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SOME NEW TOPOLOGICAL CARDINAL INEQUALITIES

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ABSTRACT. The aim of this paper is to generalize, using the language of elementary submodels, three cardinal inequalities. In particular, we will show that $|X| \leq g(X)^{L(X)t(X)}$ for every topological space. This result is a generalization of the following result due to Alan Dow: If X is a compact Hausdorff space with countable tightness, then $|X| \leq g(X)^{\omega}$ [*Closures of discrete sets in compact spaces*, Studia Sci. Math. Hungar. **42** (2005), no. 2, 227–234].

1. INTRODUCTION

A natural way of exploring the properties of a given space is studying its small (in some sense) subspaces and deciding whether they reflect the properties of the whole. For example, in [1], Ofelia T. Alas, Vladimir V. Tkachuk, and Richard G. Wilson obtain several results of the form that if the closure of each discrete subspace of X has a property $\mathcal{P}$, then X has $\mathcal{P}$. For Alas, Tkachuk, and Wilson, *small subspace* means it is the closure of a discrete subspace. In their paper they ask,

Let X be a compact space such that the closure of each discrete subspace of X has cardinality $\leq 2^{\omega}$. Is it true that $|X| \leq 2^{\omega}$?

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Recently, Alan Dow [6] proved that if X is a Hausdorff compact space with $t(X) = \omega$, then $|X| \leq g(X)^\omega$. (We will define $g(X)$ later in the paper.) This result gives a partial positive answer to the above question by Alas, Tkachuk, and Wilson. On the other hand, in [6], Dow asks,

Can each compact space of countable tightness be written as a union of at most 2^ω many sets, each of which has a dense discrete subset?

In [13], I. Juhász and Z. Szentmiklóssy proved '2^κ is a finite successor of κ for every cardinal κ' implies that in any countably tight compactum X there is a discrete subspace D with $|\overline{D}| = |X|$. This result yields that under the same assumption this is a positive consistent answer to Dow's question.

In this paper, we use the language of elementary submodels (see [5]) with two purposes: (1) to give a partial positive answer to Dow's question and strengthen Dow's inequality above and (2) to generalize two cardinal inequalities.

2. Notations and definitions

The reader should refer to [9] and [12] for definitions and terminology on cardinal functions not explicitly given here. Let L, s, t, χ, ψ, and ψ_c denote the standard cardinal functions of Lindelöf degree, spread, tightness, character, pseudocharacter, and closed pseudocharacter, respectively.

We now define additional cardinal functions that are not so well known. The following cardinal invariant, for Hausdorff spaces, was introduced by R. Hodel.

Definition 2.1 ([10])**.** The Hausdorff pseudocharacter of X, denoted $H\psi(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\mathcal{B}_x$ of open neighborhoods of x, such that

(1) $|\mathcal{B}_x| \leq \kappa$;
(2) if $x \neq y$, there are $V_x \in \mathcal{B}_x$ and $V_y \in \mathcal{B}_y$, with $V_x \cap V_y = \emptyset$.

We call $\mathcal{B}_x$ a Hausdorff pseudobase.

Clearly, $\psi_c(X) \leq H\psi(X) \leq \chi(X)$, for every Hausdorff space X.

In [15], Dimitrina N. Stavrova introduced a similar invariant for Urysohn spaces. Recall that a space X is a Urysohn space if for each $x, y \in X$ such that $x \neq y$, there are open neighborhoods U of x and V of y with $\overline{U} \cap \overline{V} = \emptyset$.

Definition 2.2 ([15])**.** The Urysohn pseudocharacter of X, denoted $U\psi(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\mathcal{B}_x$ of open neighborhoods of x, such that

(1) $|\mathcal{B}_x| \leq \kappa$;
(2) if $x \neq y$, there are $V_x \in \mathcal{B}_x$ and $V_y \in \mathcal{B}_y$, with $\overline{V_x} \cap \overline{V_y} = \emptyset$.

