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PARTITION RELATIONS AND POWER HOMOGENEITY

N. A. CARLSON AND G. J. RIDDERBOS

Abstract. Applying the Erdös-Rado Theorem, we prove that the cardinality of any power homogeneous Hausdorff space X is bounded by $2^{c(X)\pi\chi(X)}$. This answers a question of Jan van Mill and provides a new proof of van Douwen's Theorem. We also give an improvement of a bound proved by Ryszard Frankiewicz on the size of $H(X)$, the group of autohomeomorphisms of X.

1. Introduction

All spaces under consideration are Hausdorff. A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism h of X such that $h(x) = y$. A space X is called *power homogeneous* if X^μ is homogeneous for some cardinal number μ. By $\pi\chi(X)$ and $\pi w(X)$, we denote the π-*character* and π-*weight* of a space X. By $d(X)$, $c(X)$, and $\chi(X)$, we denote *density*, *cellularity*, and *character*, respectively.

In 1978, Eric K. van Douwen [3] proved that if X is power homogeneous, then the size of X is bounded by $2^{\pi w(X)}$. In 2005, Jan van Mill [7] showed that if X is in addition compact, then the size of X is bounded by $2^{c(X)\pi\chi(X)}$. Since always $c(X)\pi\chi(X) \leq \pi w(X)$, and strict inequality is possible, this improves van Douwen's result for the class of compact spaces. Van Mill asked in [7] if his inequality

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is also true without the assumption of compactness. A positive answer for regular spaces was provided by G. J. Ridderbos in [9], and Nathan A. Carlson showed in [2] that the assumption of regularity may be weakened to either quasi-regular or Urysohn.

In this note we prove that these separation axioms can be dropped entirely and that van Mill's inequality is valid for arbitrary power homogeneous Hausdorff spaces. We also give an improvement of a cardinality bound proved by Ryszard Frankiewicz [4]. All of our results depend on the well-known Erdös-Rado Theorem. This partition theorem was previously used by A. Hajnal and I. Juhász in [5] to show that the size of any space X is bounded by $2^{c(X)\chi(X)}$. So in the presence of power homogeneity, the character can be replaced by the π-character in this cardinality bound.

To the best of our knowledge, the use of a partition relation in the proof of cardinality bounds on homogeneous spaces is new. It should be noted that our results provide a new proof of van Douwen's Theorem.

2. Power homogeneous spaces

All product spaces carry the standard product topology. Whenever μ is a cardinal number and $A \subseteq \mu$, then by π_A we denote the projection of X^μ onto X^A. If $\alpha \in \mu$, then we write π_α for $\pi_{\{\alpha\}}$, which is the projection on the α-*th* coordinate. This notation is ambiguous because α is also a subset of μ. As a rule, we will always use π_α and π_β as projections on the respective coordinates and for $\kappa \subseteq \mu$, we will use π_κ for the projection onto X^κ. Finally, if $x \in X^\mu$, then we write x_A instead of $\pi_A(x)$, and π is always the projection onto the first coordinate, i.e. $\pi = \pi_0$.

By $\Delta(X, \kappa)$, we denote the *diagonal* in X^κ which is given by

$$\big\{x \in X^\kappa : \forall\, \alpha, \beta \in \kappa\, (x_\alpha = x_\beta)\big\}.$$

We will call a space X^μ Δ-*homogeneous* if for all points $x, z \in \Delta(X, \mu)$ there is a homeomorphism of X^μ mapping x onto z. A space X is power homogeneous if and only if there is a cardinal μ such that X^μ is Δ-homogeneous. This was proved by Ridderbos in [10].

The set of all autohomeomorphisms of a space X is denoted by $H(X)$ and we let $\operatorname{tpe}(x, X) = \{h(x) : h \in H(X)\}$ be the *type* of x in X. Recall that a set $R \subseteq X$ is *regular open* if $\operatorname{Int}\operatorname{Cl} R = R$. We denote the collection of regular open subsets of X by $\mathrm{RO}(X)$. The *semiregularization* of a space X, denoted by X_s, is the space with underlying set X and $\mathrm{RO}(X)$ as a basis for its topology. A space X is *semiregular* if $X = X_s$.

