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KELLEY REMAINDERS OF $[0, \infty)$

ROBBIE A. BEANE AND WŁODZIMIERZ J. CHARATONIK

Abstract. We investigate Kelley continua that arise as the remainder of Kelley compactifications of $[0, \infty)$. We call such continua *Kelley remainders*. The main results of the paper are that all arc-like Kelley continua and all Kelley arc continua are Kelley remainders. We also show that being such a remainder is preserved under confluent mappings.

1. Introduction and Preliminaries

A continuum X is said to be a *Kelley continuum* (or alternately, to have the *property of Kelley*), provided that for each point $p \in X$, for each subcontinuum K of X containing p, and for each sequence $\{p_n\}_{n=1}^{\infty}$ converging to p, there is a sequence of subcontinua $\{K_n\}_{n=1}^{\infty}$ converging to K such that $p_n \in K_n$ for each n (see [3, p. 167]).

Kelley continua were introduced by J. L. Kelley as Property 3.2 in [6] to investigate contractibility of hyperspaces. Robert W. Wardle in [11] investigated basic properties of Kelley continua and showed, among other things, that homogeneous continua are Kelley, that Kelley continua are preserved under confluent mappings, and that hereditarily indecomposable continua are Kelley. Since then, Kelley continua have become interesting in their own right and have provided a useful tool in topological characterizations of

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certain continua. In [1], Gerardo Acosta and Alejandro Illanes investigated hereditarily Kelley continua, i.e. continua in which each subcontinuum is Kelley.

A *Kelley compactification* of $[0,\infty)$ is a compactification of $[0,\infty)$ which is also a Kelley continuum. In this paper, we investigate continua that arise as remainders of Kelley compactifications of $[0,\infty)$. We will call such continua *Kelley remainders.* Acosta and Illanes showed that such continua are hereditarily Kelley and atriodic [1, theorems 6.2 and 6.3], and Patricia Pellicer-Covarrubias showed that any compactification of $[0,\infty)$ by a hereditarily indecomposable continuum is Kelley, and that, in fact, this characterizes hereditarily indecomposable continua [10, Corollary 7.2]. Verónica Martínez-de-la-Vega showed that there are uncountably many such compactifications with the pseudo-arc as the remainder [7, Theorem 10].

The main results of this paper are the theorems showing that all arc-like Kelley continua and Kelley arc continua are Kelley remainders. Even if some necessary conditions and some sufficient conditions are given for a continuum to be a Kelley remainder, we are lacking a characterization. We provide an example of an atriodic, hereditarily Kelley continuum which is not a Kelley remainder, answering a question posed by the referee, and also provide an example showing that being a Kelley remainder is not a hereditary property. Finally, we show that being a Kelley remainder is preserved by confluent mappings, which answers a question posed to the authors by Acosta.

All spaces considered in this paper are assumed to be metric. A *continuum* is a compact, connected metric space, and a *mapping* is a continuous function. A *ray* is a homeomorphic image of the real half-line $[0,\infty)$. A continuum is said to be *indecomposable* if it is not the union of two of its proper subcontinua. A continuum is *hereditarily indecomposable* provided that each nondegenerate subcontinuum of it is indecomposable. A continuum X is *atriodic* if there is no subcontinuum P of X such that $P \backslash X$ has at least three components. We say X is *homogeneous* if for any two points $p, q \in X$, there is a homeomorphism from X onto itself such that $h(p) = q$. For a continuum X, a *chain in* X is a nonempty, finite collection $\mathcal{C} = \{U_1, ..., U_n\}$ of open subsets U_i of X such that two members U_i and U_j intersect if and only if $|i-j| \leq 1$. The members of a chain $\mathcal{C}$ are called *links of* $\mathcal{C}$. For a given $\varepsilon > 0$, a chain $\mathcal{C}$ is an *ε-chain* if $\text{mesh}(\mathcal{C}) < \varepsilon$. A continuum X is said to be *chainable* provided that for any $\varepsilon > 0$, there is an ε-chain covering X. Chainable continua are also referred to as *arc-like*, and it is known that a continuum is arc-like if and only if it is homeomorphic to the inverse limit of an inverse sequence of arcs. Given two points $x, y \in X$, if there is a unique arc irreducible between x and y, we will denote this arc by xy. A continuum is *arcwise connected* if given two points $x, y \in X$, there is an arc A in X which is irreducible between x and y. If such an arc A is unique, we say that X is *uniquely arcwise connected.* An *arc continuum* is a continuum in which each nondegenerate proper subcontinuum is an arc. Given a continuum X, we denote by $C(X)$ the hyperspace of all subcontinua of X (see [3, Definition 1.6, p. 6]). A *Whitney map* for $C(X)$ is a mapping $\omega : C(X) \to [0,\infty)$ such that

(1) $\omega(A) < \omega(B)$ for any $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$;
(2) $\omega(A) = 0$ if and only if A is a singleton.