Clearly, $U\psi(X) \leq \chi(X)$ for every Urysohn space X.

Definition 2.3 ([16])**.** Let X be a topological space.

(1) The *almost Lindelöf degree of* X, denoted $aL(X, X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of X, there is $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ such that $X = \bigcup\{\overline{V} : V \in \mathcal{V}\}$.
(2) The *almost Lindelöf degree of* X *with respect to closed sets* $aL_c(X)$ is the least infinite cardinal κ such that for each closed set F in X and for each open collection $\mathcal{U}$ in X such that $F \subseteq \bigcup \mathcal{U}$, there is $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ such that $F \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$.

Note that $aL(X, X) \leq aL_c(X) \leq L(X)$ for every space X and equality holds for regular spaces. S. Willard and U. N. B. Dissanayake, in [16], give an example of a Hausdorff space, such as $aL(X, X) < aL_c(X) < L(X)$ (see also [11]).

Let X be a topological space and let A be a subset of X; $\overline{A}$ is the closure of A in X. For any set X and cardinal κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with cardinality $\leq \kappa$; $[X]^{<\kappa}$ and $[X]^{\kappa}$ are defined analogously.

A. V. Arhangelskiĭ introduced the following cardinal invariant.

Definition 2.4 ([2])**.** The depth of a topological space X, denoted $g(X)$, is defined as

$$g(X) = sup\{|\overline{D}| : D \text{ is discrete in } X\} + \omega.$$

It is clear that $s(X) \leq g(X)$ for every topological space (X).

Let X be a topological space and let κ be an infinite cardinal. A sequence $\{x_\alpha : 0 \leq \alpha < \kappa\}$ in X is a *free sequence of length* κ if for all $\beta < \kappa$, $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset$. (See [9], [12], [3].)

Theorem 2.5 ([3]). *If $L(X) \leq \kappa$ and $t(X) \leq \kappa$, then X does not have a free sequence with length κ^+.*

3. Main results

Before presenting the first result of our paper, we need the following definition.

Definition 3.1. Let X be a topological space and $Y \subseteq X$; we define the discrete κ-closure of Y, denoted as $[Y]^d_\kappa$, as the set $\bigcup\{\overline{D} : D \in [Y]^{\leq\kappa}$ such that D is discrete in $X\}$.

The following lemma, extracted from the middle of Theorem 2.5 in [6], is the key for our results in the first part of this paper.

Lemma 3.2 ([6], for compact Hausdorff spaces). *If X is a topological space with $L(X) \leq \kappa$ and $t(X) \leq \kappa$, then there is $Y \subseteq X$ with $|Y| \leq g(X)^\kappa$ such that $X = [Y]^d_\kappa$.*

Proof: Assume that $\mathfrak{M}$ is an elementary submodel of the universe such that $\{X, \tau, g(X)\} \cup g(X) \subseteq \mathfrak{M}$, $|\mathfrak{M}| \leq g(X)^\kappa$, and every subset of $\mathfrak{M}$ with cardinality less than or equal to κ is a member of $\mathfrak{M}$.

Let $Y = X \cap \mathfrak{M}$; it is clear that $|Y| \leq g(X)^\kappa$. The proof will be complete if we can show that $X = [Y]^d_\kappa$. Suppose it is not and let $p \in X \setminus [Y]^d_\kappa$. Let $\mathcal{U}$ denote the collection of all open $U \in \mathfrak{M}$ such that $p \notin \overline{U}$.