The well-known Erdös-Rado Theorem states that if $f : [X]^2 \to \kappa$ is a function and $|X| > 2^\kappa$, then there is some subset Y of X with $|Y| \geq \kappa^+$ such that $f(y) = f(z)$ for all $y, z \in [Y]^2$. We shall use this theorem to prove our main result. To indicate the use of this partition relation in the presence of homogeneity, we first prove the following proposition.

Proposition 2.1. *Suppose $f : X \to Y$ is an open and continuous map. If A is some type in X, then*

$$|f[A]| \leq 2^{\pi\chi(X)c(Y)}.$$

Proof: Let $\kappa = \pi\chi(X)c(Y)$. We may fix $B \subseteq A$, such that $f{\restriction}B : B \to f[A]$ is a bijection. Fix $p \in B$, and for every $x \in B$, fix a homeomorphism $h_x \in H(X)$ such that $h_x(p) = x$. Let $\mathcal{U}$ be a local π-base at p in X with $|\mathcal{U}| \leq \kappa$ and fix a well-ordering on $\mathcal{U}$. We now define a map $G : [B]^2 \to \mathcal{U}$ by

$$G(\{x, z\}) = \min\{U \in \mathcal{U} : fh_x[U] \cap fh_z[U] = \emptyset\},$$

where $\{x, z\} \in [B]^2$. We invite the reader to check that this is well-defined. We now prove that the size of B is bounded by 2^κ by contradiction; so assume that $|B| > 2^\kappa$. Then it follows from the Erdös-Rado Theorem that there is some set $Z \subseteq B$ and $U \in \mathcal{U}$ such that $G(\{x, z\}) = U$ for all $x, z \in Z$. This means that the collection $\mathcal{C} = \{fh_z[U] : z \in Z\}$ is a family of pairwise disjoint open subsets of Y. Since $|\mathcal{C}| = |Z| \geq \kappa^+$ and $c(Y) \leq \kappa$, this is impossible. □

It follows from the previous proposition that the size of homogeneous spaces X is bounded by $2^{c(X)\pi\chi(X)}$. We now turn towards proving that this inequality is also valid for power homogeneous spaces. We fix a power homogeneous space X and a cardinal number μ such that X^μ homogeneous. We also fix a cardinal number κ with $\pi\chi(X) \leq \kappa$. Without loss of generality, we may assume that $\kappa \leq \mu$. Fix $p \in \Delta(X, \mu)$ and a local π-base $\mathcal{U}$ at $\pi(p)$ in X. For $B \subseteq A \subseteq \mu$, let $\pi_{A\to B}$ be the projection of X^A onto X^B. For $A \subseteq \mu$, define $\mathcal{U}(A)$ by

$$\Big\{\pi^{-1}_{A\to B}\Big[\prod\nolimits_{b\in B} U_b\Big] : B \in [A]^{<\omega}, \forall\, b \in B\ (U_b \in \mathcal{U})\Big\}.$$

Observe that $\mathcal{U}(A)$ is a local π-base at p_A in X^A; see [9] for details. We shall need the following lemma which is a consequence of Theorem 2.2 in [1].

Lemma 2.2. *For every $x \in \Delta(X,\mu)$ there is a homeomorphism $h_x : X^\mu \to X^\mu$ such that $h_x(p) = x$ and the following conditions are satisfied.*

(1) *For all $z \in X^\mu$, if $z_\kappa = p_\kappa$ then $\pi(h_x(z)) = \pi(x)$,*
(2) *For all $U \in \mathcal{U}(\kappa)$, there is a point $q(U) \in \pi_\kappa^{-1}[U]$ and a basic open neighborhood U_x of $h_x(q(U))_\kappa$ in X^κ such that*
   (a) *$q(U)_\alpha = p_\alpha$ for all $\alpha \in \mu \setminus \kappa$ and*
   (b) *$\pi_\kappa^{-1}[U_x] \subseteq h_x[\pi_\kappa^{-1}[U]]$.*

Proof: Since X^μ is homogeneous, we pick $h : X^\mu \to X^\mu$ such that $h(p) = x$. Applying [1, Theorem 2.2], we find $A \in [\mu]^{\leq\kappa}$ such that (1) is satisfied for A instead of κ. Next for all $U \in \mathcal{U}(A)$, we pick $q(U) \in \pi_A^{-1}[U]$ as in (2)(a),where κ is replaced by A. For (2)(b),we may just pick a basic open neighborhood of $h(q(U))$ in X^μ which is contained in $h[\pi_A^{-1}[U]]$. Since $|\mathcal{U}(A)| \leq \kappa$, we obtain a set B of at most κ many coordinates such that all the basic open sets obtained in this way depend only on the coordinates in B.

