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# SOME REMARKS ON MAXIMAL CONNECTED HAUSDORFF SPACES

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Abstract. We prove that there exists a connected Hausdorff space containing a maximal connected subspace for which every open superset is dense and which, under the restriction that its maximal connected subspace remains connected, does not have a maximal connected expansion. We also prove that there exists an uncountable connected separable Hausdorff submaximal space with a dispersion point.

## 1. Introduction

In [6], G. J. Kennedy and S. D. McCartan pose the question of whether every space containing a maximal connected subspace for which every open superset is dense has a maximal connected expansion. We prove that there exists a connected Hausdorff space having these properties which, under the restriction that its maximal connected subspace remains connected, does not have a maximal connected expansion. The required space is obtained by attaching two specific spaces. The first is a countable connected Hausdorff submaximal space, having a dispersion point. The use of a space with a dispersion point is that such a space does not have a maximal connected expansion [4]. The second space is the Katětov extension of any maximal connected Hausdorff space.

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A space is called *submaximal* [2], if every dense subset of it is open. A point $p$ of a connected space $Y$ is called (1) a *dispersion point* [7] if the subspace $Y \setminus \{p\}$ is totally disconnected, (2) a *cut point* if the subspace $Y \setminus \{p\}$ is disconnected. The Katětov extension and its properties are described in detail by Jack R. Porter and R. Grant Woods in [10]. The existence of maximal connected Hausdorff spaces of every cardinality is proved by A. G. El'kin in [3]. The concept of a maximal connected topology was introduced by J. Pelham Thomas in [12].

In [1], A. V. Arhangel'skiĭ and P. J. Collins pose the problems of whether there exists a dense-in-itself uncountable separable Hausdorff (Tychonoff) submaximal space. Both these problems have been answered by Ronnie Levy and Porter in [8]. Based on [8] and [2], we observe that if we start with any countable Hausdorff space without isolated points and consider any regularly open-maximal topology [9], we get a countable Hausdorff submaximal space without isolated points whose Katětov extension answers the first of the above problems. It is obvious that if the initial space is a countable connected Hausdorff space, then any regularly open-maximal topology is a countable connected Hausdorff submaximal space whose Katětov extension is an uncountable connected separable Hausdorff submaximal space. We prove that this result can be extended to the Hausdorff submaximal spaces with a dispersion point.

## 2. Main Results

The following space $(Y, \tau)$ is due to Prabir Roy [11]. It is a countable connected Hausdorff space having the point $p$ as a dispersion point.

Let $C_i, i = 1, 2, ...$ be a countable collection of disjoint dense subsets of the rationals $Q$. On the set $Y = \{p\} \cup \{(r, i) \in Q \times Z^+ : r \in C_i\}$, we define a topology $\tau$ as follows.

For every point of the form $(r, 2n)$, a basis of open neighborhoods includes the sets $U_\epsilon(r, 2n) = \{(t, 2n) : |t - r| < \epsilon\}$ for every $\epsilon > 0$.

For every point of the form $(r, 2n-1)$, a basis of open neighborhoods includes the sets $U_\epsilon(r, 2n-1) = \{(t, m) : |t - r| < \epsilon, m = 2n-2, 2n-1, 2n\}$ for every $\epsilon > 0$.

For the point $p$, a basis of open neighborhoods includes the sets $U_n(p) = \{(s, i) \in Y : i \geq 2n\}, n = 1, 2, ....$

On each $C_i, i = 1, 2, ...,$ we consider two disjoint dense subsets $L_i, M_i$ such that $C_i = L_i \cup M_i$. We set $L = \bigcup_{i=1}^{\infty} L_i$ and $M = \bigcup_{i=1}^{\infty} M_i$. Obviously, the sets $L, M$ are disjoint and dense. We consider the simple expansion $\tau \vee \tau_M$ (where $\tau_M$ is the topology generated by $\{M\}$) and any regularly open-maximal topology equivalent to $\tau \vee \tau_M$. We denote this topology by $\sigma$.