See [3, Definition 13.1, p. 105]. It is known that for any continuum X, Whitney maps exist for $C(X)$ [3, Theorem 13.4, p. 107]. If $\{K_n\}_{n=1}^{\infty}$ is a sequence of subcontinua of X and K is a subcontinuum of X, then the symbol $\text{Lim}\, K_n = K$ means that $\{K_n\}_{n=1}^{\infty}$ converges to K with respect to the Hausdorff metric.

Now we formally state the definition of a Kelley remainder.

Definition 1.1. A continuum X is a Kelley remainder if there is a Kelley compactification of $[0,\infty)$ with X as the remainder.

2. Arc-like continua

The main result of this paper is that every arc-like Kelley continuum is a Kelley remainder. This result has also been obtained independently by Carlos Islas using inverse limit techniques. His proof is to appear in his Ph.D. dissertation, *Continuos 2-equivalentes.*

Before proving this theorem, we will need to establish Lemma 2.2. Let X be a Kelley continuum and let $Y = X \cup R$ be a compactification of $R = [0,\infty)$ with remainder X. In Lemma 2.2, we will consider such spaces Y which satisfy the following property.

Property 2.1. For any $p \in X$, any proper subcontinuum K of X containing p, and any sequence $\{p_n\}_{n=1}^{\infty}$ in R converging to p, there is a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of R converging to K such that $p_n \in K_n$ for all $n \in \{1, 2, 3, ...\}$.

Lemma 2.2. *Let X be a Kelley continuum and let $Y = X \cup R$ be a compactification of $R = [0, \infty)$. Then Y is a Kelley continuum if and only if it satisfies Property 2.1.*

Proof: Notice that if Y is Kelley, then it follows directly from the definition of a Kelley continuum that Y has Property 2.1. So now assume that Y satisfies Property 2.1. We wish to show that this is, in fact, enough for Y to be Kelley, so to that end, let $p \in Y$, let K be a subcontinuum of Y containing p, and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in Y converging to p. We will consider the following cases.

Case 1: Assume that $p \in R$. For each n, define K_n to be an irreducible continuum containing p_n and K. Notice that all but finitely many of the points in the sequence $\{p_n\}_{n=1}^{\infty}$ are in R, and for any m such that $p_m \in R$, we have that $K_m = K \cup pp_n$. It follows that the sequence $\{K_n\}_{n=1}^{\infty}$ converges to K.

Case 2: Assume that $p \in X$ and that $X \subset K$. For each n, define K_n to be an irreducible continuum containing p_n and K. It can be seen that $\{K_n\}_{n=1}^{\infty}$, thus defined, must converge to K.

Case 3: Assume that $p \in X$ and that K is a proper subset of X. First consider the case in which all but finitely many of the points in the sequence $\{p_n\}_{n=1}^{\infty}$ are in X. Let $\{p_{n_i}\}_{i=1}^{\infty}$ denote the subsequence of $\{p_n\}_{n=1}^{\infty}$ that is contained in X. In this case, we can appeal to X being a Kelley continuum to define a sequence $\{K_{n_i}\}_{i=1}^{\infty}$ of continua converging to K, such that $p_{n_i} \in K_{n_i}$ for all i. Taking arbitrary subcontinua of Y containing the members of $\{p_n\}_{n=1}^{\infty}$ not in X, we form a sequence $\{K_n\}_{n=1}^{\infty}$ of continua converging to K such that $p_n \in K_n$ for all n.

In the case where all but finitely many of the points in the sequence $\{p_n\}_{n=1}^{\infty}$ are in R, we can proceed in a similar manner, though appealing to Y satisfying Property 2.1, instead of X being a Kelley continuum.

Finally, if infinitely many elements of $\{p_n\}_{n=1}^{\infty}$ are in X and infinitely many are in R, then we appeal to X being Kelley for the subsequence of $\{p_n\}_{n=1}^{\infty}$ contained in X and appeal to Y satisfying Property 2.1 for the subsequence of $\{p_n\}_{n=1}^{\infty}$ contained in R. Taking the union of the two sequences of continua so obtained, we get our desired sequence of continua $\{K_n\}_{n=1}^{\infty}$.

