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by

Alejandro Illanes

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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A TREE-LIKE CONTINUUM WHOSE HYPERSPACE OF SUBCONTINUA ADMITS A FIXED-POINT-FREE MAP

ALEJANDRO ILLANES

Abstract. We show that if X is a simple triod with a ray surrounding it, then its hyperspace of subcontinua does not have the fixed point property. This answers an old question by James T. Rogers, Jr.

1. Introduction

A *continuum* is a compact, connected metric space with more than one point. For a continuum X, we consider the following hyperspaces:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\} \text{ and, for a positive integer } n,$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}.$$

All these hyperspaces are considered with the Hausdorff metric.

A *Whitney map for* $C(X)$ is a continuous function $\mu : C(X) \to [0,1]$ such that (a) $\mu(\{p\}) = 0$ for each $p \in X$, and (b) if $A, B \in C(X)$ and $A \subset B \neq A$, then $\mu(A) < \mu(B)$. It is known (see [4, Theorem 13.4]) that for each continuum X, there exist Whitney maps for $C(X)$.

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A topological space Y is said to have *the fixed point property* (fpp) provided that for each continuous function $f : Y \to Y$ there exists a point $p \in Y$ such that $f(p) = p$.

Let $\mathbb{R}^2$ denote the Euclidean plane. Let $D_0 = \{z \in \mathbb{R}^2 : |z| = 1\} \cup \{(\frac{t+1}{t})(\cos(t), \sin(t)) \in \mathbb{R}^2 : t \in [1, \infty)\}$ and $D_1 = D_0 \cup \{z \in \mathbb{R}^2 : |z| \leq 1\}$. A *simple triod* is a continuum that is the union of three arcs emanating from a single point and otherwise disjoint from one another. Let X_0 denote the continuum which is the union of a simple triod T_0 and a ray R_0 surrounding it. The continuum X_0 is illustrated in Figure 1 (see page 4).

A discussion on what is known about the fpp of hyperspaces can be found in [4, Chapter VI] and [7, Chapter VII]. We mention here only some facts.

In 1952, Bronisław Knaster posed the following question: If X is a continuum with the fpp, then does $C(X)$ has the fpp? ([2, Problem 186]). A fundamental example on the theory of the fpp is the cone over D_0 (cone(D_0)) since Ronald J. Knill showed that this cone does not have the fpp ([5]). In [9], James T. Rogers, Jr. showed that cone(D_0) and $C(D_0)$ are homeomorphic; thus, $C(D_0)$ does not have the fpp. Hence, D_0 was the first example for which the hyperspace of subcontinua does not have the fpp. In [8], Sam B. Nadler, Jr. and Rogers answered Knaster's question by showing that $C(D_1)$ does not have the fpp.

In the direction of trying to find stronger properties under which Knaster's question has a positive answer, Rogers asked the following question: If X is a tree-like continuum, does $C(X)$ have the fpp? ([10, p. 283]). This question has also appeared in [7, Question 7.8], [1, Question 199], [11, Question 8, p. 207], and [6, Problem 109]. Another related question was posed by Nadler: Does $C(X)$ have the fpp for every one-dimensional continuum X with the fpp? [7, p. 297].

In this paper, we answer questions posed by Nadler and Rogers by showing that $C(X_0)$ does not have the fpp.

Recently, the continuum X_0 has played another important role in the fpp theory on continua. Answering an old question on this topic, the author has shown in [3] that cone(X_0) does not have the fpp.

The idea of the proof that $C(X_0)$ does not have the fpp is similar to the following one: Take a Whitney map μ for $C(X_0)$. Then there exist a number $t \in (0, \mu(T_0))$ and a homeomorphism $h : \text{cone}(D_1) \to \mu^{-1}([t, \mu(X_0)])$ such that $h(\text{base of cone}(D_1)) = \mu^{-1}(t)$. Let C_0 be the half upper part of the cone(D_1), that is, $C_0 = \{(z, s) \in \text{cone}(D_1) : z \in D_1 \text{ and } \frac{1}{2} \leq s \leq 1\}$, where the vertex of cone(D_1) is given by the points of the form $(z, 1)$ of cone(D_1). Since C_0 is homeomorphic to cone(D_1), cone(D_0) is a retract of cone(D_1), and cone(D_0) does not have the fpp, we obtain that C_0 does not have the fpp. Thus, if we can show that $h(C_0)$ is a retract of $C(X_0)$, then $C(X_0)$ does not have the fpp. In order to see that $h(C_0)$ is a retract of $C(X_0)$, notice that if we shrink the closed set $\mu^{-1}([0, t])$, of $C(X_0)$, to a point, we obtain the continuum $C(X_0)/\mu^{-1}([0, t])$ which is homeomorphic to the space obtained by shrinking the closed set $\mu^{-1}(t)$, of $\mu^{-1}([t, 1])$, to a point. Thus, $C(X_0)/\mu^{-1}([0, t])$ is homeomorphic to cone$(D_1)/$(base of cone(D_1)). Hence, $C(X_0)/\mu^{-1}([0, t])$ is homeomorphic to the suspension of D_1. Since C_0 can be seen as a retract of suspension of D_1, it follows that $h(C_0)$ is a retract of $C(X_0)$. Therefore, $C(X_0)$ does not have the fpp.

The difficult part of the proof is the formal construction of the homeomorphism h. In fact, instead of constructing h, we construct a map F which is not a homeomorphism but it has some nice properties. This construction is made in the following section by using a very specific Whitney map μ.

The rest of the paper is devoted to constructing the necessary formulae to give the formal proof that $C(X_0)$ does not have the fpp.

2. Auxiliary constructions

Let θ be the origin of the Euclidean plane $\mathbb{R}^2$. Given two points p and q in $\mathbb{R}^2$ (or $\mathbb{R}^3$), let pq denote the convex segment joining them. For each $i \in \{1, 2, 3\}$, let $v_i = (\cos(\frac{2\pi i}{3}), \sin(\frac{2\pi i}{3}))$.

Let $T_0 = \theta v_1 \cup \theta v_2 \cup \theta v_3$.

Let $\varphi : [2, \infty) \to \mathbb{R}^2$ be the piecewise linear function defined by the following conditions.

$\varphi(6n) = (1+\frac{1}{n+1})v_1$, $\varphi(6n+1) = -\frac{1}{n+1}v_3$, $\varphi(6n+2) = (1+\frac{1}{n+1})v_2$, $\varphi(6n+3) = -\frac{1}{n+1}v_1$, $\varphi(6n+4) = (1+\frac{1}{n+1})v_3$, and $\varphi(6n+5) = -\frac{1}{n+1}v_2$, where $n \geq 0$ is an integer for which the corresponding point lies in the interval $[2, \infty)$.

Let $X_0 = T_0 \cup \varphi([2, \infty))$. Notice that X_0 is a compactification of the ray $[2, \infty)$ with T_0 as remainder. Notice that X_0 is a tree-like continuum. The continuum X_0 is illustrated in Figure 1.

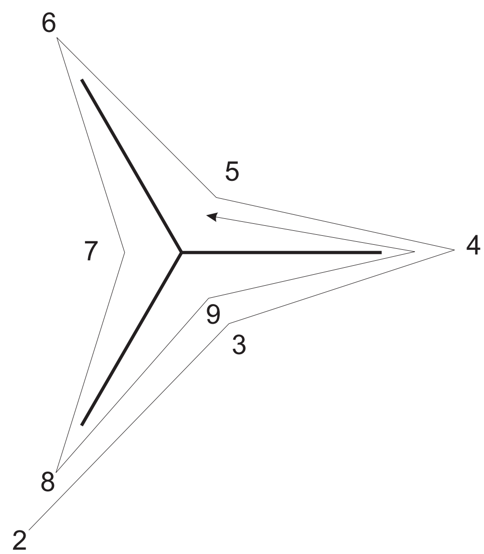

FIGURE 1.

For each $t \in [0, 1]$, let $\Delta(t)$ be the convex hull in $\mathbb{R}^3$ of the set $\{((1-t)v_1, t), ((1-t)v_2, t), ((1-t)v_3, t)\}$. Let $\mathfrak{T} = \bigcup\{\Delta(t) : t \in [0, 1]\}$. Notice that $\mathfrak{T}$ is the tetrahedron with vertices $(v_1, 0)$, $(v_2, 0)$, $(v_3, 0)$, and $(\theta, 1)$. Moreover, $\mathfrak{T}$ is the geometric cone of the solid triangle $\Delta(0)$ with vertex at the point $(\theta, 1)$.

For each $t \in [0, 1]$, let $\psi(t) : [1, \infty) \to \mathbb{R}^3$ be the piecewise linear map defined by the following conditions.

$\psi(t)(6n) = ((1-t)(v_1 - \frac{1}{n+1}v_2), t)$; $\psi(t)(6n+1) = ((1-t)(v_1 - \frac{1}{n+1}v_3), t)$; $\psi(t)(6n+2) = ((1-t)(v_2 - \frac{1}{n+1}v_3), t)$; $\psi(t)(6n+3) = ((1-t)(v_2 - \frac{1}{n+1}v_1), t)$; $\psi(t)(6n+4) = ((1-t)(v_3 - \frac{1}{n+1}v_1), t)$;

$\psi(t)(6n+5) = ((1-t)(v_3 - \frac{1}{n+1}v_2), t)$, where $n \geq 0$ is an integer for which the corresponding point lies in the interval $[1, \infty)$.

