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A TOPOLOGICAL ANALOGUE OF THE LIPSCHITZ CONDITION

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ABSTRACT. The paper shows that a notion, similar to the notion of a Lipschitz function, can be defined in a purely topological way. A concept of a topologically Lipschitz function is defined and some properties of topologically Lipschitz functions are investigated. Possible applications to fixed-point theory are discussed.

1. INTRODUCTION

The classical notion of a Lipschitz function is defined with the aid of a metric. There are some other notions, such as contractivity, uniform convergence, and measure of noncompactness, that seem to work only in metric spaces or in uniform spaces. Nevertheless, a topological analogue of contractivity was defined in [7] and generalized in [4]. The concept of strong convergence from [5] serves as a topological analogue of uniform convergence and in [6], a kind of measure of noncompactness was defined in purely topological terms. In the papers mentioned above, open covers are often used to describe and define new notions.

In this paper, we define the notion of a topologically Lipschitz function using only the language of topological spaces. Although this theory could be applied very easily in uniform spaces, we omit

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this part because it is too obvious and we prefer to concentrate on investigating some important properties of topologically Lipschitz functions.

In what follows, we will use these notions concerning topological spaces and functions: a net of points, a limit of a net, a net of functions, pointwise convergence (see e. g., [1] or [2]).

We use the following notation: if p is an open cover of a topological space X and x is an element of X we denote

$$st_p(x) = \bigcup_{V \in p, x \in V} V.$$

The set $st_p(x)$ is called the *star of x in the cover p.*

By induction, we can define the *n-th star of x* by

$$st_p^n(x) = \bigcup_{V \in p, V \cap st_p^{n-1}(x) \neq \emptyset} V.$$

If x, y are elements of a topological space X and p is an open cover of X, then $x \in st_p^n(y)$ is equivalent to $y \in st_p^n(x)$.

It is quite easy to see that a topological space X is connected if and only if for each x, y from X and each open cover p of X, there exists an integer n such that $x \in st_p^n(y)$. (This was shown, for example, in [8].)

2. Main results

We will use the following notation: If X is a topological space, we denote the set of all open covers of X by P_X. We denote the set of all real numbers with its usual topology by $\mathbb{R}$.

For each $\varepsilon > 0$, we define $p_\varepsilon = \{(a + \frac{\varepsilon}{2}, a - \frac{\varepsilon}{2}); a \in \mathbb{R}\}$, and we denote $E = \{p_\varepsilon; \varepsilon > 0\}$.

Now we are going to define a notion of scale. A scale will be a set of open covers that will measure or, more exactly, determine whether a function is topologically Lipschitz or not.

Definition 2.1. Let (X, T) be a topological space. Let P_X be the set of all open covers of X. If C is a nonempty subset of P_X, we call it a *scale.*

If

(i) for all $x \in X$ and for all $V \in T$ such that $x \in V$, there exists $p \in C$ such that $st_p(x) \subset V$, we call C a *simple scale.*

If

(ii) for all $x \in X$ and for all $V \in T$ such that $x \in V$, and for all $m \in \mathbb{N}$ there exists $p \in C$ such that $st_p^m(x) \subset V$,

we call C a *fine scale.*

Remark 2.2. Using the notation defined above, we can see that E is a fine scale in $\mathbb{R}$, so it is a simple scale in $\mathbb{R}$, too. Of course, similar scales, created by open balls of the same radius, can be defined in metric spaces. Analogously, we see that fine scales exist in uniform spaces, too. It would be interesting to find a characterization of spaces having fine scales, but this is not the goal of the present article.

Definition 2.3. Let X be a topological space and let C be a scale in X. Let $f : X \to X$ be a function and $x \in X$ a point.

We say that f is $1 - C$ Lipschitz at x if, for all $p \in C$,

$$f(st_p(x)) \subset st_p(f(x)).$$

If m, n are positive integers, we say that f is $(m, n) - C$ Lipschitz at x if, for all $p \in C$,

$$f(st_p^n(x)) \subset st_p^m(f(x)).$$

We say that f is $1 - C$ Lipschitz if it is $1 - C$ Lipschitz at all x from X. Analogously, we say that f is $(m, n) - C$ Lipschitz if it is $(m, n) - C$ Lipschitz at all x from X.