By applying suitable coordinate changes (see [9, §3]), we obtain h_x as required. □

We point out that the points $q(U)$ from the previous lemma depend on x. In the proof of the following theorem we will not write $q(x,U)$ to express this dependence because we only consider points of the form $h_x(q(U))$. We will implicitly assume that in this notation, the point $q(U)$ is the point $q(x,U)$.

Theorem 2.3. *If X is power homogeneous, then $|X| \leq 2^{\pi\chi(X)c(X)}$.*

Proof: We let $\kappa = \pi\chi(X)c(X)$ and we fix $\mu \geq \kappa$ such that X^μ is homogeneous. For every $x \in \Delta(X,\mu)$, we fix a homeomorphism $h_x : X^\mu \to X^\mu$ as in the previous lemma. For $x \in \Delta(X,\mu)$ and $U \in \mathcal{U}(\kappa)$, the open set U_x is a basic open subset of X^κ, so we may fix a collection $\{U_{x,\alpha} : \alpha \in \kappa\}$ of open subsets of X such that

$$U_x = \bigcap_{\alpha<\kappa} \pi_\alpha^{-1}[U_{x,\alpha}].$$

For every $\alpha \in \kappa$, we also fix a local π-base $\{V(x,U,\alpha,\beta) : \beta < \kappa\}$ of the point $h_x(q(U))_\alpha$ in X. We first observe the following claim.

Claim 1. Whenever $x, y \in \Delta(X, \mu)$ are different, there are some $U \in \mathcal{U}(\kappa)$ and $\alpha, \beta < \kappa$ such that

$$V(x, U, \alpha, \beta) \subseteq U_{x,\alpha} \setminus \overline{U}_{y,\alpha}.$$

Proof of Claim: Since $\pi(x) \neq \pi(y)$, we have that $h_y^{-1}(x)_\kappa \neq p_\kappa$. Fix an open neighborhood W of p_κ in X^κ such that $h_y^{-1}(x)_\kappa \notin \overline{W}$ and let

$$\mathcal{W} = \{U \in \mathcal{U}(\kappa) : U \subseteq W\}.$$

Note that $\mathcal{W}$ is a local π-base at p_κ in X^κ and $h_y^{-1}(x)_\kappa \notin \mathrm{Cl} \bigcup \mathcal{W}$. So we have

$$x \in \mathrm{Cl}\{h_x(q(U)) : U \in \mathcal{W}\},$$

and since $\pi_\kappa^{-1}[U_y] \subseteq h_y \pi_\kappa^{-1}[U]$ for all $U \in \mathcal{W}$, we have

$$x \notin \mathrm{Cl} \bigcup \{\pi_\kappa^{-1}[U_y] : U \in \mathcal{W}\}.$$

But this means that there is some $U \in \mathcal{W}$ such that

$$h_x(q(U))_\kappa \notin \overline{U}_y.$$

Since U_y is a basic open subset of X^κ, it follows that there is some $\alpha < \kappa$ such that

$$h_x(q(U))_\alpha \notin \overline{U}_{y,\alpha}.$$

Since $\{V(x, U, \alpha, \beta) : \beta < \kappa\}$ is a local π-base at $h_x(q(U))_\alpha$ in X and $h_x(q(U))_\alpha \in U_{x,\alpha}$, we may pick $\beta < \kappa$ such that $V(x, U, \alpha, \beta) \subseteq U_{x,\alpha} \setminus \overline{U}_{y,\alpha}$ and this completes the proof of the claim.

