, $\underline{x} \in U(\underline{p_0})$. If $k > 0$, then there is a positive integer M so that if $i \geq M$, then $s_i(\underline{x}) = k$. That is, if $i \geq M$, then $x_i \in [p_k, p_{k-1})$. This implies that $\lim_{i\to\infty} x_i = p_k$, and therefore, $\underline{x} \in U(\underline{p_k})$.

Next, to see (2), suppose that $\lim_{i\to\infty} s_i(\underline{x})$ does not exist. Then, for each positive integer M, there is a positive integer i so that $x_i \in [0, p_{M-1})$. Therefore, $\lim_{i\to\infty} x_i = 0$ which implies that $\underline{x} \in U(\underline{p_\infty})$. □

The following theorem describes the composant structure of (I, f).

Theorem 3.2. *Let $f : I \to I$ be the function defined above. Suppose that the two points $\underline{x} = (x_0, x_1, x_2, \ldots)$ and $\underline{y} = (y_0, y_1, y_2, \ldots)$ are in (I, f). Then $\underline{x}$ and $\underline{y}$ are in the same composant if and only if there is a positive integer N such that one of the following conditions is satisfied.*

(1) *If $i \geq N$, then $s_i(\underline{x}) = s_{i+1}(\underline{x})$ and $s_i(\underline{y}) = s_{i+1}(\underline{y})$.*
(2) *If $i \geq N$, then $s_{i+1}(\underline{x}) = s_i(\underline{x}) + 1$ and $s_{i+1}(\underline{y}) = s_i(\underline{y}) + 1$.*
(3) *If $i \geq N$, then $s_i(\underline{x}) = s_i(\underline{y}) = n_i$ and $q_{n_i - 1}$ is not in $[x_i, y_i]$.*

We prove the theorem by establishing the following propositions, which imply the conclusion of the theorem.

The first proposition describes the composant of $\underline{p_0}$.

Proposition 3.3. *Suppose that* $\underline{a} = (a_0, a_1, a_2, \ldots) \in (I, f)$*. Then,* $\underline{a} \in C(\underline{p_0})$ *if and only if there exists a positive integer* k *so that* $\lim_{i\to\infty} s_i(\underline{a}) = k$*.*

Proof: First, we show that if $\lim_{i\to\infty} s_i(\underline{a}) = k$, then $\underline{a} \in C(\underline{p_0})$. If $\lim_{i\to\infty} s_i(\underline{a}) = k$, then for each positive integer $i = 0, 1, 2, \ldots$, $a_i \in [p_k, p_0]$. Also, we have that $f([p_k, p_0]) = [p_k, p_0]$, which implies that there exists an H, a proper subcontinuum of (I, f), so that for each $i = 0, 1, 2, \ldots$, $\prod_i(H) = [p_k, p_0]$. Clearly, $\underline{p_0}$, $\underline{a} \in H$, and therefore, $\underline{a} \in C(\underline{p_0})$.

Next, we show that if $\lim_{i\to\infty} s_i(\underline{a})$ is not equal to a positive integer, then $\underline{a}$ is not in $C(\underline{p_0})$. Notice that if $\lim_{i\to\infty} s_i(\underline{a}) \neq k$ for some positive integer k, then $\lim_{i\to\infty} s_i(\underline{a}) = \infty$; that is, $\lim_{i\to\infty} a_i = 0$. Now, let K be a subcontinuum of (I, f) that contains both $\underline{a}$ and $\underline{p_0}$. Then, for each $i = 0, 1, 2, \ldots$, $p_0 = 1 \in \prod_i(K)$. We want to show that for each $i = 0, 1, 2, \ldots$, $0 \in \prod_i(K)$, which implies that $K = (I, f)$, and that would complete the proof. Since $\lim_{k\to\infty} p_k = 0$, it suffices to show that for each $j = 0, 1, 2, \ldots$ and each $m = 0, 1, 2, \ldots$, $p_m \in \prod_j(K)$. Let integers j, $m \geq 0$ be given. Since $\lim_{i\to\infty} a_i = 0$, we have an integer $m_i \geq j$ so that $a_{m_i} \leq p_m$, which implies that $p_m \in \prod_{m_i}(K)$. But $f(p_m) = p_m$ and therefore, $p_m \in \prod_j(K)$. □

Notice that Proposition 3.3 implies that if $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $\underline{b} = (b_0, b_1, b_2, \ldots)$ are such that $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$, then they are in the same composant.