We have defined a topological analogue of the Lipschitz condition. Now we show that topologically Lipschitz functions and classical Lipschitz functions have some similar properties.

The proofs of the following two theorems are easy and very similar. Therefore, we provide only the proof of the second theorem.

Theorem 2.4. *Let X be a topological space, x a point from X, and $f : X \to X$ a function. Let C be a simple scale in X. If f is $1 - C$ Lipschitz at x, then f is continuous at x.*

Theorem 2.5. *Let X be a topological space, x a point from X, and $f : X \to X$ a function. Let C be a fine scale in X. Let m, n be positive integers. If f is $(m, n) - C$ Lipschitz at x, then f is continuous at x.*

Proof: Let V be an open subset of X such that $f(x) \in V$. Since C is a fine scale, there exists an open cover p from C such that $st_p^m(f(x)) \subset V$. The function f is $(m,n) - C$ Lipschitz at x, so

$$f(st_p^n(x)) \subset st_p^m(f(x)).$$

Denote $U = st_p(x)$. We see that U is an open neighborhood of x. Since $U \subset st_p^n(x)$, we obtain $f(U) \subset V$. The function f is proved to be continuous at x. $\square$

It is known that if a differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a derivative 0, then f is constant. The following theorem shows that a topological analogue of this is true, too.

Theorem 2.6. *Let X be a connected T_1 space and C a simple scale in X. Let $f : X \to X$ be a function and z an element of X. If f is $(1,n) - C$ Lipschitz at z for every positive integer n, then f is a constant function.*

Proof: We will prove that for all x from X, $f(x) = f(z)$. The space X is connected; so, for each $p \in C$, we have $X = \bigcup_n st_p^n(z)$ for some positive integer n. Thus, we have $f(X) \subset st_p(f(z))$ for each open cover p from C. As C is a simple scale and X is T_1, we have $f(X) \subset \bigcap_{p \in C} st_p(f(z)) = \{f(z)\}$. Hence, $f(X) = \{f(z)\}$, and so f is constant. $\square$

3. Pointwise limits of topologically Lipschitz functions

In this section, we show that a pointwise limit of a net of topologically Lipschitz functions is a topologically Lipschitz function.

Theorem 3.1. *Let X be a topological space and C be a scale on X. Let $\{f_\gamma\}_{\gamma \in \Gamma}$ be a net of $(m,n) - C$ Lipschitz functions from X to X converging pointwise to a function $f : X \to X$. Then f is $(m+2,n) - C$ Lipschitz.*

Proof: It suffices to show that for f, C, and an arbitrary z from X, the following statement is true.

(*) For all $p \in C$ and for all $x \in X$, if $x \in st_p^n(z)$, then $f(x) \in st_p^{m+2}(f(z))$.

Let z be an arbitrary element of X. Let p be an element of C and let $x \in st_p^n(z)$. Fix U_1 and U_2 from p such that $f(x) \in U_1$ and $f(z) \in U_2$ and fix an index α such that $f_\alpha(x) \in U_1$ and $f_\alpha(z) \in U_2$ (such an index exists because $\{f_\gamma\}_{\gamma\in\Gamma}$ converges to f pointwise). The following are true.

(1) $f(x) \in st_p(f_\alpha(x))$.
(2) $f_\alpha(x) \in st_p^m(f_\alpha(z))$.
(3) $f_\alpha(z) \in st_p(f(z))$.

Together, (1), (2), and (3) imply $f(x) \in st_p^{m+2}(f(z))$. Since z was arbitrary, f is proved to be $(m+2, n) - C$ Lipschitz. □

4. Contractivity

Topological methods are used largely when solving a fixed point problem (see e. g., [3], [4], and [7]). Intuitively, we feel that a topologically Lipschitz function f with a small Lipschitz constant should be contractive. In that case, we should be able to obtain some results concerning the existence of fixed points for f. In this section, we suggest one way that results of this kind could be obtained. We will need some further research into the properties of scales to be able to prove a fixed point theorem similar to Theorem 4.3.