Claim. $\mathcal{U}$ is an open cover of $X \cap \mathfrak{M}$. Indeed, let $y \in Y$ and $\mathcal{U}_y \in \mathfrak{M}$ denote the collection of open sets $U \subseteq X$ such that $y \notin \overline{U}$. According to Šapirovskiĭ's lemma (see [12], 4.12), there is a discrete set $D_y \in \mathfrak{M}$ and a subcollection $\mathcal{W}_y$ with cardinality equal to that of D_y such that $X \setminus \{y\} = \overline{D_y} \cup \bigcup \mathcal{W}_y$. Since $|\mathcal{W}| = |D_y| \leq g(X)$ and $\mathcal{W}_y \in \mathfrak{M}$, we have that $\mathcal{W}_y \subseteq \mathfrak{M}$. In addition, since D_y is discrete and $t(X) \leq \kappa$, $p \notin \overline{D_y}$; hence, there is some $W_y \in \mathcal{W}_y$ such that $p \in W_y$. Since $y \notin \overline{W_y}$, we have that there is an open $U_y = X \setminus \overline{W_y} \in \mathfrak{M}$ such that $y \in U_y$ and $p \notin \overline{U}_y$. We have shown that $\mathcal{U}$ is an open cover of $X \cap \mathfrak{M}$.

We begin choosing points $\{y_\alpha : \alpha \in \kappa^+\} \subseteq Y$ and collections $\mathcal{U}_\alpha \subseteq \mathcal{U}$ with $|\mathcal{U}_\alpha| \leq \kappa$ such that

(1) $\{y_\beta : 0 \leq \beta < \alpha\}$ is discrete, for $0 < \alpha < \kappa$;
(2) $\overline{\{y_\beta : 0 \leq \beta < \alpha\}} \subseteq \mathcal{U}_\alpha$;
(3) $y_\alpha \in Y \setminus \bigcup_{\beta \leq \alpha} \mathcal{U}_\beta$.

Let $\mathcal{U}_0$ be any subset of $\mathcal{U}$ with $|\mathcal{U}| \leq \kappa$. Since $p \notin \bigcup \mathcal{U}_0$ and $\mathcal{U}_0 \in \mathfrak{M}$, we may, by elementarity, select some $y_0 \in Y \setminus \bigcup \mathcal{U}_0$. Let $0 < \alpha < \kappa^+$ and suppose that for each $\beta < \alpha$, we have chosen $y_\beta \in Y$ such that (1) and (2) hold. Clearly, $p \notin \overline{\{y_\beta : \beta < \alpha\}}$. Because of our assumptions on $\mathfrak{M}$, $\{y_\beta : \beta < \alpha\}$ is a member of $\mathfrak{M}$ and so is $\overline{\{y_\beta : \beta < \alpha\}}$. Since $|\overline{\{y_\beta : \beta < \alpha\}}| \leq g(X)$, then $\overline{\{y_\beta : \beta < \alpha\}} \subseteq \mathfrak{M}$. Since $L(X) \leq \kappa$, then there is a collection $\mathcal{U}_\alpha \subseteq \mathcal{U}$ with $|\mathcal{U}_\alpha| \leq \kappa$ which covers it. We may choose $y_\alpha \in Y \setminus \bigcup_{\beta \leq \alpha} \bigcup \mathcal{U}_\beta$. This completes the construction.

The sequence $\{y_\alpha : \alpha < \kappa^+\}$ we are constructing would be a free sequence in X, which would contradict Theorem 2.5. □

From Lemma 3.2, we can obtain several results. The first result is a partial answer to Dow's question mentioned in the introduction of our paper.

Theorem 3.3. *If X is a compact T_1 space with $t(X) = \omega$ and $\psi(X) = \omega$, then X is a union of at most $\mathfrak{c}$ many sets, each of which has a dense discrete subset.*

Proof: By A. A. Gryslov's theorem [8], we have that $g(X) \leq 2^\omega$; hence, by Lemma 3.2, there is $Y \subseteq X$ with $|Y| \leq 2^\omega$ such that $X = [Y]^d_\omega$. □

The following result is a generalization of Dow's inequality.