The next proposition describes the composant of $\underline{p_\infty}$.

Proposition 3.4. *Suppose that* $\underline{a} = (a_0, a_1, a_2, \ldots) \in (I, f)$*. Then* $\underline{a} \in C(\underline{p_\infty})$ *if and only if there is a positive integer* N *so that if* $i \geq N$*, then* $s_{i+1}(\underline{a}) = s_i(\underline{a}) + 1$*.*

Proof: First, we suppose that there is a positive integer N so that if $i \geq N$, then $s_{i+1}(\underline{a}) = s_i(\underline{a}) + 1$, and we show that $\underline{a} \in C(\underline{p_\infty})$. We define $\{J_i\}_{i=0}^{\infty}$, a sequence of closed intervals, as follows. For each $i = 0, 1, 2, \ldots, N$, let $J_i = [0, 1]$, and for each $i = N+1, N+2, N+3, \ldots$, let $J_i = [0, \frac{1}{2^{i-N}}]$. Clearly, $f(J_{i+1}) = J_i$ for each $i = 0, 1, 2, \ldots$ and therefore, there exists a proper subcontinuum K of (I, f) so that for each $i = 0, 1, 2, \ldots$, $\prod_i(K) = J_i$. Clearly, $\underline{p_\infty} \in K$. Now, in order to show that $\underline{a} \in K$, we notice that since for each $i \geq N$, $s_{i+1}(\underline{a}) = s_i(\underline{a}) + 1$, we have $s_{N+1}(\underline{a}) = s_N(\underline{a}) + 1 \geq 1$ $s_{N+2}(\underline{a}) = s_{N+1}(\underline{a}) + 1 \geq 2$, ..., and inductively, for each $i = 0, 1, 2, \ldots$, $s_{N+i+1}(\underline{a}) = s_{N+i}(\underline{a}) + 1 \geq i + 1$. Now it is easy to see that for each $i = 0, 1, 2, \ldots$, $a_i \in J_i$ and therefore, $\underline{a} \in K$. This completes the proof that if $\underline{a}$ satisfies the condition in the proposition, then $\underline{a} \in C(\underline{p_\infty})$.

Next, we suppose that, for each positive integer N, there is an $m \geq N$ so that $s_{m+1}(\underline{a}) \neq s_m(\underline{a}) + 1$, and we show that $\underline{a}$ is not in $C(\underline{p_\infty})$. If m is any positive integer such that $s_{m+1}(\underline{a}) \neq s_m(\underline{a}) + 1$, then either $s_{m+1}(\underline{a}) = s_m(\underline{a})$ or $s_{m+1}(\underline{a}) = s_m(\underline{a}) + 2$. Notice that the latter is possible for at most one value of m. It follows that, for each positive integer N, there is an $m \geq N$ so that $s_{m+1}(\underline{a}) = s_m(\underline{a})$. Let H be a subcontinuum of (I, f) with $\underline{a}$, $\underline{p_\infty} \in H$. First, we show that $\prod_0(H) = [0, 1]$. Clearly, $0 \in \prod_0(H)$. Suppose that $s_0(\underline{a}) = l$. Then $p_{l+1} \in \prod_0(H)$. Find positive integers $m, m_1, m_2, \ldots, m_{l+1}$ so that for each $i = 1, 2, \ldots, l + 1$,

(1) $m_i \leq m$;
(2) $i \neq j$, then $m_i \neq m_j$; and
(3) $s_{m_i}(\underline{a}) = s_{m_i - 1}(\underline{a})$.

Now we claim that $s_m(\underline{a}) \leq l + m - (l + 1) = m - 1$. To see this, notice that for each $j = 0, 1, 2, \ldots$, $s_{j+1}(\underline{a}) \leq s_j(\underline{a}) + 1$ and that for each $j = m_1 - 1, m_2 - 1, \ldots, m_{l+1} - 1$, $s_{j+1}(\underline{a}) = s_j(\underline{a})$. Recall that $\{s_i(\underline{a})\}_{i=0}^{\infty}$ is nondecreasing. But $s_m(\underline{a}) - s_0(\underline{a}) \leq m - (l + 1)$ since, for at least $l + 1$ values of j that are less than m, we have that $s_{j+1}(\underline{a}) = s_j(\underline{a})$. Therefore, $s_m(\underline{a}) \leq m - 1$, as we claimed. Since $s_m(\underline{a}) \leq m - 1$ and $0 \in \prod_m(H)$, $p_m \in \prod_m(H)$. But $f^m(p_m) = p_0 = 1$, which implies that $1 = p_0 \in \prod_0(H)$. Therefore, $\prod_0(H) = [0, 1]$.