The main result of this section will concern some results from [4].

Definition 4.1 ([4, Definition 2]). Let (X, T) be a topological space. Let $f : X \to X$ be a function. Then f is said to be *feebly topologically contractive* if for every open cover p of X and for every pair of points $x, y \in X$, there exists $k \in \mathbb{N}$ and $U \in p$ such that $f^k(x) \in U$ and $f^k(y) \in U$ holds.

Remark 4.2. We can see that a function $f : X \to X$ is feebly topologically contractive if and only if for every open cover p of X and for every pair of points $x, y \in X$, there exists $k \in \mathbb{N}$ such that $f^k(x) \in st_p(f^k(y))$ holds.

In [4], the following result was obtained.

Theorem 4.3 ([4, part of Theorem 2]). *Let X be a T_1 topological space. Let $f : X \to X$ be a feebly topologically contractive function with closed graph. Then f has a unique fixed point.*

Now we are going to show that in connected topological spaces, some topologically Lipschitz functions will have a contractivity property.

Theorem 4.4. *Let X be a connected topological space, C a scale in X, and $f : X \to X$ a function. Let f be $(1, n) - C$ Lipschitz for some integer $n \geq 2$. Then*

(contr) *for all $p \in C$ and for all $x, y \in X$, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $f^m(x) \in st_p(f^m(y))$.*

Proof: Since f is $(1, n) - C$ Lipschitz, (contr) will be shown to be true if we prove that the following is true.

(4) For all $p \in C$ and for all $x, y \in X$, there exists $m_0 \in \mathbb{N}$ such that $f^{m_0}(x) \in st_p(f^{m_0}(y))$.

Suppose, contrary to what we wish to prove, that

(5) there exists $p \in C$ and there exists $x, y \in X$ such that for all $m \in \mathbb{N}$, $f^m(x) \notin st_p(f^m(y))$.

From now on, we consider a fixed open cover p and two chosen points x and y for which (5) is true. Statement (5) implies

(6) for all $m \in \mathbb{N}$, $f^m(x) \notin st_p^n(f^m(y))$.

To see this, if (6) were not true, then for some m, $f^m(x) \in st_p^n(f^m(y))$ would be true and, since f is $(1, n) - C$ Lipschitz, $f^{m+1}(x) \in st_p(f^{m+1}(y))$ would follow and this would be a contradiction.

Now for each $m \in \mathbb{N}$, define

$$k_m = min\{k; f^m(x) \in st_p^k(f^m(y))\}.$$

The numbers k_m exist because of the connectedness of X.

Put $l = min\{k_m; m \in \mathbb{N}\}$. Take o such that $l = k_o$. We see immediately that $l > n$; otherwise, $f^o(x) \in st_p^n(f^o(y))$ would be true and this would imply $f^{o+1}(x) \in st_p(f^{o+1}(y))$. Therefore, we have

$$l = rn + s \text{ for some } r \in \mathbb{N}, s \in \mathbb{N}.$$

Denote $u = f^o(x)$ and $v = f^o(y)$. Since $u \in st_p^l(v)$, there exist points $u_0, u_1, \dots u_r$ from X such that $u_0 = u$, $u_r = v$ and

$$\text{for all } i = 0, 1, \dots, r-1, \ u_i \in st_p^n(u_{i+1})$$

is true. Since f is $(1, n) - C$ Lipschitz, we obtain

$$\text{for all } i = 0, 1, \ldots, r-1, \; f(u_i) \in st_p(f(u_{i+1})).$$

This implies $f(u) \in st_p^{r+1}(f(v))$; so, $f^{o+1}(x) = f(u) \in st_p^{r+1}(f(v)) = st_p^{r+1}(f^{o+1}(y))$ is true. But we can see that $r + 1 < l$ and this contradicts the minimality of l.

We have proved by contradiction that (4) is true. □

The results obtained above imply, in fact, that every $(1, n) - P_X$ Lipschitz function in connected Hausdorff spaces has a unique fixed point. But the system P_X is too large and it can be shown that such a function is constant. To obtain an interesting fixed point theorem, we would need to work with a smaller system of open covers, probably with a fine scale with special properties.

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