The behavior of $\psi(t)$ is illustrated in Figure 2.

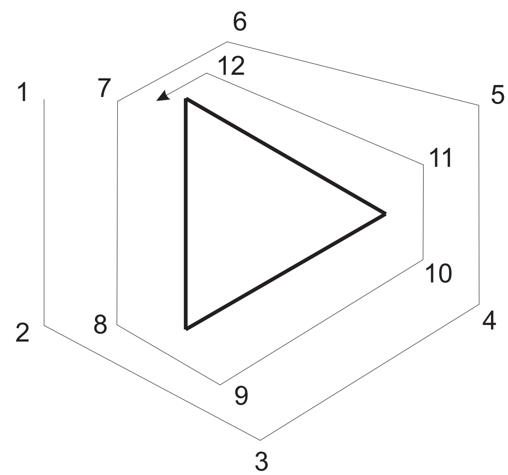

FIGURE 2.

Define $\psi : [0,1] \times [1, \infty) \to \mathbb{R}^3$ by $\psi(t,s) = \psi(t)(s)$.

Let $\mathfrak{D} = \mathfrak{T} \cup \psi([0,1] \times [1, \infty))$. It is easy to show that the following properties hold.

(A) ψ is continuous. Given $(t,s), (u,v) \in [0,1] \times [1, \infty)$, $\psi(t,s) = \psi(u,v)$ if and only if $(t,s) = (u,v)$ or $t = u = 1$. The set $\mathfrak{D}$ is the geometric cone of the set $\mathfrak{T} \cup \psi(\{0\} \times [1, \infty))$, with vertex at $(\theta, 1)$, and $\mathfrak{D}$ is homeomorphic to cone(D_1).

Given a convex segment J in $\mathbb{R}^2$, let $l(J)$ denote the length of J, we extend the definition of l by putting $l(\{p\}) = 0$ for each $p \in \mathbb{R}^2$ and $l(\emptyset) = 0$. Given $A \in C(T_0)$ and $i \in \{1,2,3\}$, let $f_i(A) = l(A \cap \theta v_i)$. Then f_i is a continuous function.

Let $\omega : C(T_0) \to [0,3]$ be given by $\omega(A) = f_1(A) + f_2(A) + f_3(A)$. It is easy to show that ω is a Whitney map for $C(T_0)$ with the following property: for each $A \in \omega^{-1}([\frac{5}{2}, 3])$ and each $i \in \{1,2,3\}$, $A \cap \theta v_i \neq \emptyset$.

Let $f : \omega^{-1}([\frac{5}{2}, 3]) \to \mathfrak{T}$ be given by $f(A) = ((2 - 2f_1(A))v_1 + (2 - 2f_2(A))v_2 + (2 - 2f_3(A))v_3, 2\omega(A) - 5)$.

Given $A \in \omega^{-1}([\frac{5}{2}, 3))$, let $t = 2\omega(A) - 5$, then $t \in [0, 1)$ and $f(A) = (\frac{2-2f_1(A)}{1-t}(1-t)v_1 + \frac{2-2f_2(A)}{1-t}(1-t)v_2 + \frac{2-2f_3(A)}{1-t}(1-t)v_3, t)$. This implies that $f(A)$ is a convex combination of the vectors $((1-t)v_1, t)$, $((1-t)v_2, t)$, and $((1-t)v_3, t)$. Thus, $f(A) \in \mathfrak{T}$. In the case that $\omega(A) = 3$, we have that $A = T_0$ and $f(A) = (\theta, 1) \in \mathfrak{T}$. We have shown that $f(\omega^{-1}([\frac{5}{2}, 3])) \subset \mathfrak{T}$.

Clearly, f is continuous; using the formula in the paragraph above, it is easy to check that f is one to one. Now we prove that f is onto. Let $p = (x_1(1-t)v_1 + x_2(1-t)v_2 + x_3(1-t)v_3, t) \in \mathfrak{T}$, where $t, x_1, x_2, x_3 \in [0, 1]$ and $x_1 + x_2 + x_3 = 1$. Since $f(T_0) = (\theta, 1)$, we assume that $t < 1$. For each $i \in \{1, 2, 3\}$, let $s_i = 1 - \frac{x_i(1-t)}{2}$, then $s_i \in [\frac{1}{2}, 1]$. Let A_i be a subarc of θv_i such that $\theta \in A_i$ and $l(A_i) = s_i$. Let $A = A_1 \cup A_2 \cup A_3$. Then $A \in C(T_0)$, $\omega(A) = s_1 + s_2 + s_3 = \frac{5+t}{2}$ and, for each $i \in \{1, 2, 3\}$, $2 - 2f_i(A) = 2 - 2s_i = x_i(1-t)$. Thus, $f(A) = p$. Hence, f is onto.

We have proved that f is a homeomorphism.

Let $g : X_0 \to T_0$ be the piecewise linear map defined for the conditions: $g|_{T_0} =$ identity of T_0, $g(-\frac{1}{n}v_i) = \theta$ and $g((1+\frac{1}{n})v_i) = v_i$, for each $i \in \{1, 2, 3\}$, where n is an integer such that the respective point is in X_0. Let $G : C(X_0) \to C(T_0)$ be the induced map defined by $G(A) = g(A)$ (the image of A under g), for each $A \in C(X_0)$.

Let $\mathfrak{A} = \varphi([2, \infty))$ and $C(\mathfrak{A}) = \{A \in C(X_0) : A \subset \mathfrak{A}\}$. Define $N, M : C(\mathfrak{A}) \to [2, \infty)$ by $N(A) = \min(\varphi^{-1}(A))$ and $M(A) = \max(\varphi^{-1}(A))$. Clearly, N and M are continuous and $A = \varphi([N(A), M(A)])$ for each $A \in C(\mathfrak{A})$.

For each $i \in \{1, 2, 3\}$, let $\mathfrak{R}_i = \{A \in C(\mathfrak{A}) : G(A) \cap \{v_1, v_2, v_3\} = \{v_i\}$ and $\omega(G(A)) \geq \frac{5}{2}\}$. Let $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2 \cup \mathfrak{R}_3$ and $\mathfrak{Q} = \text{cl}_{C(X_0)}(\mathfrak{R}) \cup C(T_0)$. Then $\mathfrak{Q}$ is a closed subset of $C(X_0)$.

We extend ω by defining $\mu : \mathfrak{Q} \to [0, 3]$ as $\mu(A) = \omega(G(A))$ for each $A \in \mathfrak{Q}$. Clearly, μ is continuous.

Next, we check that if $A, B \in \mathfrak{Q}$ and $A \subset B \neq A$, then $\mu(A) < \mu(B)$. If $B \subset T_0$, since ω is a Whitney map and $G(A) = A$, $G(B) = B$, we have $\mu(A) < \mu(B)$. Suppose then that $B \in \text{cl}_{C(X_0)}(\mathfrak{R})$. Then there exists a sequence $\{B_n\}_{n=1}^{\infty}$ in $\mathfrak{R}$ converging to B. Suppose, for example, that $B_n \in \mathfrak{R}_1$ for each $n \geq 1$. It is easy to see that for each $n \geq 1$, there exists a positive integer r_n such that $B_n \subset \varphi([6r_n - 2, 6r_n + 2])$. In the case that the sequence $\{r_n\}_{n=1}^{\infty}$

is unbounded, we have that $B \subset T_0$. Since we have analyzed this case, we may assume that there exists an integer r such that $B_n \subset \varphi([6r-2, 6r+2])$ for each $n \geq 1$. Then $B \subset \varphi([6r-2, 6r+2])$. Since $\omega(G(B)) \geq \frac{5}{2}$, there exist $s_2, s_3 \in (0,1]$ such that $G(B) = \theta v_1 \cup \theta(s_2 v_2) \cup \theta(s_3 v_3)$, where $s_3 v_3 = g(N(B))$ and $s_2 v_2 = g(M(B))$. Since $A \subset B \neq A$, $N(B) \notin \varphi^{-1}(A)$ or $M(B) \notin \varphi^{-1}(A)$. Thus, $s_2 v_2 \notin G(A)$ or $s_3 v_3 \notin G(A)$. Hence, $G(A) \subset G(B) \neq G(A)$. Therefore, $\mu(A) < \mu(B)$.

By Ward's Extension Theorem for Whitney Maps (see [4, Theorem 16.10]), there exists a Whitney map for $C(X_0)$, which we can also denote by μ, $\mu : C(X_0) \to [0, 4]$ such that $\mu(A) = \omega(G(A))$ for each $A \in \mathfrak{Q}$ and $\mu(X_0) = 4$.

Let $\mathfrak{G} = \mu^{-1}([\frac{5}{2}, 3])$. We are going to extend the map f to a map $F : \mathfrak{G} \to \mathfrak{D}$.