**Proposition 2.1.** *The space $(Y, \sigma)$ has the following properties.*

1. *It is countable connected Hausdorff having the point $p$ as a dispersion point.*
2. *It is submaximal.*
3. *The subset $L$ is a closed discrete subspace for which every open superset is dense.*

*Proof:* (1) That $(Y, \sigma)$ is countable Hausdorff is obvious. Since the topologies $\tau, \tau \vee \tau_M, \sigma$ have the same regularly open subsets, then $(Y, \sigma)$ is connected. That $p$ remains the dispersion point is obvious.

(2) By [9, 2.2], every regularly open-maximal topology is submaximal.

(3) The subset $M$ is open dense in $\tau \vee \tau_M$, hence open in $\sigma$. By [9, 1.1(3)], it follows that $Cl_{\tau \vee \tau_M} M = Cl_\sigma M$; that is, $M$ is dense in $\sigma$. Since $Y$ is submaximal, $L$ is a closed discrete subspace of $(Y, \sigma)$. Let $U \in \sigma$ and $L \subseteq U$. By [9, 1.1(2)], $Cl_\tau U = Cl_\sigma U$. Since $L$ is dense in $\tau$, $U$ is dense in $\sigma$. □

The following proposition is mentioned by El'kin in [3] without proof. That every maximal connected space is submaximal is proved by Thomas in [13].

**Proposition 2.2.** *Let $(X, \tau)$ be a Hausdorff maximal connected space. Then the Katětov extension $\kappa X$ of $X$ is maximal connected.*

*Proof:* We first prove that $\kappa X$ is submaximal. Let $D$ be a dense subset of $\kappa X$. Since $X$ is open, $D \cap X$ is dense in $X$ and hence is also dense in $\kappa X$. As $D \cap X \subseteq D \subseteq \kappa X = Cl_{\kappa X}(D \cap X)$, $D$ is open in $\kappa X$.

Suppose there exists a connected topology $\tau''$ strictly finer than the Katětov extension of $X$. We denote by $\tau'$ the induced topology by $\tau''$ on $(X, \tau)$. Since $X$ is maximal connected, there exists $U \in \tau''$ such that the subset $A = U \cap X$ is open and closed in $\tau'$. Since $\kappa X$

is submaximal, it follows that $Int_{\kappa X}U \neq \emptyset$. For if $Int_{\kappa X}U = \emptyset$, then $U$ is a closed discrete subspace of $\kappa X$; hence, $U$ is open and closed in $\tau''$, which is impossible, because $\tau''$ is connected.

Let $s \in Cl_{\tau''}U \setminus U$. By the definition of the Katětov extension and since $U \cap X$ is open and closed in $\tau'$, it follows that $s \in \kappa X \setminus X$ and $s \in Cl_{\tau''}A$. Hence, $s \in Cl_{\kappa X}A$ and therefore, $s$ is an accumulation point of $Int_\tau A$ in $\kappa X$. Thus, $s$ belongs to the open subset $Int_\tau A \cup \{q \in \kappa X \setminus X : Int_\tau A \in q\}$ in $\kappa X$. Since $A$ is open in $\tau''$, the subset $Cl_{\tau''}U = A \cup \{q \in \kappa X \setminus X : Int_\tau A \in q\}$ is open and closed in $\tau''$, contradicting the fact that $\tau''$ is connected. □

We attach the spaces $(Y, \sigma)$ and $\kappa X$ by attaching only the closed discrete subspace $L$ of $Y$ to the closed discrete subspace $\kappa X \setminus X$ of $\kappa X$ and the spaces $Y \setminus L$ and $X$ remaining disjoint. That is, we consider a one-to-one function $f$ of $L$ into $\kappa X \setminus X$, and we identify each $l \in L$ to $f(l) \in \kappa X \setminus X$. The remaining points of $(\kappa X \setminus X) \setminus f(L)$ may be removed.