Theorem 3.4. *If X is a topological space, then $|X| \leq g(X)^{L(X)t(X)}$.*

Proof: Let $L(X)t(X) = \kappa$; by Lemma 3.2, there is $Y \subseteq X$ with $|Y| \leq g(X)^\kappa$ such that $X = [Y]^d_\kappa$. Then $|X| \leq |\bigcup\{\overline{D} : D \in [Y]^{\leq\kappa}, \text{ D discrete}\}| \leq g(X)^\kappa$. □

Corollary 3.5 (Dow [6])**.** *If X is a compact Hausdorff space with countable tightness, then $|X| \leq g(X)^\omega$.*

Now we turn to the second part of this paper. In [10], Hodel introduced the Hausdorff pseudocharacter and proved that $|X| \leq 2^{L(X)H\psi(X)}$ for every Hausdorff space X. Since $aL_c(X) \leq L(X)$ for every topological space, it is natural to ask if L can be replaced by aL_c in this inequality. In [14], Stavrova obtains an affirmative answer to this question. We will use the elementary submodels technique to show it. But first, we need a result due to Hodel.

Let κ be an infinite cardinal and let X be a set. Suppose that for each $x \in X$, $\mathcal{V}_x$ is a family of subsets of X which contain x. For every $L \subseteq X$, let $L^* = \{x \in X : V \cap L \neq \emptyset, \text{for all } V \in \mathcal{V}_x\}$ be as defined by Hodel in [10].

Theorem 3.6 ([10])**.** *Let κ be an infinite cardinal and let X be a set. If for each $x \in X$, $\mathcal{V}_x = \{V_\gamma(x) : \gamma < \kappa\}$ is a family of subsets of X which contain x such that if $x \neq y$, there exists $\gamma \in \kappa$ such that $V_\gamma(x) \cap V_\gamma(y) = \emptyset$. Then*

(1) $|L^*| \leq |L|^\kappa$;
(2) *if $L = \bigcup_{\alpha<\kappa^+} E^*_\alpha$, where $\{E_\alpha : 0 \leq \alpha < \kappa^+\}$ is a sequence of subsets of X with $\bigcup_{\beta<\alpha} E^*_\beta \subseteq E_\alpha$ for all $\alpha < \kappa^+$, then $L^* = L$.*

Now we are ready to state and prove Stavrova's theorem.

Theorem 3.7 ([14])**.** *If X is a Hausdorff space, then $|X| \leq 2^{aL_c(X)H\psi(X)}$.*

Proof: Let $\kappa = aL_c(X) \cdot H\psi(X)$, $\lambda = 2^\kappa$, and assume that $\mathfrak{M}$ is an elementary submodel of some sufficiently large fragment of the universe such that X and τ are members of $\mathfrak{M}$, the cardinal λ is a subset of $\mathfrak{M}$, $|\mathfrak{M}| \leq 2^\kappa$, and every subset of $\mathfrak{M}$ with cardinality less than or equal to κ is a member of $\mathfrak{M}$.

For each $x \in X$, fix an open neighborhood collection $\mathcal{B}_x$ of X such that $|\mathcal{B}_x| \leq \kappa$, and such that if $x \neq y$, then there are $V_x \in \mathcal{B}_x$ and $V_y \in \mathcal{B}_y$ such that $V_x \cap V_y = \emptyset$.

Claim. $X \cap \mathfrak{M} = (X \cap \mathfrak{M})^* = \{x \in X : V \cap L \neq \emptyset, \text{for all } V \in \mathcal{B}_x\}$. Indeed, it is clear that $X \cap \mathfrak{M} \subseteq (X \cap \mathfrak{M})^*$. Let $x \in (X \cap \mathfrak{M})^*$; then for each $U \in \mathcal{B}_x$, we have $U \cap (X \cap \mathfrak{M}) \neq \emptyset$, so there is $y_U \in U \cap (X \cap \mathfrak{M})$. Let $A = \{y_U : U \in \mathcal{B}_x\}$; since $A \subseteq \mathfrak{M}$ and $|A| \leq \kappa$, then $A \in \mathfrak{M}$, so $A^* \in \mathfrak{M}$. Since $|A| \leq \kappa$, then $|A^*| \leq 2^\kappa$ for Lemma 3.6; thus, $A^* \subseteq \mathfrak{M}$. Since $x \in A^*$, then $x \in \mathfrak{M}$. Therefore, $X \cap \mathfrak{M} = (X \cap \mathfrak{M})^*$.