Given a real number x, let $[x]$ denote the maximum integer less than or equal to x. For each integer $m \geq 1$, define $n_m = [\frac{m+4}{6}]$ and $k_m = [\frac{m+6}{6}]$ and let $(m)_3 \in \{1, 2, 3\}$ be such that $(m)_3 \equiv m (\operatorname{mod} 3)$. Notice that, for each even integer $m \geq 2$, $g(\varphi(m)) = v_{(\frac{m+2}{2})_3}$.

In order to define F, we need to prove properties (B) and (C).

(B) If $A \in \mathfrak{G}$ and there exists an even integer $m \geq 2$ such that $\varphi^{-1}(A) \subset [m, m+4]$, then $A \in \operatorname{cl}_{C(X_0)}(\mathfrak{R}_{(\frac{m+4}{2})_3})$, $m \leq N(A) \leq m + \frac{1}{2} < m + \frac{7}{2} \leq M(A) \leq m+4$, and $\mu(A) = \mu(G(A))$.

In order to prove (B), we analyze the case that $m \equiv 4 (\operatorname{mod} 6)$. The cases $m \equiv 0$ or $2 (\operatorname{mod} 6)$ are similar. Let $n \geq 0$ be an integer such that $m = 6n+4$. Notice that $(\frac{m+4}{2})_3 = 1$. First, we show that $A \in \operatorname{cl}_{C(X_0)}(\mathfrak{R}_1)$. Given $\varepsilon \in (0, \frac{1}{2})$, let $A_\varepsilon = \varphi([6n+4+\varepsilon, 6n+\frac{15}{2}+\varepsilon])$, then $G(A_\varepsilon) \in C(T_0)$, $l(G(A_\varepsilon) \cap \theta v_3) = 1-\varepsilon$, $l(G(A_\varepsilon) \cap \theta v_1) = 1$, $l(G(A_\varepsilon) \cap \theta v_2) = \frac{1}{2}+\varepsilon$, $G(A_\varepsilon) \cap \{v_1, v_2, v_3\} = \{v_1\}$, so $\omega(G(A_\varepsilon)) = \frac{5}{2}$. Thus, $A_\varepsilon \in \mathfrak{R}_1$. Hence, for each $\varepsilon \in [0, \frac{1}{2}]$, the continuum $A_\varepsilon = \varphi([6n+4+\varepsilon, 6n+\frac{15}{2}+\varepsilon])$ belongs to $\operatorname{cl}_{C(X_0)}(\mathfrak{R}_1) \subset \mathfrak{Q}$ and $\mu(A_\varepsilon) = \frac{3}{2}$. Hence, $A \subsetneq A_0$ and $A \subsetneq A_{\frac{1}{2}}$. Since $A \subset \varphi([6n+4, 6n+8])$, $N(A) \leq m + \frac{1}{2}$ and $m + \frac{7}{2} \leq M(A)$. Since A can be approximated by sets of the form $\varphi([N(A) \pm \delta, M(A) \pm \eta])$, $\delta, \eta > 0$, it follows that $A \in \operatorname{cl}_{C(X_0)}(\mathfrak{R}_1)$ and $\mu(A) = \omega(G(A)) = \mu(G(A))$.

(C) Let $A \in \mathfrak{G} \cap C(\mathfrak{A})$ be such that $\varphi^{-1}(A)$ is not contained in any interval of the form $[n, n+4]$, for an even integer $n \geq 2$.

Let m be the maximum even integer such that $m \leq N(A)$. Then $m < N(A) < m+2 < m+4 < M(A) < m+6$.

In order to prove (C), let $n \geq 2$ be an even integer. Notice that the continuum $B_n = \varphi([n, n+4])$ has the following properties:

$$G(B_n) = T_0,\ B_n \in cl_{C(X_0)}(\mathfrak{R}),\ \text{and}\ \mu(B_n) = \omega(T_0) = 3.$$

In the case that there exists an even integer $n \geq 2$ such that $B_n \subset A$, since $3 \leq \mu(B_n) \leq \mu(A) \leq 3$, we have $B_n = A$. Thus, $\varphi^{-1}(A) = [n, n+4]$, contrary to our assumption. We have shown that $\varphi^{-1}(A)$ does not contain and is not contained in any set of the form B_n. If $m = N(A)$, then $B_m \subset \varphi^{-1}(A)$ or $\varphi^{-1}(A) \subset B_m$, a contradiction. Thus, $m < N(A)$. By the definition of m, $N(A) < m+2$. Since $\varphi^{-1}(A)$ is not contained in B_{m+2}, $m+4 < M(A)$, and since B_{m+2} is not contained in $\varphi^{-1}(A)$, $M(A) < m+6$. This completes the proof of (C).

For each positive even integer m, let $\mathfrak{K}_m = \{A \in \mathfrak{G} : \varphi^{-1}(A) \subset [m, m+4]\}$ and $\mathfrak{L}_m = \{A \in \mathfrak{G} : [m+2, m+4] \subset \varphi^{-1}(A)\}$. Clearly, $\mathfrak{K}_m$ and $\mathfrak{L}_m$ are closed subsets of $\mathfrak{G}$. By property (C), $\mathfrak{G} = (C(T_0) \cap \mathfrak{G}) \cup (\bigcup\{\mathfrak{K}_m \cup \mathfrak{L}_m : m \geq 2 \text{ is an even integer}\})$. By property (B), the sets $\mathfrak{K}_1, \mathfrak{K}_2, \ldots$ are pairwise disjoint.

For each $A \in \mathfrak{G}$, let $t(A) = 2\mu(A) - 5$. Then $t(A) \in [0,1]$. We define F by the formulae:

$F(A) = ((2(4 - M(A) + m))(g(\varphi(m+4)) - \frac{1}{n_m} g(\varphi(m+2))) + 2(N(A) - m)(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))), t(A))$, if $A \in \mathfrak{K}_m$.

$F(A) = (2\frac{1-t(A)}{t(A)+3}((2 - N(A) + m)(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))) + (N(A) - m - 3 + \mu(A))(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+4))), t(A))$, if $A \in \mathfrak{L}_m$.

$F(A) = f(A)$, if $A \in C(T_0) \cap \mathfrak{G}$.

We need to prove some properties of F.

(D) F is well defined.

Suppose that $A \in \mathfrak{L}_m \cap \mathfrak{L}_r$, where $2 \leq m < r$. Then $[m+2, m+4] \cup [r+2, r+4] \subset \varphi^{-1}(A)$. In this case, $[m+2, m+6] \subset \varphi^{-1}(A)$. Let $A_0 = \varphi([m+2, m+6])$. Then $A_0 \subset A$. Since $G(A_0) = T_0$, by (B), $3 = \mu(A_0) \leq \mu(A) \leq 3$. Thus, $A = A_0$ and $t(A) = 1$. Thus, $F(A) = (\theta, 1)$ independently if we consider A either as an element of $\mathfrak{L}_m$ or as an element of $\mathfrak{L}_r$.

Now suppose that $A \in \mathfrak{K}_m \cap \mathfrak{L}_r$ for some even integers m and r. In this case, $[r+2, r+4] \subset \varphi^{-1}(A) \subset [m, m+4]$. Thus, $r \leq m \leq r+2$. We need to consider two cases.

Case 1. $m = r$.

In this case, we have that $M(A) = m+4 = r+4$, and by (B), $m \leq N(A) \leq m + \frac{1}{2}$ and $\mu(A) = \mu(G(A)) = \mu(\varphi([N(A), m+4])) = m + 1 - N(A) + 2 = m + 3 - N(A)$. Thus, if we consider A as an element of $\mathfrak{K}_m$, noticing that $4 + m - M(A) = 0$, we obtain that $F(A) = (2(N(A) - m)(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))), t(A))$, and if we consider A as an element of $\mathfrak{L}_r = \mathfrak{L}_m$, noticing that $N(A) - m - 3 + \mu(A) = 0$, $t(A) + 3 = 2(m - N(A) + 2)$ and $1 - t(A) = 2(N(A) - m)$, we obtain that $F(A) = (2\frac{1-t(A)}{t(A)+3}((2 - N(A) + m)(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))), t(A)) = (2(N(A) - m)(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))), t(A))$. So both definitions of F coincide on A.

Case 2. $m = r + 2$.

In this case, we have that $N(A) = m = r + 2$, and by (B), $m + \frac{7}{2} \leq M(A) \leq m+4$ and $\mu(A) = \mu(G(A)) = \mu(\varphi([m, M(A)])) = 2 + M(A) - (m+3) = M(A) - m - 1$. Thus, if we consider A as an element of $\mathfrak{K}_m$, noticing that $N(A) - m = 0$, we obtain that $F(A) = (2(4 + m - M(A))(g(\varphi(m+4)) - \frac{1}{n_m} g(\varphi(m+2)), t(A))$, and if we consider A as an element of $\mathfrak{L}_r = \mathfrak{L}_{m-2}$, noticing that $N(A) - m = 0$, $g(\varphi(m-2)) = g(\varphi(m+4))$, $n_m = k_{m-2}$, $t(A) + 3 = 2(\mu(A) - 1)$ and $1 - t(A) = 6 - 2\mu(A) = 2(4 + m - M(A))$, we obtain that $F(A) = (2\frac{1-t(A)}{t(A)+3}(\mu(A) - 1)(g(\varphi(m-2)) - \frac{1}{k_{m-2}} g(\varphi(m+2))), t(A)) = (2(4+m-M(A))(g(\varphi(m+4)) - \frac{1}{n_m} g(\varphi(m+2)), t(A))$. So both definitions of F coincide on A.