In each case above, we construct the desired sequence $\{K_n\}_{n=1}^{\infty}$ converging to K, such that $p_n \in K_n$ for all n. Hence, Y is a Kelley continuum. □

Theorem 2.3. *Every arc-like Kelley continuum is a Kelley remainder.*

Proof: Let X be an arc-like continuum. It is known that every arc-like continuum is planar. We will begin by embedding X into $\mathbb{R}^3$ such that $X \subset \mathbb{R}^2 \times \{0\}$. Our construction will take place in $\mathbb{R}^3$. Let $\mathcal{C}_m = \{B_1^m, B_2^m, ..., B_{k_m}^m\}$ be a chain of open balls in $\mathbb{R}^3$ covering X such that $\text{mesh}(\mathcal{C}_m) < \frac{1}{2^m}$ for all $m \in \{1, 2, ...\}$. Let $\{y_m\}_{m=1}^{\infty}$ be a decreasing sequence of real numbers in $(0, 1]$ converging to 0 such that $(\mathbb{R}^2 \times \{y_m\}) \cap (\bigcup_{i=1}^{k_m} B_i^m)$ is connected and $(\mathbb{R}^2 \times \{y_m\}) \cap B_i^m \neq \emptyset$ for each $0 \leq i \leq k_m$. Given $m \in \{1, 2, ...\}$, choose $a_m, b_m \in \mathbb{R}^2 \times \{y_m\}$ such that $a_m \in B_1^m$ and $b_m \in B_{k_m}^m$. Without loss of generality, we may assume (by taking subsequences, if necessary) that the sequences $\{a_m\}_{m=1}^{\infty}$ and $\{b_m\}_{m=1}^{\infty}$ converge to points $a \in X$ and $b \in X$, respectively. It is worth noting that a and b may not be distinct. Now, for each m, let R_m be an arc from a_m to b_m such that

(1) $R_m \subset (\mathbb{R}^2 \times \{y_m\}) \cap (\bigcup_{i=1}^{k_m} B_i^m)$;
(2) if x and y are points of R_m in B_i^m and $z \in xy \subset R_m$, then $z \in B_i^m$.

Also, for each m, we define S_m to be a straight line segment from b_m to b_{m+1} if m is odd and from a_m to a_{m+1} if m is even. Note that $\text{Lim}\, S_{2m} = \{a\}$ and $\text{Lim}\, S_{2m+1} = \{b\}$. Now let $R = \bigcup_{m=1}^{\infty}(R_m \cup S_m)$ and let $Y = X \cup R$. We can see that R is a ray and that Y is a compactification of R with X as the remainder.

We have left to show that Y is a Kelley continuum. Let p be any point in X, let K be any proper subcontinuum of X containing p, and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in R converging to p. By Lemma 2.2, we need to find a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of R converging to K such that $p_n \in K_n$ for every $n \in \{1, 2, 3, ...\}$.

For each n, there is an m_n such that $p_n \in R_{m_n} \cup S_{m_n}$. First, let us assume that eventually $p_n \in R_{m_n}$ for all n. For all such n, there exists some i_n such that $p_n \in B^{m_n}_{i_n}$. Choose $q_n \in (X \cap B^{m_n}_{i_n})$ for each n. Since $p_n \to p$ and $\text{diam}(B^{m_n}_{i_n}) \to 0$, the sequence $\{q_n\}_{n=1}^{\infty}$ converges to p. As X is a Kelley continuum, there exists a sequence $\{L_n\}_{n=1}^{\infty}$ of subcontinua of X converging to K such that $q_n \in L_n$ for each n. Let $\mathcal{D}_{m_n} = \{B^{m_n}_i \in \mathcal{C}_{m_n} : B^{m_n}_i \cap L_n \neq \emptyset\}$ for all n. Since L_n is a continuum, each $\mathcal{D}_{m_n}$ is a chain. Now, define $\{K_n\}_{n=1}^{\infty}$ such that $K_n = \text{cl}(R_{m_n} \cap (\bigcup \mathcal{D}_{m_n}))$. Note that each K_n is an arc intersecting every member of $\mathcal{D}_{m_n}$. Since $\text{mesh}(\mathcal{D}_{m_n}) \to 0$ and $\text{Lim}\, L_n = K$, we have that the sequence $\{K_n\}_{n=1}^{\infty}$ converges to K.