We now prove the desired inequality. So assume that $|X| > 2^\kappa$. We fix a well-ordering $\prec$ on X and define a map $G : [X]^2 \to \mathcal{U}(\kappa) \times \kappa \times \kappa$ as follows: Let $\{x, y\} \in [X]^2$ and assume that $x \prec y$. Applying the previous claim, we may let $G(\{x, y\}) = \langle U, \alpha, \beta \rangle$ be such that

$$V(x, U, \alpha, \beta) \subseteq U_{x,\alpha} \setminus \overline{U}_{y,\alpha}.$$

Here we have identified $\Delta(X, \mu)$ with X. Note that $|\mathcal{U}(\kappa) \times \kappa \times \kappa| = \kappa$. Since $|X| > 2^\kappa$, we apply the Erdös-Rado Theorem to find $Y \subseteq X$ and $\langle U, \alpha, \beta \rangle \in \mathcal{U}(\kappa) \times \kappa \times \kappa$ such that $|Y| = \kappa^+$ and for all $\{x, y\} \in [Y]^2$, $G(\{x, y\}) = \langle U, \alpha, \beta \rangle$. By possibly removing the $\prec$-largest element from Y, we may assume that for all $y \in Y$, $V(y, U, \alpha, \beta) \subseteq U_{y,\alpha}$.

Consider the collection $\mathcal{C} = \{V(x, U, \alpha, \beta) : x \in Y\}$ of open subsets of X^κ. If $x, y \in Y$ are different with $x \prec y$, then we have

$$V(x, U, \alpha, \beta) \cap \overline{U}_{y,\alpha} = \emptyset \qquad \text{and} \qquad V(y, U, \alpha, \beta) \subseteq U_{y,\alpha},$$

and, therefore, $V(x, U, \alpha, \beta)$ and $V(y, U, \alpha, \beta)$ are disjoint. But this means that the collection $\mathcal{C}$ consists of pairwise disjoint open subsets of X. Since $|\mathcal{C}| = |Y| = \kappa^+$ and $c(X) \leq \kappa$, this is impossible. □

It is well-known that if X is regular, then its density is bounded by $\pi\chi(X)^{c(X)}$. This was proved by B. Šapirovskiĭ in [11], and Carlson, in [2], showed that this inequality is also valid for quasi-regular spaces. This bound is not true in general, (see, for example, section 4), but we have just proved that the density of power homogeneous spaces X is bounded by $2^{c(X)\pi\chi(X)}$. Since always $|\mathrm{RO}(X)| \leq \pi w(X)^{c(X)}$ we obtain the following corollary.

Corollary 2.4. *If X is power homogeneous, then*

$$|\mathrm{RO}(X)| \leq 2^{c(X)\pi\chi(X)}.$$

In [2], Carlson introduced the $\pi\theta$-character of a space X denoted by $\pi\chi_\theta(X)$ and proved that $\pi\chi_\theta(X) \leq \pi\chi(X)$ (see [2, Corollary 2.7]). It was also proved there that Theorem 2.3 is equivalent to the following result which solves Question 4.8 in [2]. By Proposition 2.1 in [2], the class of spaces with (power) homogeneous semiregularization contains the class of spaces that are (power) homogeneous.

Corollary 2.5. *If X has power homogeneous semiregularization, then $|X| \leq 2^{c(X)\pi\chi_\theta(X)}$.*

Proof: This follows from Theorem 2.3 and Observation 4.9 in [2]. □

It was proved by Ridderbos in [9, Theorem 4.5] that if X is power homogeneous and $\pi w(X) = \mu$, then X^μ is Δ-homogeneous. This fact yields a quick proof of van Douwen's Theorem, see [9, Theorem 4.6]. Note that it follows from Proposition 2.1 that if X^κ is Δ-homogeneous where $\kappa = c(X)\pi\chi(X)$, then $|X| \leq 2^\kappa$. So this raises the following question.

Question 2.6. *Suppose X is power homogeneous and let $\kappa = c(X)\pi\chi(X)$. Is X^κ Δ-homogeneous?*

3. The group of homeomorphisms $H(X)$

In this section, we study the size of the group $H(X)$ for arbitrary Hausdorff spaces X. Frankiewicz proved in [4] that the size of $H(X)$ is bounded by $2^{\pi w(X)}$. Since $|X| \leq |H(X)|$ whenever X is homogeneous, this improves van Douwen's Theorem from [3] for homogeneous spaces. In view of Theorem 2.3, it is natural to ask if the size of $H(X)$ is also bounded by $2^{c(X)\pi\chi(X)}$. We do not know the answer to this question, but using the Erdös-Rado Theorem, we will present an improvement of the Frankiewicz result.