This completes the proof of (D).

(E) $F(A) \in \mathfrak{D}$ for each $A \in \mathfrak{G}$.

There are three cases to be considered.

Case 1. $A \in C(T_0) \cap \mathfrak{G}$.

In this case, $F(A) = f(A) \in \mathfrak{T} \subset \mathfrak{D}$.

Case 2. $A \in \mathfrak{K}_m$ for some even integer $m \geq 2$.

In this case, by (B), $\varphi^{-1}(A) \subset [m, m+4]$, $A \in \mathrm{cl}_{C(X_0)}(\mathfrak{R}_{(\frac{m+4}{2})_3})$, $m \leq N(A) \leq m + \frac{1}{2} < m + \frac{7}{2} \leq M(A) \leq m+4$ and $\mu(A) = \mu(G(A))$. Notice that $\mu(G(A)) = (m + 1 - N(A)) + 1 + (M(A) - (m+3)) =$

$M(A) - N(A) - 1$ and $t(A) = 2(M(A) - N(A)) - 7$. In the case that $t(A) = 1$, $M(A) - N(A) = 4$. This implies that $M(A) = m+4$ and $N(A) = m$. Thus, $F(A) = (\theta, 1) \in \mathfrak{D}$. Thus, we may assume that $t(A) < 1$. Since $\frac{2(4+m-M(A))}{1-t(A)} + \frac{2(N(A)-m)}{1-t(A)} = 1$, $F(A)$ is a convex combination of the vectors $w_1 = ((1-t(A))(g(\varphi(m+4)) - \frac{1}{n_m} g(\varphi(m+2)), t(A))$ and $w_2 = ((1-t(A))(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))), t(A))$. If $m = 6n$, then for some $n \geq 1$, $w_1 = ((1-t(A))(v_3 - \frac{1}{n} v_2), t(A))$ and $w_2 = ((1-t(A))(v_1 - \frac{1}{n+1} v_2), t(A))$. If $m = 6n+2$, then for some $n \geq 0$, $w_1 = ((1-t(A))(v_1 - \frac{1}{n+1} v_3), t(A))$ and $w_2 = ((1-t(A))(v_2 - \frac{1}{n+1} v_3), t(A))$. If $m = 6n+4$, then for some $n \geq 0$, $w_1 = ((1-t(A))(v_2 - \frac{1}{n+1} v_1), t(A))$ and $w_2 = ((1-t(A))(v_3 - \frac{1}{n+1} v_1), t(A))$. Hence, $F(A) \in \psi([0,1] \times [1,\infty)) \subset \mathfrak{D}$.

Case 3. $A \notin C(T_0)$ and $A \notin \mathfrak{K}_r$ for any even integer $r \geq 2$.

By (C), there exists an even integer $m \geq 2$ such that $m < N(A) < m+2 < m+4 < M(A) < m+6$. Let $s = 2\frac{2-N(A)+m}{t(A)+3}$. Since $2\frac{2-N(A)+m}{t(A)+3} + 2\frac{N(A)-m-3+\mu(A)}{t(A)+3} = 1$, $F(A) = ((1-t(A))(s(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2))) + (1-s)(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+4))), t(A))$; if we prove that $0 \leq s \leq 1$, we will have that $F(A)$ is a convex combination of the vectors $w_1 = ((1-t(A))(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+2)), t(A))$ and $w_2 = ((1-t(A))(g(\varphi(m)) - \frac{1}{k_m} g(\varphi(m+4))), t(A))$. Clearly, $s \geq 0$. If $m + \frac{1}{2} \leq N(A)$, since $\frac{5}{2} \leq \mu(A)$, we obtain $0 \leq N(A) - m - 3 + \mu(A)$. Since $t(A) + 3 = 2(\mu(A) - 1) > 0$, we conclude that $0 \leq 1 - s$ and $s \leq 1$. Thus, we suppose that $N(A) < m + \frac{1}{2}$. Let $B = \varphi([N(A), m+4]) \subset A$. Then $\omega(G(B)) = m + 1 - N(A) + 2 = m + 3 - N(A) \geq \frac{5}{2}$. It is easy to see that $B \in \mathrm{cl}_{C(X_0)}(\mathfrak{R}_{(\frac{m+4}{2})3})$, so $\mu(B) = \omega(G(B)) = m + 3 - N(A)$. Hence, $m + 3 - N(A) \leq \mu(A)$. This implies that $0 \leq 1 - s$. This completes the proof that $F(A)$ is a convex combination of the vectors w_1 and w_2.

If $m = 6n$, then for some $n \geq 1$, $w_1 = ((1-t(A))(v_1 - \frac{1}{n+1} v_2), t(A))$ and $w_2 = ((1-t(A))(v_1 - \frac{1}{n+1} v_3), t(A))$. If $m = 6n+2$, then for some $n \geq 0$, $w_1 = ((1-t(A))(v_2 - \frac{1}{n+1} v_3), t(A))$ and $w_2 = ((1-t(A))(v_2 - \frac{1}{n+1} v_1), t(A))$. If $m = 6n+4$, then for some $n \geq 0$, $w_1 = ((1-t(A))(v_3 - \frac{1}{n+1} v_1), t(A))$ and $w_2 = ((1-t(A))(v_3 - \frac{1}{n+1} v_2), t(A))$. Hence, $F(A) \in \psi([0,1] \times [1,\infty)) \subset \mathfrak{D}$.

This completes the proof of (E).

(F) F is onto.

Let $z \in \mathfrak{D}$. If $z \in \mathfrak{T}$, since f is onto, there exists $A \in C(T_0) \cap \mathfrak{G}$ such that $F(A) = f(A) = z$. Thus, we may assume that $z = \psi(t,s)$, for some $t \in [0,1]$ and $s \in [1,\infty)$.

We analyze only the cases that, for some integer $n \geq 1$, $s \in [6n, 6n+1]$ or $s \in [6n+1, 6n+2]$; the cases in which s belongs to intervals of the form $[6n+2, 6n+3]$ or $[6n+3, 6n+4]$ are similar to the first one and the cases in which s belongs to intervals of the form $[6n+4, 6n+5]$ and $[6n+5, 6n+6]$ are similar to the second one.

In the case that $s \in [6n, 6n+1]$, $z = \psi(t,s) = (6n+1-s)((1-t)(v_1 - \frac{1}{n+1}v_2), t) + (s-6n)((1-t)(v_1 - \frac{1}{n+1}v_3), t)$. Let $m = 6n$ and $N_1 = 6n+2-(6n+1-s)(\frac{t+3}{2})$. Note that $m \leq N_1 \leq m+2$. Since $\mu(\varphi([N_1,\infty) \cup T_0)) > \mu(T_0) = 3 \geq \frac{t+5}{2}$, there exists $M_1 > N_1$ such that $\mu(\varphi([N_1, M_1])) = \frac{t+5}{2}$. Let $A = \varphi([N_1, M_1]) \in \mathfrak{G}$. We claim that $m+4 \leq M_1$. Suppose to the contrary that $M_1 < m+4$. Then $\varphi^{-1}(A) \subset [m, m+4]$. By (B), $m \leq N_1 \leq m + \frac{1}{2} < m + \frac{7}{2} \leq M_1 \leq m+4$ and $\mu(A) = \mu(G(A)) = m+1-N_1+1+M_1-(m+3) = M_1 - N_1 - 1$. Thus, $M_1 = \frac{t+5}{2} + 6n + 2 - (6n+1-s)(\frac{t+3}{2}) + 1 = (\frac{t+3}{2})(s-6n) + 6n + 4 \geq m+4$, a contradiction. Therefore, $m+4 \leq M_1$.

Hence, $m \leq N(A) \leq m+2 < m+4 \leq M(A)$ and $A \in \mathfrak{L}_m$. Since $t(A) = t$, $F(A) = (2\frac{1-t}{t+3}((2-N_1+m)(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))) + (N_1-m-3+\mu(A))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+4))), t)$. Since $2-N_1+m = (6n+1-s)(\frac{t+3}{2})$, $N_1-m-3+\mu(A) = -1-(6n+1-s)(\frac{t+3}{2}) + \frac{t+5}{2} = (s-6n)(\frac{t+3}{2})$, and $k_m = n+1$, we obtain that $F(A) = \psi(t,s) = z$.

