TOPOLOGY PROCEEDINGS
**Volume 32, 2008**
Pages 245–252

**http://topology.auburn.edu/tp/**

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Electronically published on July 17, 2008

**Topology Proceedings**

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
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Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

http://topology.auburn.edu/tp/

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Volume 32 (2008)
Pages 245-252

E-Published on July 17, 2008

# SELECTIVE SCREENABILITY AND THE HUREWICZ PROPERTY

LILJANA BABINKOSTOVA

Abstract. We characterize the Hurewicz covering property in metrizable spaces in terms of properties of the metrics of the space. Then we show that a weak version of selective screenability, when combined with the Hurewicz property, implies selective screenability.

## 1. Definitions and notation

Let $X$ be an infinite set and let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of $X$. The selection principle $\mathsf{S}_c(\mathcal{A},\mathcal{B})$, introduced in [2], states that for each sequence $(A_n : n < \infty)$ of elements of the family $\mathcal{A}$ there exists a sequence $(B_n : n < \infty)$ such that for each $n$, $B_n$ is a pairwise disjoint family refining $A_n$ and $\bigcup_{n<\infty} B_n$ is a member of the family $\mathcal{B}$. For $X$, topological space $\mathcal{O}$ denotes the collection of all open covers of $X$ and $\mathcal{O}_{fin}$ denotes the collection of all finite open covers of $X$. For a positive integer $n$, let $\mathcal{O}_n$ denote the collection of open covers consisting of at most $n$ sets. David F. Addis and John H. Gresham introduced the instance $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ of the selection principle in [1], where it was called property $C$. It is a selective version of the screenability property introduced by R. H. Bing in [4].

2000 *Mathematics Subject Classification.* Primary 54D20, 54D45, 55M10; Secondary 03E20.

*Key words and phrases.* Haver property, Hurewicz property, Menger property, selection principle, selective screenability, $\sigma$-totally bounded, totally bounded.

As was shown in [1], $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is a natural generalization of finite covering dimension to the infinite. Alexandroff's notion of weakly infinite-dimensional is also a natural generalization of finite covering dimension and is equivalent to $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$. Witold Hurewicz's notion of countable dimensionality is another natural generalization of finite covering dimension: $X$ is countable dimensional if it is a union of countably many finite dimensional subspaces. The following implications hold. (See [1].)

$$\text{countable dimensional} \Rightarrow \mathsf{S}_c(\mathcal{O},\mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}_2,\mathcal{O}).$$

The Hilbert cube, $[0,1]^{\mathbb{N}}$, does not have property $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$ [1]. Piet Borst proved in [5] that there exists a compact separable metric space $X$ which has property $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$, but not property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Since for compact spaces $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O}) \Leftrightarrow \mathsf{S}_c(\mathcal{O},\mathcal{O})$, Borst's example shows that $\mathsf{S}_c(\mathcal{O}_2,\mathcal{O})$ does not imply $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$. Roman Pol [14] constructed a compact metric space which has property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ but is not countable dimensional. If $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ implies $\mathsf{S}_c(\mathcal{O},\mathcal{O})$, it is an open problem. (See [6, Question 3.10].) We expect that the answer to this question is "No," and state a conjecture about it near the end of this paper. In [6], a class of spaces which does not distinguish $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is identified. In this paper, we will extend this to a larger class of separable metric spaces which does not distinguish $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Examples show that the class we describe properly extends the class from [6].

First, we first give a convenient characterization of the Hurewicz property in metrizable spaces. Next, we show that metrizable spaces with the Hurewicz property do not distinguish $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Then, we connect this with Borst's work in [6], and finally, we state a conjecture.

## 2. Characterizing the Hurewicz property in metrizable spaces

A topological space $X$ has the *Hurewicz property* [11] if for each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of $X$ there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each $n$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each element of $X$ is in all but finitely many of the sets $\cup\mathcal{V}_n$. The metrizable space $X$ is said to be *Haver* [9] with respect to a metric $d$ if for each sequence $(\epsilon_n : n < \infty)$ of positive reals there is a sequence $(\mathcal{V}_n : n < \infty)$ where each $\mathcal{V}_n$ is a pairwise disjoint family of open sets, each of $d$-diameter less than $\epsilon_n$, such that $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of $X$.

A metric space $(X, d)$ is totally bounded if for each $\epsilon > 0$ there is a finite set $F \subset X$ such that $X \subseteq \bigcup_{f \in F} B_d(f, \epsilon)$, where $B_d(f, \epsilon) = \{x \in X : d(x, f) < \epsilon\}$. A metric space is $\sigma$-totally bounded if it is a union of countably many subsets, each totally bounded.