On the set $Z = X \cup Y$, we define a topology $\rho$ as follows. $X$ and $Y \setminus L$ keep their own topology. For every point $l \in L$, a basic open neighborhood is the set $B_Z(l) = B_{\kappa X}(f(l)) \cup B_Y(l)$, where $B_{\kappa X}(f(l))$ is a basic open neighborhood of $f(l)$ in $\kappa X$ and $B_Y(l)$ is a basic open neighborhood of $l$ in $(Y, \sigma)$.

**Proposition 2.3.** *The space $(Z, \rho)$ has the following properties.*

(1) *It is Hausdorff connected submaximal.*
(2) *The subspaces $Z \setminus X \cup L$ and $Z \setminus Y$ are open.*
(3) *$L$ is a closed discrete subspace of $Z$.*
(4) *The subspace $X \cup L$ is maximal connected.*
(5) *Every open superset of $X \cup L$ is dense.*

*Proof:* (1), (2), and (3) follow from the definition of the space $Z$.

(4) By [5, Theorem 7], every connected subspace of a maximal connected space is maximal connected. Hence, $X \cup L$ is maximal connected.

(5) Every open superset $V$ of $X \cup L$ is of the form $X \cup U$, where $U$ is open in $(Y, \sigma)$ containing $L$. Hence, by Proposition 2.1.(3), $V$ is dense. □

**Proposition 2.4.** *If the space $(Z, \rho)$ has a maximal connected expansion $\rho_m$, then the subspace $(X, \rho_m \mid X)$ is not connected.*

*Proof:* Suppose $\rho$ has a maximal connected expansion $\rho_m$ in which $X$ remains connected. By [4, Theorem 14], the space $(Z, \rho_m)$ has a cut point in the boundary of every nondense open subset. Let $A$ be an open and closed subset of $Y$ in $\sigma$. The subset $A \setminus L$ is open in $\sigma$, hence open in $\rho$ and therefore open in $\rho_m$. For the open $A \setminus L$, the cut points of $(Z, \rho_m)$ are either the point $p$ or a point of $L$.

Suppose $p$ is a cut point. Then there exist two open subsets $F, G$ in $\rho_m$ such that $F \cap G = \emptyset$ and $F \cup \{p\} \cup G = Z$. Obviously, the subsets $F \cup \{p\}$ and $G \cup \{p\}$ are closed connected in $\rho_m$ and hence connected in $\rho$. Since $p \notin X$, the point $p$ cannot cut the connected subspace $X$. Hence, one of the above subsets contains $X$. Let $X \subseteq G \cup \{p\}$. Then for the connected subspace $F \cup \{p\}$ of $(Z, \rho_m)$, it holds that

(a) $F \cup \{p\}$ is maximal connected in $(Z, \rho_m)$ because it is a connected subspace of a maximal connected space, by [5, Theorem 7];

(b) $F \cup \{p\}$ is connected in $(Z, \rho)$, having the point $p$ as a dispersion point, because $F \cup \{p\} \subseteq Y$.

Hence, $F \cup \{p\}$ has a maximal connected expansion which is impossible by [4, Theorem 15]. Therefore, the point $p$ cannot be a cut point.

Suppose $l \in L$ is a cut point. Therefore, similarly as above, there exist two open subsets $S, T$ in $\rho_m$ such that $S \cap T = \emptyset$ and $S \cup \{l\} \cup T = Z$. Hence, one of the closed connected subsets $S \cup \{l\}$, $T \cup \{l\}$ in $\rho_m$ contains $X$. Let $X \subseteq T \cup \{l\}$. But then $S \cup \{l\} \subseteq Y$; that is, the subspace $S \cup \{l\}$, which is connected with a dispersion point in $(Z, \rho)$, is expanded in a maximal connected space, which is impossible. □

**Proposition 2.5.** *Let $X$ be a countable connected Hausdorff submaximal space with a dispersion point $p$. Then the Katětov extension $\kappa X$ of $X$ is an uncountable connected separable Hausdorff submaximal space having the point $p$ as a dispersion point.*

*Proof:* That the space $\kappa X$ is connected separable Hausdorff is obvious.