In the case that $s \in [6n+1, 6n+2]$, $z = \psi(t,s) = (6n+2-s)((1-t)(v_1 - \frac{1}{n+1}v_3), t) + (s-6n-1)((1-t)(v_2 - \frac{1}{n+1}v_3), t)$. Let $m = 6n+2$, $N_1 = m + \frac{(s-6n-1)(1-t)}{2}$, and $M_1 = m+4-\frac{(6n+2-s)(1-t)}{2}$. Note that $m \leq N_1 \leq m + \frac{1}{2} < m + \frac{7}{2} \leq M_1 \leq m+4$. Let $A = \varphi([N_1, M_1])$. Then $\varphi^{-1}(A) \subset [m, m+4]$ and $\mu(G(A)) = m+1-N_1+1+M_1-m-3 = M_1-N_1-1 = 3 - \frac{1-t}{2}$. Thus, $\mu(G(A)) \in [\frac{5}{2}, 3]$. It easy to see that $A \in \mathrm{cl}_{C(X_0)}(\mathfrak{R})$. Hence,

$\mu(A) = \mu(G(A)) = \frac{t+5}{2}$. So $t = t(A)$. Since $n_m = n+1 = k_n$, $F(A) = \psi(t,s) = z$.

This completes the proof of (F).

(G) If $A, B \in \mathfrak{G}$ and $F(A) = F(B)$, then $A = B$ or $F(A) = (\theta, 1)$.

Let $A, B \in \mathfrak{G}$ be such that $F(A) = F(B)$. In the case that $A, B \in C(T_0)$, since f is one to one, we obtain that $A = B$. Now, if $A \in C(T_0)$ and $B \notin C(T_0)$, then $F(A) \in \mathfrak{T}$ and $F(B) \in \psi([0,1] \times [1,\infty))$. Thus, $F(A) = (\theta, 1)$. Hence, we may assume that $A, B \subset \mathfrak{A}$ and $F(A) \neq (\theta, 1)$. We consider three cases.

Case 1. $A \in \mathfrak{K}_m$ and $B \in \mathfrak{K}_r$ for some even integers $m, r \geq 2$.

As we saw in Case 2 in the proof of (E), $F(A)$ is in the convex segment w_1w_2, where $w_1 = ((1-t(A))(g(\varphi(m+4)) - \frac{1}{n_m}g(\varphi(m+2)), t(A))$ and $w_2 = ((1-t(A))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))), t(A))$ and $F(B)$ is in a similar convex segment. Since segments of this kind are disjoint for different values of m, we obtain that $m = r$. Thus, $((2(4-M(A)+m))(g(\varphi(m+4)) - \frac{1}{n_m}g(\varphi(m+2))) + 2(N(A)-m)(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))), t(A)) = F(A) = F(B) = ((2(4-M(B)+m))(g(\varphi(m+4)) - \frac{1}{n_m}g(\varphi(m+2))) + 2(N(B)-m)(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))), t(B))$. This implies that $2(4-M(A)+m) = 2(4-M(B)+m)$ and $2(N(A)-m) = 2(N(B)-m)$. This implies that $M(A) = M(B)$ and $N(A) = N(B)$. Thus, $A = B$.

Case 2. $A, B \notin C(T_0)$, $A \notin \mathfrak{K}_r$, and $B \notin \mathfrak{K}_r$ for any even integer $r \geq 2$.

Notice that $F(A) = F(B) \neq (\theta, 1)$ implies that $t(A) = t(B) < 1$. By (C), there exist even integers $m, r \geq 2$ such that $m < N(A) < m+2 < m+4 < M(A) < m+6$ and $r < N(B) < r+2 < r+4 < M(B) < r+6$. Moreover, as we saw in Case 3 in the proof of (E), $F(A)$ is in the segment joining the points $w_1 = ((1-t(A))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2)), t(A))$ and $w_2 = ((1-t(A))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+4))), t(A))$, and $F(B)$ is in a similar convex segment. Since segments of this kind are disjoint for different values of m, we obtain that $m = r$. Thus, $(2\frac{1-t(A)}{t(A)+3}((2-N(A)+m)(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))) + (N(A)-m-3+\mu(A))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+4))), t(A)) = F(A) = F(B) = (2\frac{1-t(B)}{t(B)+3}((2-N(B)+$

$m)(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))) + (N(B) - m - 3 + \mu(B))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+4))), t(B))$. This implies that $2 - N(A) + m = 2 - N(A) + m$. This implies that $N(A) = N(B)$. Thus, one of the sets $[N(A), M(A)]$ and $[N(B), M(B)]$ is contained in the other and the same happens with $A = \varphi([N(A), M(A)])$ and $B = \varphi([N(B), M(B)])$. Since $\mu(A) = \mu(B)$, we conclude that $A = B$.

Case 3. $A \in \mathfrak{K}_m$ for some even integer $m \geq 2$, $B \notin C(T_0)$ and $B \notin \mathfrak{K}_r$ for any even integer $r \geq 2$.

Notice that $F(A) = F(B) \neq (\theta, 1)$ implies that $t(A) = t(B) < 1$ and $\mu(A) = \mu(B)$. As we saw in Case 2 in the proof of (E), $F(A)$ is in the convex segment w_1w_2, where $w_1 = ((1 - t(A))(g(\varphi(m+4)) - \frac{1}{n_m}g(\varphi(m+2)), t(A))$ and $w_2 = ((1 - t(A))(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))), t(A))$. By (C), there exists an even integer $r \geq 2$ such that $r < N(B) < r + 2 < r + 4 < M(B) < r + 6$. Moreover, as we saw in Case 3 in the proof of (E), $F(B)$ is in the segment joining the points $z_1 = ((1 - t(B))(g(\varphi(r)) - \frac{1}{k_r}g(\varphi(r+2)), t(B))$ and $z_2 = ((1 - t(B))(g(\varphi(r)) - \frac{1}{k_r}g(\varphi(r+4))), t(B))$. Clearly, these segments can be intersected only in their end points. Thus, $\{w_1, w_2\} \cap \{z_1, z_2\} \neq \emptyset$. We consider the possible subcases.

3.1. $w_1 = z_1$.

Here, $g(\varphi(m+4)) - \frac{1}{n_m}g(\varphi(m+2)) = g(\varphi(r)) - \frac{1}{k_r}g(\varphi(r+2))$. That is, $v_{(\frac{m}{2})_3} - \frac{1}{n_m}v_{(\frac{m+4}{2})_3} = v_{(\frac{r+2}{2})_3} - \frac{1}{k_r}v_{(\frac{r+4}{2})_3}$, so $v_{(\frac{m}{2})_3} - v_{(\frac{r+2}{2})_3} = \frac{1}{n_m}v_{(\frac{m+4}{2})_3} - \frac{1}{k_r}v_{(\frac{r+4}{2})_3}$. This implies that $v_{(\frac{m}{2})_3} = v_{(\frac{r+2}{2})_3}$, $\frac{1}{n_m}v_{(\frac{m+4}{2})_3} = \frac{1}{k_r}v_{(\frac{r+4}{2})_3}$, $(\frac{m}{2})_3 = (\frac{r+2}{2})_3$, and $(\frac{m+4}{2})_3 = (\frac{r+4}{2})_3$. Thus, $\frac{m}{2} \equiv \frac{r+2}{2} (\operatorname{mod} 3)$ and $\frac{m+4}{2} \equiv \frac{r+4}{2} (\operatorname{mod} 3)$, a contradiction. Hence, this subcase is impossible.

3.2. $w_1 = z_2$.

Here, $v_{(\frac{m}{2})_3} - \frac{1}{n_m}v_{(\frac{m+4}{2})_3} = v_{(\frac{r+2}{2})_3} - \frac{1}{k_r}v_{(\frac{r}{2})_3}$. This implies that $v_{(\frac{m}{2})_3} = v_{(\frac{r+2}{2})_3}$ and $\frac{1}{n_m}v_{(\frac{m+4}{2})_3} = \frac{1}{k_r}v_{(\frac{r}{2})_3}$, so $(\frac{m}{2})_3 = (\frac{r+2}{2})_3$ and $n_m = k_r$. Thus, $\frac{m}{2} \equiv \frac{r+2}{2} (\operatorname{mod} 3)$, $m \equiv r + 2 (\operatorname{mod} 6)$, $m + 4 \equiv r + 6 (\operatorname{mod} 6)$, and $[\frac{m+4}{6}] = [\frac{r+6}{6}]$. This implies that $m = r + 2$. Moreover, $F(A) = F(B)$ implies that $F(A) = w_1$ and $F(B) = z_2$. Since $F(A) = ((2(4 - M(A) + m))(g(\varphi(m+4)) - \frac{1}{n_m}g(\varphi(m+2))) + 2(N(A) - m)(g(\varphi(m)) - \frac{1}{k_m}g(\varphi(m+2))), t(A))$ and $F(B) =$

$(2\frac{1-t(B)}{t(B)+3}((2-N(B)+r)(g(\varphi(r))-\frac{1}{k_r}g(\varphi(r+2)))+(N(B)-r-3+\mu(B))(g(\varphi(r))-\frac{1}{k_r}g(\varphi(r+4))), t(B))$, we obtain that $2(N(A)-m)=0$ and $2-N(B)+r=0$, so $N(A)=m=r+2=N(B)$. This implies that $A\subset B$ or $B\subset A$. Since $\mu(A)=\mu(B)$, we conclude that $A=B$.

3.3. $w_2=z_1$.