It follows from the previous claim that $X \cap \mathfrak{M}$ is closed.

The proof will be complete if we can show $X = X \cap \mathfrak{M}$. Assume that there is $p \in X \setminus \mathfrak{M}$. Let $y \in X \cap \mathfrak{M}$; since $\mathcal{B}_y \in \mathfrak{M}$ and $|\mathcal{B}| \leq \kappa$, then $\mathcal{B}_y \subseteq \mathfrak{M}$. Since $p \neq y$, then there are $U_y \in \mathcal{B}_y$ and $V_p \in \mathcal{B}_p$ such that $U_y \cap V_p = \emptyset$; thus, $p \notin \overline{U_y}$.

Let $\mathcal{U} = \{U_y \in \mathcal{B}_y : y \in X \cap \mathfrak{M}\}$; it is clear that $\mathcal{U} \subseteq \mathfrak{M}$ is an open collection in X which covers the closed set $X \cap \mathfrak{M}$. Since

$aL_c(X) \leq \kappa$, there is $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ such that $X \cap \mathfrak{M} \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$. Since each $V \in \mathfrak{M}$, then $\overline{V} \in \mathfrak{M}$; thus, $\{\overline{V} : V \in \mathcal{V}\} \subseteq \mathfrak{M}$ and $|\{\overline{V} : V \in \mathcal{V}\}| \leq \kappa$, so we have that $\{\overline{V} : V \in \mathcal{V}\} \in \mathfrak{M}$, so $\bigcup\{\overline{V} : V \in \mathcal{V}\} \in \mathfrak{M}$. Then $\bigcup\{\overline{V} : V \in \mathcal{V}\}$ covers X, which is a contradiction because $p \notin \bigcup\{\overline{V} : V \in \mathcal{V}\}$. Therefore, $X = X \cap \mathfrak{M}$. Thus, $|X| \leq 2^{aL_c(x)H\psi(X)}$. □

Corollary 3.8 ([10])**.** *If X is a Hausdorff space, then $|X| \leq 2^{L(X)H\psi(X)}$.*

Corollary 3.9 (Arhangel'skiĭ)**.** *If X is a T_2-space, then $|X| \leq 2^{L(X)\chi(X)}$.*

Definition 3.10 ([3])**.** Let X be a topological space and let κ be an infinite cardinal. We say that X is strictly κ-quasi-Lindelöf if for every closed subset F of X and every collection $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ of open sets in X with $F \subseteq \bigcup\{\bigcup\mathcal{U}_\alpha : \alpha < \kappa\}$, we have that for each $\alpha < \kappa$, there is $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{\leq\omega}$ such that $F \subseteq \bigcup\{\overline{\bigcup\mathcal{V}_\alpha} : \alpha < \kappa\}$. If $\kappa = \omega$, we say strictly quasi-Lindelöf instead of strictly ω-quasi-Lindelöf.

It was proved in [3] that if X is a first countable Hausdorff strictly quasi-Lindelöf space, then $|X| \leq 2^\omega$. Since $H\psi(X) \leq \chi(X)$ for every topological space, it is natural to ask if χ can be replaced by $H\psi$ in this inequality. The next theorem gives an affirmative answer to this question.