Now assume that eventually $p_n \in S_{m_n}$ for all n. Since every subsequence of $\{S_m\}_{m=1}^{\infty}$ converges to a singleton, the subsequence $\{S_{m_n}\}_{n=1}^{\infty}$ must converge to $\{p\}$. For each n, let $g_n \in R$ such that $\{g_n\} = R_{m_n} \cap S_{m_n}$. We proceed as in the previous case with $\{g_n\}_{n=1}^{\infty}$ playing the role of $\{p_n\}_{n=1}^{\infty}$, but instead defining $\{K_n\}_{n=1}^{\infty}$ by $K_n = S_{m_n} \cup \text{cl}((R_{m_n} \cap (\bigcup \mathcal{D}_{m_n})))$. Since $\{S_{m_n}\}_{n=1}^{\infty}$ converges to $\{p\}$, K_n converges to K.

Therefore, Y is a Kelley continuum, and hence, X is a Kelley remainder. □

3. Arc continua

Next we show that any atriodic, Kelley arc continuum is a Kelley remainder. The Ingram continuum [4] is an example of an atriodic, arc continuum, which is not arc-like. It was additionally shown to be a Kelley continuum in [5].

Theorem 3.1. *Every Kelley arc continuum is a Kelley remainder.*

Proof: Let X be a Kelley arc continuum. Recall that a point $p \in X$ is said to be an endpoint of X in the classical sense if p is an endpoint of each arc in X which contains p. Choose a point $a \in X$ which is not an endpoint of X in the classical sense, and choose arcs A_0 and A_1 such that each arc has a as an endpoint and $A_0 \cap A_1 = \{a\}$. Consider the families $\mathcal{C}_0$ and $\mathcal{C}_1$ of all arcs Q having a as an endpoint such that $Q \cap A_i$ is a non-trivial arc for $i \in \{0, 1\}$. Then either $\text{cl}(\bigcup \mathcal{C}_i)$ is the whole X or it is an arc. If both $\text{cl}(\bigcup \mathcal{C}_0)$ and $\text{cl}(\bigcup \mathcal{C}_1)$ are arcs, then X is an arc, which is certainly a Kelley remainder. So, assume that $\text{cl}(\bigcup \mathcal{C}_0) = X$. Let $\omega : C(X) \to [0, 1]$

be a Whitney map such that $\omega(X)=1$, and let $R\subset X\times[0,1]$ be defined by $R=\{\langle x,1-\omega(ax)\rangle\in X\times[0,1]:x\in\bigcup\mathcal{C}_0\}$. Then R is a ray and $Y=X\cup R$ is a compactification of R having X as the remainder.

We will now show that Y is a Kelley continuum. Let p be any point in Y, let K be any proper subcontinua of X containing p, and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in R converging to p. By Lemma 2.2, we need to find a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of R converging to K such that $p_n\in K_n$ for every $n\in\{1,2,3,...\}$. Let $\pi:Y\to X$ be the projection mapping. Then $\pi(p_n)$ are points in X converging to p, so we may find continua $L_n\subset X$ converging to K such that $\pi(p_n)\in L_n$. Finally, define K_n as the component of $\pi^{-1}(L_n)$ containing p_n. By the construction, the sequence $\{K_n\}_{n=1}^{\infty}$ converges to K. Since $p_n\in K_n$ for each n, Y is a Kelley continuum, and hence, X is a Kelley remainder. □

4. Additional results

In this section, we give some additional results relating to Kelley remainders. Theorems 4.2 and 4.3 were pointed out to the authors by the referee.

Theorem 4.1. *Up to homeomorphism, there exists exactly one Kelley compactification of $[0,\infty)$ having a simple closed curve as the remainder.*

Proof: Let R be a homeomorphic image of $[0,\infty)$, let S be a simple closed curve, and let $X=S\cup R$ be a Kelley compactification of R with S as the remainder. By [8, Lemma 4.4], we may assume that X is a planar continuum and that S is the standard unit circle S^1, which is expressed in polar coordinates as $S^1=\{\langle r,\theta\rangle:r=1\}$. Note that throughout this proof, any time we make reference to the coordinates of a point in $\mathbb{R}^2$, they are given in polar coordinates. Let $e:[1,\infty)\to R$ be a homeomorphism. We will assume that R is equipped with a natural linear order, as provided by e. Define $\sigma:X\to S^1$ by $\sigma(\langle r,\theta\rangle)=\langle 1,\theta\rangle$. Since R is contractible, there is a continuous function $\phi:R\to\mathbb{R}$ such that $\sigma(x)=\langle 1,\phi(x)\rangle$.