Proceeding as in subcase 3.2, we obtain that $m=r$, $2(N(A)-m)=1-t(A)=2\frac{1-t(B)}{t(B)+3}((2-N(B)+r)$, and $N(B)-r-3+\mu(B)=0$. Since $t(B)=2\mu(B)-5$, using that $N(B)-r-3+\mu(B)=0$, we have that $1-t(B)=2(3-\mu(B))=2(N(B)-m)$ and $t(B)+3=2(\mu(B)-1)=2(m+2-N(B))$. Hence, $N(A)-m=N(B)-m$ and $N(A)=N(B)$. This implies that $A=B$.

3.4. $w_2=z_2$.

Proceeding as in subcase 3.1, we obtain that the equality $w_2=z_2$ is impossible.

This ends the proof of (G).

(H) F is continuous.

In order to prove (H), observe that, for each even positive integer $m\geq 2$, the sets $\mathfrak{K}_m$ and $\mathfrak{L}_m$ are closed in $\mathfrak{G}$ and $F|\mathfrak{K}_m$, $F|\mathfrak{L}_m$ are continuous. Since F is well defined and the family $\{\mathfrak{K}_m : m$ is an even positive integer$\}\cup\{\mathfrak{L}_m : m$ is an even positive integer$\}$ is locally finite in $\mathfrak{G}-C(T_0)$ (see (B) and (C)), we obtain that F is continuous in $\mathfrak{G}-C(T_0)$. Since $F|C(T_0)\cap\mathfrak{G}$ is clearly continuous, in order to check that F is continuous, we need only to take a sequence of elements $\{A_r\}_{r=1}^{\infty}$ in $\mathfrak{G}-C(T_0)$ such that $\lim A_r=A$, for some $A\in C(T_0)$, and prove that there exists a subsequence $\{B_s\}_{s=1}^{\infty}$ of $\{A_r\}_{r=1}^{\infty}$ such that $\lim F(B_s)=F(A)$. Thus, we need only to consider two cases.

Case 1. For each $r\geq 1$, $A_r\in\mathfrak{K}_{m_r}$ for some even integer $m_r\geq 2$.

By (B), $m_r\leq N(A_r)\leq m_r+\frac{1}{2}<m_r+\frac{7}{2}\leq M(A_r)\leq m_r+4$, $A_r\in \mathrm{cl}_{C(X_0)}(\mathfrak{R}_{(\frac{m_r+4}{2})_3})$ (so $v_{(\frac{m_r+4}{2})_3}\in G(A_r)$), and $\mu(A_r)=\mu(G(A_r))$, for each $r\geq 1$. Notice that $\lim m_r=\infty$, $\lim n_{m_r}=\infty$, and $\lim k_{m_r}=\infty$. Since $(\frac{m_r}{2})_3\in\{1,2,3\}$, we may assume that there exists $s\in\{1,2,3\}$ such that, for each $r\geq 1$, $(\frac{m_r}{2})_3=s$. Then $F(A_r)=((2(4-M(A_r)+m_r))(v_s-\frac{1}{n_{m_r}}v_{(s+2)_3})+2(N(A_r)-$

$m_r)(v_{(s+1)_3}-\frac{1}{k_{m_r}}v_{(s+2)_3},t(A_r))$. Notice that $G(A_r)=G(\varphi([N(A_r), M(A_r)]))=G(\varphi([N(A_r),m_r+1]))\cup G(\varphi(m_r+1,m_r+3]))\cup G(\varphi([m_r+3,M(A_r)]))=\theta((m_r+1-N(A_r))v_{(s+1)_3})\cup\theta v_{(s+2)_3}\cup\theta((M(A_r)-m_r-3)v_s)$. Thus, $f_{(s+1)_3}(G(A_r))=m_r+1-N(A_r)$, $f_{(s+2)_3}(G(A_r))=1$, $f_s(G(A_r))=M(A_r)-m_r-3$, and $\mu(G(A_r))=m_r+1-N(A_r)+1+M(A_r)-m_r-3=M(A_r)-N(A_r)-1$. Since G and each f_i are continuous, $\lim m_r+1-N(A_r)=f_{(s+1)_3}(A)$, $f_{(s+2)_3}(A)=1$, $\lim M(A_r)-m_r-3=f_s(A)$, and $\lim M(A_r)-N(A_r)-1=\lim\mu(A_r)=\mu(A)$. Thus, $\lim F(A_r)=(2(1-f_s(A))v_s+2(1-f_{(s+1)_3}(A))v_{(s+1)_3}+2(1-f_{(s+2)_3}(A))v_{(s+2)_3},t(A))=F(A)$.

Case 2. For each $r\geq 1$, $A_r\notin\mathfrak{K}_k$ for any even integer $k\geq 2$.

By (C), for each $r\geq 1$, there exists an even integer m_r such that $m_r<N(A_r)<m_r+2<m_r+4<M(A_r)<m_r+6$. Notice that $\lim m_r=\infty$, $\lim n_{m_r}=\infty$, and $\lim k_{m_r}=\infty$. Since $(\frac{m_r}{2})_3\in\{1,2,3\}$, we may assume that there exists $s\in\{1,2,3\}$ such that, for each $r\geq 1$, $(\frac{m_r}{2})_3=s$. Then $F(A_r)=(2\frac{1-t(A_r)}{t(A_r)+3}((2-N(A_r)+m_r)(v_{(s+1)_3}-\frac{1}{k_{m_r}}v_{(s+2)_3})+(N(A_r)-m_r-3+\mu(A_r))(v_{(s+1)_3}-\frac{1}{k_{m_r}}v_s),t(A_r))$. Since, for each $r\geq 1$, $0<2-N(A_r)+m_r<2$ and $\frac{1}{2}\leq\frac{2}{t(A_r)+3}\leq\frac{2}{3}$, we may assume that $\{2(\frac{2-N(A_r)+m_r}{t(A_r)+3})\}_{n=1}^{\infty}$ converges to $t_1\in[0,\frac{4}{3}]$. Since $2(\frac{2-N(A_r)+m_r}{t(A_r)+3})+2(\frac{N(A_r)-m_r-3+\mu(A_r)}{t(A_r)+3})=1$, $\lim 2(\frac{N(A_r)-m_r-3+\mu(A_r)}{t(A_r)+3})=1-t_1$. Thus, $\lim F(A_r)=((1-t(A))v_{(s+1)_3},t(A))$. Since $\theta v_{(s+2)_3}\cup\theta v_s\subset G(\varphi([m_r+2,m_r+4]))\subset G(A_r)$, $\theta v_{(s+2)_3}\cup\theta v_s\subset A$. Hence, $f(A)=((2-2f_{(s+1)_3}(A))v_{(s+1)_3}, t(A))$. Since $\mu(A)=f_1(A)+f_2(A)+f_3(A)=2+f_{(s+1)_3}(A)$, $1-t(A)=6-2\mu(A)=2-2f_{(s+1)_3}(A)$. Therefore, $\lim F(A_r)=F(A)$.

This completes the proof of (H).

(I) $\mu^{-1}(3)=F^{-1}((\theta,1))$ is an arc, $\mu^{-1}(\frac{5}{2})=F^{-1}(\Delta(0)\cup\psi(\{0\}\times[1,\infty)))$, and $F^{-1}((\theta,1))\cap\mathrm{cl}_{C(X_0)}(F^{-1}(\mathfrak{T}-\{(\theta,1)\}))=\{T_0\}$.

We prove (I). The equality $\mu^{-1}(3)=F^{-1}((\theta,1))$ follows from the fact that $\mu(A)=3$ if and only if $t(A)=1$. The equality $\mu^{-1}(\frac{5}{2})=F^{-1}(\Delta(0)\cup\psi(\{0\}\times[1,\infty)))$ follows from the fact that $\mu(A)=\frac{5}{2}$ if and only if $t(A)=0$. In order to show that $\mu^{-1}(3)$ is an arc, let $\sigma:[1,\infty)\to[0,1)$ be a homeomorphism and let $\gamma:\mu^{-1}(3)\to[0,1]$ be given by $\gamma(A)=\sigma(N(A))$, if $A\in C(\mathfrak{A})$ and $\gamma(T_0)=1$. It is easy to check that γ is a homeomorphism. Given $A \in F^{-1}((\theta,1)) \cap \mathrm{cl}_{C(X_0)}(F^{-1}(\mathfrak{T} - \{(\theta,1)\}))$, we have that $\mu(A) = 3$ and $A = \lim A_n$ where, for each $n \geq 1$, $F(A_n) \in \mathfrak{T} - \{(\theta,1)\}$. Since $f(\omega^{-1}([\frac{5}{2},3])) = \mathfrak{T}$, by (G), $A_n \subset T_0$. Hence, $A \subset T_0$ and $\mu(A) = 3$, so $A = T_0$. Clearly, $T_0 \in F^{-1}((\theta,1)) \cap \mathrm{cl}_{C(X_0)}(F^{-1}(\mathfrak{T} - \{(\theta,1)\}))$. This finishes the proof of (I).