We say that a subset Z of X *separates* a subset $\mathcal{G}$ of $H(X)$, if for all $f, g \in \mathcal{G}$ with $f \neq g$ there is some $z \in Z$ with $f(z) \neq g(z)$. The *separation degree* of X, denoted by $sd(X)$, is defined as

$$sd(X) = \min\{|Z| : Z \text{ separates } H(X)\}.$$

It is always the case that $sd(X) \leq d(X)$. Carlson observed in [2] that if X is (power) homogeneous then X_s is (power) homogeneous. As a corollary, we have the following simple observation.

Observation 3.1. *If X is any space, then $H(X) \subseteq H(X_s)$ and therefore $sd(X) \leq sd(X_s)$. In particular it follows that $sd(X) \leq d(X_s)$.*

Proof: The first statement is proved by Carlson in [2, Proposition 2.1]. The inequalities are immediate consequences. □

Proposition 3.2. *The size of $H(X)$ is bounded by $2^{c(X)\pi\chi(X)sd(X)}$.*

Proof: Let $\kappa = c(X)\pi\chi(X)sd(X)$ and fix a subset Z of X which separates $H(X)$ such that $|Z| \leq \kappa$. For every $z \in Z$, we fix a local π-base $\mathcal{U}_z$ at z of size $\leq \kappa$ and we set $\mathcal{U} = \bigcup\{\mathcal{U}_z : z \in Z\}$. Note that $|\mathcal{U}| \leq \kappa$. Fix a well-ordering on $\mathcal{U}$ and define a map $G : [H(X)]^2 \to \mathcal{U}$ as

$$G(\{f, g\}) = \min\{U \in \mathcal{U} : f(U) \cap g(U) = \emptyset\},$$

where $f, g \in H(X)$ are different. It is routine to check that G is well-defined; this follows from the fact that Z separates $H(X)$. As before, it follows from the Erdös-Rado Theorem that $|H(X)| \leq 2^\kappa$. □

In the next section, we shall provide examples to show that the previous proposition really improves Frankiewicz's result from [4]. However, we do not know whether the separation degree can be dropped from the bound in Proposition 3.2, so we ask the following question.

Question 3.3. *Is the size of $H(X)$ always bounded by $2^{c(X)\pi\chi(X)}$?*

On close inspection of the proof of Theorem 1 in [4], it is not hard to realize that it is actually shown that the size of $H(X)$ is always bounded by $d(X)^{\pi\chi(X)sd(X)}$. The separation degree cannot be dropped from this bound; simply observe that if X is a discrete space of size $\mathfrak{c}$, then any bijection on X is a homeomorphism and therefore, $|H(X)| = 2^{\mathfrak{c}}$, whereas $d(X)^{\pi\chi(X)} = \mathfrak{c}$.

4. Examples

In this section, we construct spaces in which the π-character and separation degree are both strictly less than the density. An example of a space X for which $sd(X) < d(X)$ can be found in [2, Example 4.4]. We shall modify the example from [2] to also get the π-character less than the density.

Proposition 4.1. *Fix an infinite cardinal number κ and let Z be any space of size κ^+ such that $\pi\chi(Z) \leq \kappa$. Furthermore, assume that $|U| = \kappa^+$ whenever U is a non-empty open subset of Z. Then there is a space X such that $X_s \approx Z_s$, $\pi\chi(X) = \kappa$ and $d(X) = \kappa^+$.*

Proof: We enumerate Z faithfully by $\{x_\alpha : \alpha < \kappa^+\}$. Of course, the underlying set of X will be Z, but we modify the topology on it. We have just defined a well-ordering on X. If $x \in X$ and $x = x_\alpha$, then we write $pred(x) = \{x_\beta : \beta < \alpha\}$. Consider the following collection of subsets of X:

$$\mathcal{B} = \big\{U \setminus pred(x) : x \in U \ \&\ U \text{ is an open subset of } Z\big\}.$$

This collection may serve as a basis for a topology on X; suppose $B = U \setminus pred(x)$ and $B' = U' \setminus pred(x')$ are both members of $\mathcal{B}$ such that $y \in B \cap B'$. In order to show that $\mathcal{B}$ may serve as a basis, we need to show that $y \in E \subseteq B \cap B'$ for some $E \in \mathcal{B}$. But since $y \in B \cap B'$, it follows that $y \notin (pred(x) \cup pred(x'))$ and therefore $pred(x) \cup pred(x') \subseteq pred(y)$. Now let $E = (U \cap U') \setminus pred(y)$. Then $y \in E$ since $y \in U \cap U'$ and we have just shown that $E \subseteq B \cap B'$.