Our first goal is to show that since X is Kelley, the function ϕ must not be bounded. To that end, assume to the contrary that ϕ is bounded. Then both $\liminf\phi(e(t))$ and $\limsup\phi(e(t))$ must exist and be finite. It can be seen that since $S^1 \subset \mathrm{cl}(R)$, the limit $\lim \phi(e(t))$ does not exist, and thus, $\operatorname{liminf} \phi(e(t)) \neq \operatorname{limsup} \phi(e(t))$. Let $a, b \in \mathbb{R}$ such that $\operatorname{liminf} \phi(e(t)) < a < b < \operatorname{limsup} \phi(e(t))$ and $b - a < \pi$. The function ϕ must take each of the values a and b infinitely many times. Let $\{t_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of real numbers in $[1, \infty)$ and let $a_n = e(t_n)$ for all $n = \{1, 2, ...\}$ such that for all such n, $\phi(a_n) = a$, and there exists a $b_n \in a_n a_{n+1}$ satisfying $\phi(b_n) = b$. For each $k = \{1, 2, ...\}$, choose $p_k \in a_{2k-1} a_{2k}$ such that $\phi(p_k)$ is an absolute maximum value for $\phi|_{a_{2k-1}a_{2k}}$. The sequence $\{p_k\}_{k=1}^{\infty}$ has a convergent subsequence; therefore, without loss of generality, we will assume that $\{p_k\}_{k=1}^{\infty}$ itself converges to a point $p \in S^1$. Let $l = b - a$. It can be seen that for each k, $\mathrm{length}(\phi(a_{2k-1} p_k)) \geq l$ and $\mathrm{length}(\phi(p_k a_{2k})) \geq l$. For every k, there exist arcs $I_k \subset a_{2k-1} p_k$ and $J_k \subset p_k a_{2k}$ such that $I_k \cap J_k = \{p_k\}$ and $\mathrm{length}(\phi(I_k)) = \mathrm{length}(\phi(J_k)) = l$. Note that $I = \mathrm{Lim}\, I_k = \mathrm{Lim}\, J_k$ is a proper subcontinuum of S^1 with p as an endpoint. Let $K = \mathrm{cl}(S^1 \setminus I)$. There is no sequence of arcs in R containing the points p_n and converging to K, contradicting the fact that X is a Kelley continuum. So the function ϕ is unbounded either above or below. Without loss of generality, we will assume that ϕ is unbounded above.

Define $\psi : R \to \mathbb{R}$ by $\psi(x) = \max\{\psi(y) : y \leq x\}$ for all $x \in R$. Then ψ is an increasing (though not necessarily strictly increasing) mapping. We wish to show that $\lim(\psi(e(t)) - \phi(e(t))) = 0$. Assume that this limit is nonzero. Then there exists a strictly increasing sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers and $\omega > 0$ such that $\psi(e(t_n)) - \phi(e(t_n))) > \omega$. Furthermore, it can be shown that this sequence can be chosen in such a way that each t_n is a local minimum for $\phi \circ e$. Let $p_n = e(t_n)$ for all n. Since $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence, we will assume that $\{p_n\}_{n=1}^{\infty}$ itself converges to a point p, which must be in S^1. Proceeding in a manner similar to the argument used in the previous paragraph, for each n, we may find arcs I_n and J_n in R such that $I_n \cap J_n = \{p_n\}$, $\mathrm{length}(\phi(I_n)) = \mathrm{length}(\phi(J_n)) = \omega$, and $I = \mathrm{Lim}\, I_k = \mathrm{Lim}\, J_k$ is a proper subcontinuum of S^1 with p as an endpoint. As before, these conditions lead us to conclude that X is not Kelley, which is a contradiction. Therefore, we have that $\lim(\psi(e(t)) - \phi(e(t))) = 0$. Next, define $g : R \to \mathbb{R}$ by $g(x) = \psi(x) - \frac{1}{e^{-1}(x)}$ for all $x \in R$. Then g is a strictly increasing mapping. It is clear that $\lim(\psi(e(t)) - g(e(t))) = 0$ and hence that $\lim|\phi(e(t)) - g(e(t))| = 0$.

Now note that $g(R)$ is a half-line in $\mathbb{R}$. Define $f : g(R) \to \mathbb{R}^2$ by $f(t) = \langle 1 + \frac{1}{t}, t\rangle$, and let $T = f(g(R))$. The mapping f is a homeomorphism from $g(R)$ to T. Let $Y = S^1 \cup T$. Then Y is a compactification of T having S^1 as the remainder. Furthermore, Y is homeomorphic to the continuum referred to as $(SP)_1$ in [8], which is a Kelley continuum. We wish to show that X and Y are homeomorphic. To that end, we define $h : X \to Y$ by $h(x) = f(g(x))$ for all $x \in R$ and $h(x) = x$ for all $x \in S^1$. It is immediate from the definitions of h and Y that h is a bijection, so we have only to show that h is continuous to establish that it is a homeomorphism.