Theorem 3.11. *Let X be a Hausdorff strictly κ-quasi-Lindelöf space. If $H\psi(X) \leq \kappa$, then $|X| \leq 2^\kappa$.*

Proof: Let $\lambda = 2^\kappa$; assume that $\mathfrak{M}$ is an elementary submodel of some sufficiently large fragment of the universe such that $\{X, \tau, \lambda\} \cup \lambda \subseteq \mathfrak{M}$, $|\mathfrak{M}| \leq 2^\kappa$, and every subset of $\mathfrak{M}$ with cardinality less than or equal to κ is a member of $\mathfrak{M}$.

For each $x \in X$, we consider a Hausdorff pseudobase $\mathcal{B}_x$ with $|\mathcal{B}_x| \leq \kappa$.

Note that $X \cap \mathfrak{M} = (X \cap \mathfrak{M})^* = \{x \in X : V \cap L \neq \emptyset, \text{for all } V \in \mathcal{V}_x\}$; hence, $X \cap \mathfrak{M}$ is closed.

Claim. $X = X \cap \mathfrak{M}$. Assume that there is $p \in X \setminus \mathfrak{M}$. For each $V \in \mathcal{B}_p$, let $\mathcal{U}_V = \{U \in \mathcal{B}_x : U \cap V = \emptyset,\, x \in X \cap \mathfrak{M}\} \subseteq \mathfrak{M}$; since $x \in \mathfrak{M}$, then $\mathcal{B}_x \subseteq \mathfrak{M}$ because $|\mathcal{B}_x| \leq \kappa$. It is clear that $X \cap \mathfrak{M} \subseteq \bigcup\{\bigcup\mathcal{U}_V : V \in \mathcal{B}_p\}$; then for each $V \in \mathcal{B}_p$, there is a

$\mathcal{V}_V \in [\mathcal{U}_V]^{\leq\omega}$ such that $X \cap \mathfrak{M} \subseteq \bigcup\{\overline{\bigcup \mathcal{V}_V} : V \in \mathcal{B}_p\}$. Since $\mathcal{V}_V \subseteq \mathfrak{M}$ and $|\mathcal{V}_V| \leq \kappa$, then $\mathcal{V}_V \in \mathfrak{M}$, so that $\bigcup \mathcal{V}_V \in \mathfrak{M}$ and $\overline{\bigcup \mathcal{V}_V} \in \mathfrak{M}$; then $\{\overline{\bigcup \mathcal{V}_V : V \in \mathcal{B}_p}\} \subseteq \mathfrak{M}$, and since $|\mathcal{B}_p| \leq \kappa$, we have that $|\{\overline{\bigcup \mathcal{V}_V : V \in \mathcal{B}_p}\}| \leq \kappa$; it is easy to prove that $\bigcup\{\bigcup \mathcal{U}_V : V \in \mathcal{B}_p\} \in \mathfrak{M}$. Then $\bigcup\{\overline{\bigcup \mathcal{V}_V : V \in \mathcal{B}_p}\}$ covers X, which is a contradiction because $p \notin \bigcup\{\overline{\bigcup \mathcal{V}_V : V \in \mathcal{B}_p}\}$. □

Corollary 3.12 ([4]). *Let X be a Hausdorff strictly κ-quasi-Lindelöf space. If $\chi(X) \leq \kappa$, then $|X| \leq 2^\kappa$.*

Alessandro Fedeli has proved in [7] that if X is a Hausdorff space and Y is an ωH-subset of X, then $|Y| \leq 2^{t_{s\theta}(X)\psi_c(X)}$. Recall that $Y \subseteq X$ is a κH-set of X if for every open family $\mathcal{U}$ in X such that $Y \subseteq \bigcup \mathcal{U}$, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $Y \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$. (The definition of $t_{s\theta}$ can be found in [7].)

The following result shows that if we consider Urysohn spaces, we can use $U\psi$ (a local cardinal function that captures the Urysohn property) which is a strengthening of ψ_c, and thus, $t_{s\theta}$ can be omitted as a hypothesis in Fedeli's inequality.