3. $C(X_0)$ does not have the fixed point property

Let $D_2 = \{z \in \mathbb{R}^2 : |z| \leq 1\}$. Let $\mathfrak{C}_1$ be the cone over D_1 (D_1 is defined on page 2), $\mathfrak{C}_2$ be the cone over D_2, $\mathfrak{B}$ be the base of $\mathfrak{C}_1$, and v be the vertex of $\mathfrak{C}_1$.

Theorem 3.1. *Suppose that Z is a continuum; A is a subcontinuum of Z; $\sigma : A \to \mathfrak{C}_1$ is an onto continuous function; W_1 and W_2 are disjoint open subsets of Z; p_0 is a point of Z; and L is an arc of Z such that $Z - A = W_1 \cup W_2$, $bd_Z(W_1) = L = \sigma^{-1}(v)$, $bd_Z(W_2) = \sigma^{-1}(\mathfrak{B})$, $\sigma|A - L$ is one to one, and $cl_Z(\sigma^{-1}(\mathfrak{C}_2 - \{v\})) \cap L = \{p_0\}$. Then Z does not have the fpp.*

Proof: Let $\lambda : [1,\infty) \to D_1$ be given by $\lambda(t) = (\frac{t+1}{t})(\cos(t), \sin(t))$.

Claim 1. There exists a continuous function $\alpha : [0,1) \to [1,\infty)$ such that $\alpha(0) = 1$ and $\alpha(t) > 1$ for each $t > 0$; moreover, if $\beta : [0,1] \to \mathfrak{C}_1$ is given by $\beta(t) = (\lambda(\alpha(t)), t)$, if $t < 1$ and $\beta(1) = v$, then β is a one to one continuous function, and if $B_1 = (\mathfrak{C}_2 - \{v\}) \cup \{(\lambda(s), t) \in \mathfrak{C}_1 : t \in [0,1) \text{ and } \alpha(t) \leq s\}$, then the set $A_1 = \sigma^{-1}(B_1) \cup \{p_0\}$ is closed in Z and $\beta([0,1]) \subset B_1 \cup \{v\}$.

In order to prove Claim 1, notice that the sets $\mathrm{cl}_Z(\sigma^{-1}(\mathfrak{C}_2 - \{v\})) - \{p_0\}$ and $L - \{p_0\}$ are separated, so there exist two disjoint open sets U and V of Z such that $\mathrm{cl}_Z(\sigma^{-1}(\mathfrak{C}_2 - \{v\})) - \{p_0\} \subset U$ and $L - \{p_0\} \subset V$. Given a point $w \in \mathfrak{C}_2 - \{v\}$, there exists a unique point $p \in A$ such that $\sigma(p) = w$. Since $p \neq p_0$, $p \in U$, so $w \notin \sigma(A - U)$. We have shown that $\mathfrak{C}_2 - \{v\} \cap \sigma(A - U) = \emptyset$. For each integer $n \geq 1$, let $E_n = \{(z,t) \in \mathfrak{C}_2 : \frac{n-1}{n} \leq t \leq \frac{n}{n+1}\}$. Since $E_n \subset \mathfrak{C}_2 - \{v\}$, the sets E_n and $\sigma(A - U)$ are compact disjoint subsets of $\mathfrak{C}_1$. Thus, we can choose a number $s_n \in (0,\infty)$ such that the compact set $F_n = E_n \cup \{(\lambda(s), t) \in \mathfrak{C}_1 : \frac{n-1}{n} \leq t \leq \frac{n}{n+1}$ and $s_n \leq s\}$ misses $\sigma(A - U)$. We can assume that $s_1 < s_2 < \ldots$.

Define $\alpha : [0,1) \to [1,\infty)$ by $\alpha(t) = n(n+1)((t - \frac{n-1}{n})s_{n+1} + (\frac{n}{n+1} - t)s_n)$, if $t \in [\frac{n-1}{n}, \frac{n}{n+1}]$ and $n \geq 2$, and $\alpha(t) = 1 - 2t + 2ts_2$, if $t \in [0, \frac{1}{2}]$. Then $\alpha(0) = 0$ and $\alpha(t) > 0$ for each $t > 0$. Let $\beta : [0,1] \to \mathfrak{C}_1$ be given by $\beta(t) = (\lambda(\alpha(t)), t)$, if $t < 1$, and $\beta(1) = v$. Clearly, β is a one to one continuous function.

Let $B_1 = (\mathfrak{C}_2 - \{v\}) \cup \{(\lambda(s), t) \in \mathfrak{C}_1 : t \in [0,1) \text{ and } \alpha(t) \leq s\}$ and $A_1 = \sigma^{-1}(B_1) \cup \{p_0\}$. Clearly, $\beta([0,1]) \subset B_1 \cup \{v\}$. In order to check that A_1 is closed in Z, let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\sigma^{-1}(B_1)$ converging to an element $z \in Z$. We need to prove that $z \in A_1$. Consider two cases.

Case 1. $\sigma(a_n) = (w_n, t_n) \in \mathfrak{C}_2 - \{v\}$, for each $n \geq 1$.

We may assume that $\lim w_n = w$ for some $w \in D_2$ and $\lim t_n = t$ for some $t \in [0,1]$. Since σ is continuous, $(w,t) = \sigma(z)$. If $t < 1$, then $(w,t) \in \mathfrak{C}_2 - \{v\}$ and $z \in A_1$. Suppose then that $t = 1$, then $z \in L$. Thus, $z \in \mathrm{cl}_Z(\sigma^{-1}(\mathfrak{C}_2 - \{v\})) \cap L = \{p_0\}$. Hence, $z \in A_1$. This finishes Case 1.

Case 2. For each $n \geq 1$, $\sigma(a_n) = (\lambda(r_n), t_n)$ for some $t_n \in [0,1)$ and $r_n \in [\alpha(t_n), \infty)$.

In this case, $\sigma(z) = \lim(\lambda(r_n), t_n)$. If $\sigma(z) \in \mathfrak{C}_2 - \{v\}$, we are done. Suppose then that $\sigma(z) \notin \mathfrak{C}_2 - \{v\}$. We consider two subcases.

2.1. $\sigma(z) \notin \mathfrak{C}_2$.

In this subcase, $\sigma(z) = (\lambda(r), t)$, for some $r \in [1, \infty)$ and $t \in [0,1)$. Then $\lim t_n = t$ and $\lim r_n = r$. Since $\alpha(t_n) \leq r_n$, we obtain that $\alpha(t) \leq r$. Thus, $z \in A_1$.

2.2. $\sigma(z) = v$.

In this subcase, $z \in L$ and $\lim t_n = 1$. Given $n \geq 1$, we may assume that $t_n > \frac{1}{2}$, so there exists an integer $m_n \geq 2$ such that $t_n \in [\frac{m_n - 1}{m_n}, \frac{m_n}{m_n + 1}]$. Since $\alpha(t_n) \leq r_n$ and $\alpha(t_n)$ is a convex combination of s_{m_n} and s_{m_n+1}, we have that $s_{m_n} \leq r_n$. Thus, $(\lambda(r_n), t_n) \in F_{m_n}$. Hence, $\sigma(a_n) = (\lambda(r_n), t_n) \notin \sigma(A - U)$. Therefore, $a_n \in A \cap U \subset A - V$. This proves that $z \in L \cap (A - V)$, so $z = p_0 \in A_1$.

This completes the proof that A_1 is closed. Hence, Claim 1 holds.

Claim 2. Let $\sigma_1 = \sigma | A_1$. Then σ_1 is a homeomorphism in its image.

To prove Claim 2, since A_1 is compact, we need only to check that σ_1 is one to one. Let $p, q \in A_1$ be such that $\sigma(p) = \sigma(q)$. If $\sigma(p) = v$, since $v \notin B_1$, we obtain that $p = p_0 = q$. If $\sigma(p) \neq v$, then $p, q \in A - L$. By hypothesis $\sigma|A - L$ is one to one, so $p = q$. This ends the proof of Claim 2.

CLAIM 3. Let $L_1 = \sigma_1^{-1}(\beta([0,1]))$. Then there exists a homeomorphism $h : A_1 \to \mathfrak{C}_1$ such that $h(L_1) = \{(\lambda(1), t) \in \mathfrak{C}_1 : t \in [0,1]\}$, $\sigma^{-1}(\mathfrak{B}) = \sigma_1^{-1}(\mathfrak{B})$ and $h|\sigma^{-1}(\mathfrak{B}) = \sigma_1|\sigma^{-1}(\mathfrak{B})$.

Let $\mathfrak{C}_0 = \{(w,t) \in \mathfrak{C}_1 : \frac{1}{2} \leq t \leq 1\}$. Let $h_0 : B_1 \cup \{v\} \to \mathfrak{C}_1$ be defined by $h_0(v) = v$, $h_0(\lambda(s), t) = (\lambda(s - \alpha(t) + 1), t)$, if $(s,t) \in [\alpha(t), \infty) \times [0,1)$, and $h_0((\cos(s), \sin(s)), t) = ((\cos(s - \alpha(t) + 1), \sin(s - \alpha(t) + 1)), t)$, if $(s,t) \in [1, \infty) \times [0,1)$. It is easy to prove that h_0 is a homeomorphism and $h_0(\mathfrak{C}_0) = \mathfrak{C}_0$.