Notice that R is open in X, both f and g are continuous, and $h|_R = f \circ g$, so h is continuous at every point in R. Let $p \in S^1$ and $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in X converging to p. We wish to show that the sequence $\{h(p_n)\}_{n=1}^{\infty}$ converges to $h(p) = p$. The only non-trivial case to consider is when all but finitely many p_n are in R. Without loss of generality, we may assume that $p_n \in R$ for all $n = \{1, 2, ..\}$. Notice that $h(p_n) = f(g(p_n)) = \langle 1 + \frac{1}{g(p_n)}, g(p_n)\rangle$. For each n, let $t_n = e^{-1}(p_n)$. Since the sequence $\{p_n\}_{n=1}^{\infty}$ converges to a point in S^1, we have that $t_n \to \infty$, from which it follows that $(1 + \frac{1}{g(p_n)}) \to 1$. By the discussion in the third paragraph of this proof, we also have that $|\phi(p_n) - g(p_n)| \to 0$. From these facts, we may deduce that $\lim(h(p_n)) = \lim(p_n) = p = h(p)$, and thus that h is continuous at p. Since p was an arbitrary point in S^1, we have shown that h is continuous everywhere on S^1 and hence on all of X. Therefore, h is, in fact, a homeomorphism from X onto Y. This concludes the proof that any Kelley compactification of $[0,\infty)$ having a simple closed curve as the remainder is homeomorphic to Y. □

We shall use the previous result in the next theorem.

Theorem 4.2. *Let X be an atriodic, arcwise connected Kelley continuum. Then, up to homeomorphisms, there exists only one compactification $Y = X \cup R$ of $R = [0,\infty)$ which is Kelley.*

Proof: Since X is an atriodic, Kelley continuum, X is hereditarily Kelley by [1, Corollary 5.2]. Then X is an arcwise connected, hereditarily Kelley continuum; so by [1, Theorem 1.1], X is hereditarily locally connected. In particular, X is an atriodic, locally connected continuum, so X is either an arc or a simple closed curve. If X is an arc, then the $\sin(\frac{1}{x})$-continuum is a Kelley compactification of $[0, \infty)$ with an arc as the remainder (and the only one, up to homeomorphism, by [9, Theorem 16.28]). If X is a simple closed curve, then the previous theorem tells us that exactly one Kelley compactification of $[0, \infty)$ has X as the remainder, and that this compactification is homeomorphic to the spiral referred to as $(SP)_1$ in [8]. □

It is worth noting that the continuum referred to as $(SP)_1$ in [8] is the continuum X in Example 4.3 of this paper.

Theorem 4.3. *Let X be a homogeneous continuum and $Y = X \cup R$ be a compactification of $R = [0, \infty)$. If Y is a Kelley continuum, then X is one-dimensional.*

Proof: If Y is Kelley, then X must be atriodic [1]. Any atriodic, homogeneous continuum is a one-dimensional Kelley continuum [2]. □

Next, we give an example of an atriodic, Kelley continuum X which is not a Kelley remainder, answering a question posed by the referee. This shows that we lack a characterization for Kelley remainders.

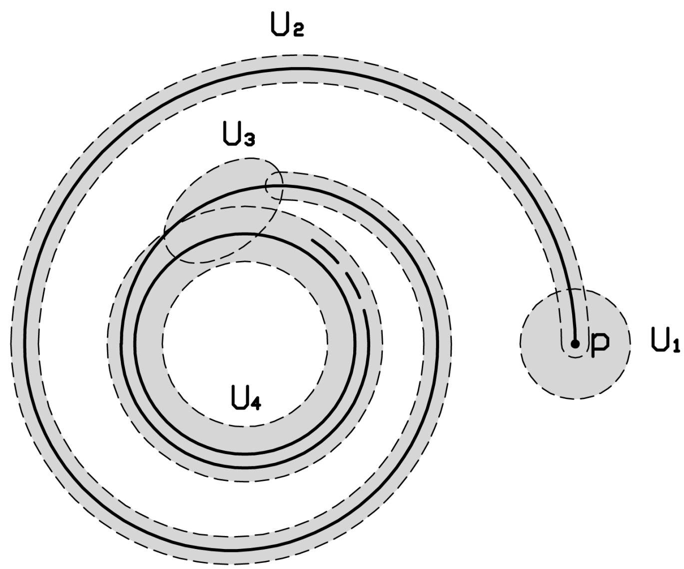
Example 4.4. *There exists an atriodic, Kelley continuum X which is not a Kelley remainder.*