From now on we will write X to denote the set X with the topology determined by $\mathcal{B}$. Since $\{x_\alpha : \alpha < \kappa^+\}$ is an enumeration of X, it follows that $d(X) = \kappa^+$; simply observe that if $D \subseteq X$

and $|D| \leq \kappa$, then by regularity of κ^+, there is some $\alpha < \kappa^+$ such that $D \subseteq pred(x_\alpha)$. But then $X \setminus pred(x_\alpha)$ is a neighborhood of x_α which misses D, showing that D is not dense in X.

To show that $\pi\chi(X) \leq \kappa$, fix $x \in X$ and let $\mathcal{U}$ be a local π-base at x in Z of size $\leq \kappa$. Since $|U| = \kappa^+$ for every $U \in \mathcal{U}$ and $|pred(x)| \leq \kappa$, we may pick $y_U \in U \setminus pred(x)$. Note that $pred(x) \subseteq pred(y_U)$. We leave it to the reader to verify that the collection $\{U \setminus pred(y_U) : U \in \mathcal{U}\}$ forms a local π-base at x in X. It remains to show that $\mathrm{RO}(X) = \mathrm{RO}(Z)$. This follows from the fact that $|U| = \kappa^+$ whenever U is a non-empty open subset of Z and it is the same as in [2, Example 4.4]. □

A space X is called Urysohn provided that distinct points of X can be separated by open neighborhoods with disjoint closures. By $\mathbb{R}$ and $\mathbb{I}$, we denote the real line and the unit interval, respectively.

Corollary 4.2. *There exists a c.c.c. Urysohn space X such that $d(X) = \omega_1$ and $sd(X)\pi\chi(X) \leq \omega$.*

Proof: Let Z be a dense subgroup of $\mathbb{R}$ of size ω_1. Note that $\pi w(Z) = \omega$. Proposition 4.1 yields a space X for which $d(X) = \omega_1$, $\pi\chi(X) = \omega$, and $X_s \approx Z_s$. Since Z is regular, we have that $Z_s \approx Z$. It follows from Observation 3.1 that $sd(X) \leq d(Z) \leq \omega$, and since $c(X) = c(X_s)$, the space X is c.c.c. Since Z is Urysohn, it follows from [8, 4K(7)] that X is Urysohn. □

So assuming CH, we obtain a Hausdorff space X such that

$$2^{c(X)\pi\chi(X)sd(X)} = \mathfrak{c} < 2^{\mathfrak{c}} = 2^{\pi w(X)}.$$

This shows that the bound provided by Proposition 3.2 is stronger than Frankiewicz's result from [4]. Note that in this case for Z, we can also take $\mathbb{R}$ or the unit circle in the plane. The previous construction also yields a Urysohn space X for which Šapirovskiĭ's inequality $d(X) \leq \pi\chi(X)^{c(X)}$ from [11] fails; assuming $\mathfrak{c}^+ = 2^{\mathfrak{c}}$, one can take $Z = \mathbb{I}^{\mathfrak{c}}$. The Urysohn space X provided by Proposition 4.1 is c.c.c. (since Z is c.c.c.); its π-character equals $\mathfrak{c}$ and $d(X) = \mathfrak{c}^+$, so, in this case, we have

$$d(X) = 2^{\mathfrak{c}} > \mathfrak{c} = \pi\chi(X)^{c(X)}.$$

This shows that [2, Corollary 2.5] cannot be proved for Urysohn spaces. Note that since Z is separable (cf. [6, 5.5]), the space X has countable separation degree, showing that the following sequence of strict inequalities is possible:

$$c(X)sd(X) < \pi\chi(X) < d(X).$$

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