By the proof of Proposition 2.2., it is submaximal. Since every countable H-closed space contains isolated points [2, Excercise 23, p. 147], $\kappa X$ is uncountable.

It remains to prove that $p$ is a dispersion point of $\kappa X$. Let $M$ be a connected subset of $\kappa X \setminus \{p\}$. Since $\kappa X$ is submaximal, it follows that $Int_{\kappa X}M \neq \emptyset$. Let $s \in \kappa X \setminus X$ and $s \in M \setminus Int_{\kappa X}M$. We set $N = M \cap X$. Since $M \setminus Int_{\kappa X}M$ is a closed discrete subspace of $\kappa X$, then $s$ is an accumulation point of $Int_X N$ in $\kappa X$. For if not, then there exists an open set $U$ in $\kappa X$ containing $s$ such that $U \cap M = \{s\}$; that is, $M$ is not connected. Hence, $s \in Int_{\kappa X}M$ and therefore, $M \setminus Int_{\kappa X}M \subseteq X$. Obviously, $Int_{\kappa X}M \cap X \subseteq Int_X N$. Since $N$ is not connected, there exist two open subsets $A, B$ in $X$ such that $A \cap N \cap B = \emptyset$ and $(A \cap N) \cup (B \cap N) = N$. Hence, the subsets $A \cap Int_X N$ and $B \cap Int_X N$ are disjoint open in $X$. Therefore, the subsets $(A \cap Int_X N) \cup \{q \in \kappa X \setminus X : A \cap Int_X N \in q\}$ and $(B \cap Int_X N) \cup \{q \in \kappa X \setminus X : B \cap Int_X N \in q\}$ are disjoint open in $\kappa X$ such that their union covers $Int_{\kappa X}M$. Since $N$ has no isolated points, it follows that the subsets $Cl_{\kappa X}(A \cap Int_X N)$ and $Cl_{\kappa X}(B \cap Int_X N)$, whose union covers $M$, are disjoint on $M$, contradicting the fact that $M$ is connected.

□

## References

[1] A. V. Arhangel'skiĭ and P. J. Collins, *On submaximal spaces*, Topology Appl. **64** (1995), no. 3, 219–241.

[2] Nicolas Bourbaki, *Elements of Mathematics: General Topology. Part 1.* Paris: Hermann; Reading, Mass.: Addison-Wesley Pub. Co., 1966.

[3] A. G. El'kin, *Maximal connected Hausdorff spaces* (Russian), Mat. Zametki, translated in Math. Notes **26** (1979), no. 6, 974–978.

[4] J. A. Guthrie, D. F. Reynolds, and H. E. Stone, *Connected expansions of topologies*, Bull. Austral. Math. Soc. **9** (1973), 259–265.

[5] J. A. Guthrie and H. E. Stone, *Spaces whose connected expansions preserve connected subsets*, Fund. Math. **80** (1973), no. 1, 91–100.

[6] G. J. Kennedy and S. D. McCartan, *Maximal connected expansions*, Proceedings of the First Summer Galway Topology Colloquium (1997). Topol. Atlas, North Bay, ON, 1998 (electronic).

[7] B. Knaster and K. Kuratowski, *Sur les ensembles connexes*, Fund. Math. **2** (1921), 206–255.

[8] Ronnie Levy and Jack R. Porter, *On two questions of Arhangel'skii and Collins regarding submaximal spaces*, Topology Proc. **21** (1996), 143–154.

[9] J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Math. Rozprawy Mat. **66** (1969) 55 pages.

[10] Jack R. Porter and R. Grant Woods, *Extensions and Absolutes of Hausdorff Spaces*. New York: Springer-Verlag, 1988.

[11] Prabir Roy, *A countable connected Urysohn space with a dispersion point*, Duke Math. J. **33** (1966), 331–333.

[12] J. Pelham Thomas, *Maximal connected topologies*, J. Austral. Math. Soc. **8** (1968), 700–705.

[13] ______, *Maximal connected Hausdorff spaces*, Pacific J. Math. **57** (1975), no. 2, 581–583.

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