Next, we let n be any positive integer and show that $\prod_n(H) = [0, 1]$, which completes the proof. We consider the subcontinuum $\hat{f}^{-n}(H)$ and the point $\hat{f}^{-n}(\underline{a})$. Clearly, $\underline{p_\infty}$, $\hat{f}^{-n}(\underline{a}) \in \hat{f}^{-n}(H)$, and it is easy to see that $\hat{f}^{-n}(\underline{a})$ satisfies the condition that for each positive integer N, there is an $m \geq N$ so that $s_{m+1}(\hat{f}^{-n}(\underline{a})) = s_m(\hat{f}^{-n}(\underline{a}))$. Therefore, $\prod_0(\hat{f}^{-n}(H)) = \prod_n(H) = [0, 1]$. (The proof of this is similar to the proof that $\prod_0(H) = [0, 1]$.) $\square$

Notice that for each $x \in (p_k, p_{k-1})$, where k is a positive integer, $f^{-1}(x)$ has two elements in $[p_{k+1}, p_k)$. But if $\underline{a} \in (I, f)$ satisfies the condition in Proposition 3.4, regardless of the choice of inverse image a_{n+1} of a_n for any $n = 0, 1, 2, \ldots$, we have that $\underline{a} \in C(\underline{p_\infty})$. Proposition 3.5 shows that the other composants are sensitive to this choice.

Proposition 3.5. *Suppose that both points $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $\underline{b} = (b_0, b_1, b_2, \ldots)$ are in (I, f) and that both $\underline{a}$ and $\underline{b}$ are not in either $C(\underline{p_\infty})$ or $C(\underline{p_0})$. Then $\underline{a}$ and $\underline{b}$ are in the same composant if and only if there is a positive integer N so that if $i \geq N$, then $n_i = s_i(\underline{a}) = s_i(\underline{b})$ and $q_{n_i - 1}$ is not in $[a_i, b_i]$.*

Notice that if $\underline{x}$ is not in $C(\underline{p_0})$, then $\lim_{i\to\infty} s_i(\underline{x}) = \infty$; that is, $\lim_{i\to\infty} x_i = 0$.

We prove Proposition 3.5 by establishing the following three lemmas.

Lemma 3.6. *Let $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $\underline{b} = (b_0, b_1, b_2, \ldots)$ be points of (I, f). Suppose that both $\underline{a}$ and $\underline{b}$ are not in either $C(\underline{p_\infty})$ or $C(\underline{p_0})$, and that for each positive integer N, there is an $m \geq N$ so that $\mid s_m(\underline{a}) - s_m(\underline{b}) \mid \geq 1$. Then $\underline{a}$ and $\underline{b}$ are not in the same composant.*

Proof: Let H be a subcontinuum of (I, f) with $\underline{a}$, $\underline{b} \in H$. It suffices to show that $0 \in \prod_0(H)$. To see that it is sufficient, first notice that for any positive integer n, $\hat{f}^{-n}(\underline{a})$, $\hat{f}^{-n}(\underline{b}) \in \hat{f}^{-n}(H)$, which is a subcontinuum of (I, f), and that $\hat{f}^{-n}(\underline{a})$ and $\hat{f}^{-n}(\underline{b})$ satisfy the same conditions that $\underline{a}$ and $\underline{b}$ do. Therefore, any argument which proves that $0 \in \prod_0(H)$ can be used to establish that $0 \in \prod_0(\hat{f}^{-n}(H)) = \prod_n(H)$. Hence, if $0 \in \prod_0(H)$, then for each $n = 0, 1, 2, \ldots$, $0 \in \prod_n(H)$, and that implies $\underline{p_\infty} \in H$. This is impossible if H is proper since $\underline{a} \in H$ and $\underline{a}$ is not in $C(\underline{p_\infty})$. This establishes our claim that it suffices to show that $0 \in \prod_0(H)$.