Theorem 3.13. *Let X be a Urysohn space. If Y is a $U\psi(X)H$-set, then $|Y| \leq 2^{U\psi(X)}$.*

Proof: Let $\kappa = U\psi(X)$ and $\lambda = 2^\kappa$. Assume that $\mathfrak{M}$ is an elementary submodel of a sufficiently large fragment of the universe such that $\{X, \tau, Y, \lambda\} \cup \lambda \subseteq \mathfrak{M}$, $|\mathfrak{M}| \leq 2^\kappa$, and every subset of $\mathfrak{M}$ with cardinality less than or equal to κ is a member of $\mathfrak{M}$.

For each $x \in X$, fix an open neighborhood collection $\mathcal{B}_x$ of X, such that $|\mathcal{B}_x| \leq \kappa$, and for each $x \neq y$, there is $V_x \in \mathcal{B}_x$ and $V_y \in \mathcal{V}_y$ such that $\overline{V_x} \cap \overline{V_y} = \emptyset$. Let $\mathcal{V}_x = \{\overline{V} : V \in \mathcal{B}_x\}$ for every $x \in X$.

Note that $X \cap \mathfrak{M} = (X \cap \mathfrak{M})^* = \{x \in X : V \cap L \neq \emptyset, \text{for all } V \in \mathcal{V}_x\}$; hence, $X \cap \mathfrak{M}$ is closed in X.

We will show that $Y \subseteq X \cap \mathfrak{M}$. Assume that there is $p \in Y \setminus \mathfrak{M}$. For each $x \in X \cap \mathfrak{M}$, let $U_x \in \mathcal{B}_x$ such that $p \notin \overline{U_x}$. Note that $\mathcal{B}_x \in \mathfrak{M}$; moreover, $\mathcal{B}_x \subseteq \mathfrak{M}$ because $|\mathcal{B}_x| \leq \kappa$. For each $x \in Y \setminus \mathfrak{M}$, let U_x be such that $\overline{U_x} \cap (X \cap \mathfrak{M}) = \emptyset$, which is possible because $X \cap \mathfrak{M} = (X \cap \mathfrak{M})^*$. Let $\mathcal{U} = \{U_x : x \in Y\}$; then Y is covered by $\mathcal{U}$. Since Y is a κH-set in X, then there is a set $C \subseteq Y$ with $|Y| \leq \kappa$, such that $Y \subseteq \bigcup\{\overline{U_x} : x \in C\}$.

If $y \in Y \cap \mathfrak{M}$, then there is an $x \in C$ such that $y \in \overline{U_x}$, so $y \in \overline{U_x} \cap \mathfrak{M}$; thus, $\overline{U_x} \cap \mathfrak{M} \neq \emptyset$, which implies that $x \in \mathfrak{M}$. Therefore, $Y \cap \mathfrak{M} \subseteq \bigcup\{\overline{U_x} : x \in C \cap \mathfrak{M}\}$.

Since $\overline{U_x} \in \mathfrak{M}$ for each $x \in C \cap \mathfrak{M}$, we have that the set $\{\overline{U_x} : x \in C \cap \mathfrak{M}\} \subseteq \mathfrak{M}$, so $\bigcup\{\overline{U_x} : C \cap \mathfrak{M}\} \in \mathfrak{M}$. Then Y is covered by $\bigcup\{\overline{U_x} : x \in C \cap \mathfrak{M}\}$, which is a contradiction because $p \notin \{\overline{U_x} : x \in C \cap \mathfrak{M}\}$. Therefore, $Y \subseteq X \cap \mathfrak{M}$ and so $|Y| \leq 2^{U\psi(X)}$. □

Corollary 3.14 ([15]). *If X is a Urysohn space, then $|X| \leq 2^{aL(X,X)U\psi(X)}$.*

However, at the moment, the authors do not know the answer to the next question.

Question 3.15. *Let Y be a κH-set in a Hausdorff space X. Is it true that $|Y| \leq 2^{H\psi(X)}$?*

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