Proof: We begin by constructing X as a subcontinuum of $\mathbb{R}^3$. In fact, the continuum will be planar, but we will need it embedded in $\mathbb{R}^3$ to proceed with the proof of the stated properties. In defining our continuum, we will use cylindrical coordinates $\langle r, \theta, z \rangle$. Let $S^1 = \{\langle r, \theta, z \rangle : r = 1, z = 0\}$ and $X = S^1 \cup \{\langle r, \theta, z \rangle : r = 1 + \frac{2\pi}{\theta}, \theta \geq 2\pi, z = 0\}$. Let Y be a compactification of $[0, \infty)$ having X as the remainder and set $R = Y \setminus X$. Now we will construct an embedding of Y into $[-2, 2]^2 \times [0, 1]$. Denote by $e : [0, \infty) \to R$ a homeomorphism from $[0, \infty)$ onto R. Since $[-2, 2]^2$ is an absolute extensor, there is a function $f : Y \to [-2, 2]^2$ such that $f|_X$ is the identity on X. Define $h : Y \to [-2, 2]^2 \times [0, 1]$ by $h(x) = x$ if $x \in X$ and $h(x) = \langle f(x), \frac{1}{1+e^{-1}(y)} \rangle$. Observe that h is one-to-one, and so we will assume from now on that Y is embedded in $[-2, 2]^2 \times [0, 1]$.

Without loss of generality, we may also assume that R does not intersect the z axis.

Define $\sigma : Y \to S^1$ by $\sigma(\langle r,\theta,z\rangle) = \langle 1,\theta,0\rangle$. Since R is contractible, there is a continuous function $\phi : R \to \mathbb{R}$ such that $\sigma(x) = \langle 1, 2\pi\phi(x), 0\rangle$. Let U_1, U_2, U_3, U_4 be open sets in $[-2,2]^2 \times [0,1]$ satisfying the following conditions (see the figure).

(1) $\{U_1, U_2, U_3, U_4\}$ is a chain covering X;
(2) $\{\langle r,\theta,z\rangle : r = 1 + \frac{2\pi}{\theta}, 2\pi \le \theta \le 4\pi, z = 0\} \subset U_2$;
(3) U_1 is an open ball such that $\langle 2, 2\pi, 0\rangle \in U_1$;
(4) $S^1 \subset U_4$;
(5) $\sigma(U_1) \cap \sigma(U_3) = \emptyset$.



Let $p = \langle 2, 2\pi, 0\rangle$. Then there exists a sequence of points $\{p_n\}_{n=1}^{\infty}$ converging to p such that

(a) $p_n \in R \cap U_1$ for all n;
(b) the arcs $p_n p_{n+1}$ converge to X;
(c) $p_n p_{n+1} \subset \bigcup_{i=1}^{4} U_i$ for all n;
(d) $p_n p_{n+1} \cap U_i \neq \emptyset$ for all $i \in \{1,2,3,4\}$ and all n.

For each n, choose $q_n \in p_np_{n+1}$ such that $\phi(q_n)$ is an absolute maximum value for $\phi|_{p_np_{n+1}}$. We may assume (by taking subsequences if necessary) that $\{q_n\}_{n=1}^{\infty}$ converges to a point q. Observe that $q \in S^1$. By the definition of the arcs p_np_{n+1} and the points q_n, we have that $\text{length}(\phi(p_nq_n)) \geq 2\pi$ and $\text{length}(\phi(q_np_{n+1})) \geq 2\pi$. For every n, there exist arcs $I_n \subset p_nq_n$ and $J_n \subset q_np_{n+1}$ such that $I_n \cap J_n = \{q_n\}$ and $\text{length}(I_n) = \text{length}(J_n) = \pi$. Note that $I = \text{Lim}\, I_n = \text{Lim}\, J_n$ is a semi-circle in S^1 with q as an endpoint. Let $K = \text{cl}(S^1 \setminus I)$. There are no arcs in R containing the points q_n and converging to K, and thus, Y is not a Kelley continuum. Since Y was an arbitrary compactification of $[0, \infty)$ having X as a remainder, we conclude that X is not a Kelley remainder. □

In the next example, we will show that being a Kelley remainder is not a hereditary property.

Example 4.5. *There exist continua X and Y such that $X \subset Y$ and Y is a Kelley remainder, though X is not.*

Proof: Let X be as in the previous example and let $Y = X \cup \{\langle 1 - \frac{2\pi}{\theta}, -\theta, 0\rangle : \theta \geq 2\pi\}$. Define $\sigma : \mathbb{R}^3 \to \mathbb{R}^2$ as the projection on the first two coordinates. Let $p_n = \langle 1 + \frac{2\pi}{2\pi n}, 2\pi n, 0\rangle$ and let $q_n = \langle 1 - \frac{2\pi}{2\pi n}, -2\pi n, 0\rangle$. For each n, define the following arcs.