In order to show that $0 \in \prod_0(H)$, suppose that n is a given positive integer and find n_k, a positive integer so that $s_{n_k}(\underline{a}) \geq n + 1$. It is possible to find such an n_k since $\underline{a} \notin C(\underline{p_0})$ implies that $\lim_{i\to\infty} s_i(\underline{a}) = \infty$. Now we find an $m_k \geq n_k$ so that $\mid s_{m_k}(\underline{a}) - s_{m_k}(\underline{b}) \mid \geq 1$. Now, since $m_k \geq n_k$, $s_{m_k}(\underline{a}) \geq s_{n_k}(\underline{a}) \geq n + 1$. Let $l_m = s_{m_k}(\underline{a})$. Then either p_{l_m} or $p_{l_m - 1}$ is in $[a_{m_k}, b_{m_k}]$ and therefore in $\prod_{m_k}(H)$. In either case, we have an $m \geq n$ so that $p_m \in \prod_{m_k}(H)$. But $f(p_m) = p_m$, and therefore, $p_m \in \prod_0(H)$. Since n is arbitrary and $\lim_{i\to\infty} p_i = 0$, $0 \in \prod_0(H)$. $\square$

Lemma 3.7. *Let $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $\underline{b} = (b_0, b_1, b_2, \ldots)$ be points of (I, f). Suppose that both $\underline{a}$ and $\underline{b}$ are not in either $C(\underline{p_\infty})$ or $C(\underline{p_0})$, and that there is a positive integer N so that if $i \geq N$, then $s_i(\underline{a}) = s_i(\underline{b})$. Also, suppose that for each positive integer $M \geq N$, there are positive integers u and v so that $u \geq M$, $s_u(\underline{a}) = s_u(\underline{b}) = v$, and $q_{v-1} \in [a_u, b_u]$. Then $\underline{a}$ and $\underline{b}$ are not in the same composant.*

Proof: Let H be a subcontinuum of (I, f) with $\underline{a}$, $\underline{b} \in H$. It suffices to prove that $0 \in \prod_0(H)$. The proof that it is sufficient is similar to the argument in the proof of Lemma 3.6. As in the proof of Lemma 3.6, we show that for any positive integer n, there is an $m \geq n$ so that $p_m \in \prod_0(H)$. Let n, a positive integer, be given. Since $\underline{a} \notin C(\underline{p_0})$ implies that $\lim_{i\to\infty} s_i(\underline{a}) = \infty$, we can find an $n_k \geq N$ so that $s_{n_k}(\underline{a}) \geq n + 2$. Also, there is an $n_l \geq n_k$ so that $q_{n_t} \in [a_{n_l}, b_{n_l}]$, where $n_t = s_{n_l}(\underline{a}) - 1$. Since $q_{n_t} \in [a_{n_l}, b_{n_l}]$, $p_{n_t-1} \in f([a_{n_l}, b_{n_l}])$. Therefore, $p_{n_t-1} \in \prod_{n_l-1}(H)$. Also, we have that $n_t - 1 = s_{n_l}(\underline{a}) - 2 \geq s_{n_k}(\underline{a}) - 2 \geq n$. Now, set $m = n_t - 1$ and notice that $f(p_m) = p_m$, which completes the proof that there is an $m \geq n$ so that $p_m \in \prod_0(H)$. □

Lemma 3.8. *Suppose that both points $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $\underline{b} = (b_0, b_1, b_2, \ldots)$ are in (I, f) and that there is a positive integer N so that if $i \geq N$, then $s_i(\underline{a}) = s_i(\underline{b}) = n_i$ and $q_{n_i-1} \notin [a_i, b_i]$. Then $\underline{a}$ and $\underline{b}$ are in the same composant of (I, f).*

Proof: Assume that $\underline{a} = (a_0, a_1, a_2, \ldots)$ and $\underline{b} = (b_0, b_1, b_2, \ldots)$ satisfy the conditions in the lemma. We define $\{J_i\}_{i=0}^{\infty}$, a sequence of closed intervals, as follows. For each $i = N, N+1, N+2, \ldots$, $J_i = [a_i, b_i]$, and for each $i = 0, 1, 2, \ldots N-1$, $J_i = f^{N-i}(J_N)$. It is easy to see that for each $i = 0, 1, 2, \ldots$, $f(J_{i+1}) = J_i$. There is a proper subcontinuum K of (I, f) so that for each $i = 0, 1, 2, \ldots$, $\prod_i(K) = J_i$, and clearly, $\underline{a}$, $\underline{b} \in K$. This implies that $\underline{a}$ and $\underline{b}$ are in the same composant. □

This completes the proof of Theorem 3.2. □

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