Define $h : A_1 \to \mathfrak{C}_1$ by $h = h_0 \circ \sigma_1$. Clearly, h is a homeomorphism, $h(p_0) = v$, and $h(L_1) = h(\sigma_1^{-1}(\beta([0,1]))) = h_0(\beta([0,1]))) = \{(\lambda(1), t) \in \mathfrak{C}_1 : t \in [0,1]\}$.

Notice that $\alpha(0) = 1$ implies that $\mathfrak{B} \subset B_1$, $\sigma^{-1}(\mathfrak{B}) = \sigma_1^{-1}(\mathfrak{B})$, and $h_0|\mathfrak{B}$ = identity of $\mathfrak{B}$. In order to show that $h|\sigma^{-1}(\mathfrak{B}) = \sigma_1|\sigma^{-1}(\mathfrak{B})$, let $w = (z, 0) \in \mathfrak{B}$ and let $p \in A$ be such that $\sigma(p) = w$. Then $h(p) = h_0(\sigma_1(p)) = h_0(\sigma(p)) = h_0(w) = w = \sigma_1(p)$. This ends the proof of Claim 3.

The following claim is easy to prove.

CLAIM 4. Let $C_1 = \{(\lambda(s), t) \in \mathfrak{C}_1 : t \in [0,1)$ and $s \leq \alpha(t)\} \cup \{v\}$. Then C_1 is closed in $\mathfrak{C}_1$ and $C_1 \cap B_1 = \beta([0,1]) - \{v\}$.

CLAIM 5. Let $A_2 = A_1 \cup W_2$ and $A_3 = W_1 \cup \sigma^{-1}(C_1)$. Then A_2 and A_3 are closed in Z, $Z = A_2 \cup A_3$, and $A_2 \cap A_3 = \sigma_1^{-1}(\beta([0,1]))$.

We prove Claim 5. Since $\mathrm{cl}_Z(A_1 \cup W_2) = A_1 \cup W_2 \cup \mathrm{bd}_Z(W_2) = A_1 \cup W_2 \cup \sigma^{-1}(\mathfrak{B}) = A_1 \cup W_2 \cup \sigma_1^{-1}(\mathfrak{B}) \subset A_1 \cup W_2$, we have that A_2 is closed in Z. Since C_1 is compact and $L = \sigma^{-1}(v) \subset \sigma^{-1}(C_1)$, $\mathrm{cl}_Z(W_1 \cup \sigma^{-1}(C_1)) = W_1 \cup \mathrm{bd}_Z(W_1) \cup \sigma^{-1}(C_1) = W_1 \cup L \cup \sigma^{-1}(C_1) \subset W_1 \cup \sigma^{-1}(C_1)$; we obtain that A_3 is closed in Z.

Notice that $\mathfrak{C}_1 = B_1 \cup C_1$. Thus, $Z = A \cup W_1 \cup W_2 = \sigma^{-1}(\mathfrak{C}_1) \cup W_1 \cup W_2 = \sigma^{-1}(B_1 \cup C_1) \cup W_1 \cup W_2 \subset A_1 \cup W_2 \cup W_1 \cup \sigma^{-1}(C_1) = A_2 \cup A_3$. Hence, $Z = A_2 \cup A_3$.

Since $\beta([0,1]) - \{v\} \subset B_1$, $\sigma^{-1}(\beta([0,1]) - \{v\}) \subset A_1$; this implies that $\sigma^{-1}(\beta([0,1]) - \{v\}) = \sigma_1^{-1}(\beta([0,1]) - \{v\})$. Since $A_1 \cup \sigma^{-1}(C_1) \subset A$, $A_2 \cap A_3 = (A_1 \cup W_2) \cap (W_1 \cup \sigma^{-1}(C_1)) = A_1 \cap \sigma^{-1}(C_1) = (\sigma^{-1}(B_1) \cup \{p_0\}) \cap \sigma^{-1}(C_1) = \sigma^{-1}(B_1 \cap C_1) \cup \{p_0\} = \sigma^{-1}(\beta([0,1]) - \{v\}) \cup \{p_0\} = \sigma_1^{-1}(\beta([0,1]) - \{v\}) \cup \sigma_1^{-1}(v) = \sigma_1^{-1}(\beta([0,1]))$.

Hence, Claim 5 is proved.

Claim 6. Let $\mathfrak{C}_0 = \{(w,t) \in \mathfrak{C}_1 : \frac{1}{2} \leq t \leq 1\}$ and $A_0 = h^{-1}(\mathfrak{C}_0)$. Then A_0 is a retract of Z and A_0 is homeomorphic to $\mathfrak{C}_1$.

By Claim 2, A_0 is homeomorphic to $\mathfrak{C}_0$. Since $\mathfrak{C}_0$ is homeomorphic to $\mathfrak{C}_1$, we conclude that A_0 is homeomorphic to $\mathfrak{C}_1$. Since an arc is an absolute retract, there exists a retraction $R_0 : A_3 \to \sigma_1^{-1}(\beta([0,1]))$. We extend R_0 to a map $R : Z \to A_2$ by defining $R(z) = z$ if $z \in A_2$, and $R(z) = R_0(z)$ if $z \in A_3$. By Claim 5, R is a well defined retraction from Z onto A_2.

Let $A_4 = A_1/\sigma_1^{-1}(\mathfrak{B})$; that is, A_4 is the continuum obtained by shrinking the subcontinuum $\sigma_1^{-1}(\mathfrak{B})$ of A_1 to a point. Notice $A_1 \cap \mathrm{cl}_Z(W_2) = A_1 \cap \mathrm{bd}_Z(W_2) = A_1 \cap \sigma^{-1}(\mathfrak{B}) = A_1 \cap \sigma_1^{-1}(\mathfrak{B}) = \sigma_1^{-1}(\mathfrak{B})$. Let $A_5 = A_2/\mathrm{cl}_Z(W_2)$. Clearly, A_4 and A_5 are homeomorphic and there exists a homeomorphism $h_1 : A_5 \to A_4$ such that $h_1(\mathrm{cl}_Z(W_2)) = \sigma_1^{-1}(\mathfrak{B})$, and if we consider $A_1 - \sigma_1^{-1}(\mathfrak{B})$ as a subspace of A_4 and $A_1 - \mathrm{cl}_Z(W_2)$ as a subspace of A_5, then h_1 has the property that $h_1|A_1 - \mathrm{cl}_Z(W_2) : A_1 - \mathrm{cl}_Z(W_2) \to A_1 - \sigma_1^{-1}(\mathfrak{B})$ is the identity map. Let $\rho : A_2 \to A_5$ be the natural projection.

By Claim 3, $h(\sigma_1^{-1}(\mathfrak{B})) = \sigma_1(\sigma^{-1}(\mathfrak{B})) = \sigma_1(\sigma_1^{-1}(\mathfrak{B})) = \mathfrak{B}$. Thus, h induces a homeomorphism $h_2 : A_4 \to \mathfrak{C}_1/\mathfrak{B}$ such that $h_2(\sigma_1^{-1}(\mathfrak{B})) = \mathfrak{B}$, and we may assume that $h_2(z) = h(z)$ for each $z \in A_1 - \sigma_1^{-1}(\mathfrak{B})$. Since $\mathfrak{C}_1/\mathfrak{B}$ is homeomorphic to the topological suspension of D_1, there is a natural retraction $R_1 : \mathfrak{C}_1/\mathfrak{B} \to \mathfrak{C}_0$.

Let $R_2 : Z \to A_0$ be given by $R_2(z) = h^{-1}(R_1(h_2(h_1(\rho(R_0(z))))))$. Then R_2 is a continuous function. Given $z \in A_0$, $z \in A_1 - \sigma_1^{-1}(\mathfrak{B}) = A_2 - \mathrm{cl}_Z(W_2)$ and $R_0(z) = z$. Thus, $h_1(\rho(z)) = z$. Hence, $h_2(z) = h(z) = h_0(\sigma_1(z))$. Since $\sigma_1(z) \in \mathfrak{C}_0$, $h_0(\sigma_1(z)) \in \mathfrak{C}_0$. Hence, $R_1(h_2(z)) = h(z)$. Thus, $R_1(z) = z$. Therefore, R_1 is a retraction.

Since $\mathfrak{C}_1$ does not have the fpp, the following claim follows immediately from Claim 6.

Claim 7. Z does not have the fpp.

And the proof of Theorem 3.1 is complete. □

Theorem 3.2. *$C(X_0)$ does not have the fpp.*

Proof: Apply Theorem 3.1 to $Z = C(X_0)$, $A = \mu^{-1}([\frac{5}{2}, 3])$, $F : \mu^{-1}([\frac{5}{2}, 3]) \to \mathfrak{D}$, $W_1 = \mu^{-1}((3,4])$, $W_2 = \mu^{-1}([0, \frac{5}{2}))$, $p_0 = T_0$, and $L = \mu^{-1}(3)$. □

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Universidad Nacional Autonoma de Mexico; Instituto de Matematicas; Area de la Investigación Científica; Circuito Exterior, Ciudad Universitaria; 04510 México, D.F

E-mail address: illanes@matem.unam.mx