(1) A_n is the arc in $\mathbb{R}^2 \times \{\frac{1}{n}\}$ such that $\sigma(A_n)$ is the unique arc joining p_1 and p_n in X.
(2) B_n is the straight line segment in $\mathbb{R}^2 \times \{\frac{1}{n}\}$ joining $\langle \sigma(p_n), \frac{1}{n}\rangle$ and $\langle \sigma(q_n), \frac{1}{n}\rangle$.
(3) C_n is the arc in $\mathbb{R}^2 \times \{\frac{1}{n}\}$ such that $\sigma(C_n)$ is the unique arc joining q_1 and q_n in X.
(4) D_n is the straight line segment joining $\langle 2, 0, \frac{1}{n}\rangle$ and $\langle 2, 0, \frac{1}{n+1}\rangle$ if n is even, and joining $\langle 0, 0, \frac{1}{n}\rangle$ and $\langle 0, 0, \frac{1}{n+1}\rangle$ if n is odd.

Finally, we define $R = \bigcup_{n=2}^{\infty}(A_n \cup B_n \cup C_n \cup D_n)$. Then $X \cup R$ is the required Kelley compactification of $[0, \infty)$ with Y as the remainder. Thus, Y is a Kelley remainder, and as shown in the previous example, X is not. □

The authors were asked by Acosta if the confluent image of a Kelley remainder is also a Kelley remainder. As our final result, we will answer this question in the affirmative.

Theorem 4.6. *Let X be a Kelley remainder and let Y be any continuum. If there exists a confluent, surjective mapping $f : X \to Y$, then Y is a Kelley remainder.*

Proof: Let $f : X \to Y$ be a confluent, surjective mapping and let Z be a Kelley compactification of $[0, \infty)$ having X as its remainder. We define the relation $\sim$ on Z by $a \sim b$ if and only if $a, b \in X$ and $f(a) = f(b)$. Notice that $X/\sim$ is homeomorphic to Y and that $Z/\sim$ is homeomorphic to a compactification of $[0, \infty)$ having Y as its remainder. Let $g : Z \to Z/\sim$ be the quotient map. It follows from the fact that if f is confluent, then g, too, is confluent. Thus, $Z/\sim$ is the confluent image of a Kelley continuum, and therefore, by [11, Theorem 4.2], $Z/\sim$ is Kelley as well. So, $Z/\sim$ is a Kelley compactification of $[0, \infty)$ having Y as its remainder, and hence, Y is a Kelley remainder. □

5. Questions

We close with some open questions relating to Kelley remainders.

Question 5.1. How can Kelley remainders be characterized?

Question 5.2. Is every circle-like Kelley continuum a Kelley remainder?

Question 5.3. Is every atriodic, tree-like Kelley continuum a Kelley remainder?

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References

[1] Gerardo Acosta and Alejandro Illanes, *Continua which have the property of Kelley hereditarily*, Topology Appl. **102** (2000), no. 2, 151–162.

[2] Charles L. Hagopian, *Atriodic homogeneous continua*, Pacific J. Math. **113** (1984), no. 2, 333–347.

[3] Alejandro Illanes and Sam B. Nadler, Jr. *Hyperspaces. Fundamentals and Recent Advances.* Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[4] W. T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. **77** (1972), no. 2, 99–107.

[5] W. T. Ingram and D. D. Sherling, *Two continua having a property of J. L. Kelley*, Canad. Math. Bull. **34** (1991), no. 3, 351–356.

[6] J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36.

[7] Verónica Martínez-de-la-Vega, *An uncountable family of metric compactifications of the ray with remainder pseudo-arc*, Topology Appl. **135** (2004), no. 1-3, 207–213.

[8] Sam B. Nadler, Jr. *Continua whose cone and hyperspace are homeomorphic*, Trans. Amer. Math. Soc. **230** (1977), 321–345.

[9] ———, *Hyperspaces of Sets. A Text with Research Questions.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49. New York-Basel: Marcel Dekker, Inc., 1978.

[10] Patricia Pellicer-Covarrubias, *The hyperspaces $K(X)$*, Rocky Mountain J. Math. **35** (2005), no. 2, 655–674.

[11] Roger W. Wardle, *On a property of J. L. Kelley*, Houston J. Math. **3** (1977), no. 2, 291–299.

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