**Theorem 1.** *Let $(X, d)$ be a metrizable space. The following are equivalent.*

(1) *$X$ has the Hurewicz property.*
(2) *$X$ is $\sigma$-totally bounded in each equivalent metric.*

*Proof:* $1 \Rightarrow 2$: For each $n$, let $\delta_n = (1/2)^n$ and $\mathcal{U}_n = \{B_d(x, \delta_n) : x \in X\}$ where $d$ is an arbitrary fixed metric of $X$. Apply the Hurewicz property to $(\mathcal{U}_n : n < \infty)$. For each $n$, choose a finite set $\mathcal{V}_n \subset \mathcal{U}_n$ such that each $x \in X$ is in all but finitely many of the sets $\cup\mathcal{V}_n$. For each $n$, define $X_n = \bigcap_{m \geq n} \cup\mathcal{V}_m$. Then for each $n$ and for $m \leq n$, $X_m \subseteq X_n$, and $\bigcup_{n<\infty} X_n$ covers $X$. We show that each $X_n$ is totally bounded in the metric $d$: Consider an $\epsilon > 0$ and consider any $X_n$. Choose $m > n$ so large that $2 \cdot (1/2)^m \leq \epsilon$. Each element of $\mathcal{V}_m$ is an open set of diameter less than $2 \cdot (1/2)^m$, and $\mathcal{V}_m$ is a finite cover of $X_n$.

$2 \Rightarrow 1$: Let $(\mathcal{U}_n : n < \infty)$ be a sequence of open covers of $X$. By [8, Remark 4, p 196], let $d$ be a metric generating the topology of $X$ such that for each $n$, $\mathcal{W}_n = \{B_d(x, 1/n) : x \in X\}$ refines $\mathcal{U}_n$. Write $X = \bigcup_{n<\infty} X_n$, where each $X_n$ is totally bounded and $X_n \subset X_{n+1}$ for every $n$. Choose for each $m$ a finite $\mathcal{F}_m \subset \mathcal{W}_m$ with $X_m \subseteq \cup\mathcal{F}_m$. Then, for each $m$, choose a finite $\mathcal{V}_m \subset \mathcal{U}_m$ such that $\mathcal{F}_m$ refines $\mathcal{V}_m$. Then, for each $x \in X$,? for all but finitely many $n$, $x \in \cup\mathcal{V}_n$. $\square$

## 3. $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ in metrizable spaces with the Hurewicz property

For easy reference, we denote the following strong form of $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ by the symbol $\mathsf{S}_c^+(\mathcal{O}_{fin}, \mathcal{O})$.

> For each sequence $(\mathcal{U}_n : n < \infty)$ of finite open covers of $X$ there are a sequence $(\mathcal{W}_n : n < \infty)$ and a sequence $m_1 < m_2 < ... < m_k < ...$ such that

(1) each $\mathcal{W}_n$ is a finite pairwise disjoint family of open sets,
(2) each $\mathcal{W}_n$ refines $\mathcal{U}_n$, and
(3) for each $x \in X$, for all but finitely many $k$, there is a $j \in [m_k, m_{k+1})$ with $x \in \cup\mathcal{W}_j$.

**Lemma 2.** *Let $(X,d)$ be a metrizable space. If $X$ has $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ and the Hurewicz property, then it has the property $\mathsf{S}_c^+(\mathcal{O}_{fin},\mathcal{O})$.*

*Proof:* Recall that $X$ has the Hurewicz property if and only if ONE has no winning strategy in the Hurewicz game (Theorem 27 of [15]). Let $(\mathcal{U}_n : n < \infty)$ be a sequence of finite open covers of $X$. Applying $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ to $(\mathcal{U}_n : n < \infty)$, choose for each $n$ a pairwise disjoint refinement $\mathcal{V}_n^1$ of $\mathcal{U}_n$ so that $F(\emptyset) = \bigcup_{n<\infty} \mathcal{V}_n^1$ covers $X$. This defines ONE's first move in the Hurewicz game. When TWO chooses a finite $T_1 \subset F(\emptyset)$, define $m_1 = min\{n : T_1 \subseteq \bigcup_{j<n} \mathcal{V}_j^1\}$. Next, apply $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ to $(\mathcal{U}_n : n \geq m_1)$ and choose for each $n \geq m_1$ a pairwise disjoint $\mathcal{V}_n^2$ that refines $\mathcal{U}_n$ consisting of open sets, so that $F(T_1) = \bigcup_{n \geq m_1} \mathcal{V}_n^2$ covers $X$. This defines ONE's response to TWO's move $T_1$. When TWO chooses a finite $T_2 \subset F(T_1)$, define $m_2 = min\{n : T_2 \subseteq \bigcup_{m_1 \leq j < n} \mathcal{V}_j^2\}$ and apply $\mathsf{S}_c(\mathcal{O}_{fin},\mathcal{O})$ to $(\mathcal{U}_n : n \geq m_2)$ to define $F(T_1,T_2)$, and so on.

Since $X$ has the Hurewicz property, $F$ is not a winning strategy for ONE. Consider an $F$-play: $F(\emptyset), T_1, F(T_1), T_2, F(T_1,T_2), T_3...$ lost by ONE. Then each $T_m$ is finite and each $x \in \cup T_m$ for all but finitely many $m$. For $j < m_1$, define $\mathcal{W}_j = \{T \in T_1 : T \in \mathcal{V}_j^1\}$. For $m_k \leq j < m_{k+1}$, define $\mathcal{W}_j = \{T \in T_{k+1} : T \in \mathcal{V}_j^{k+1}\}$. For each $j$, $\mathcal{W}_j$ is finite pairwise disjoint and refines $\mathcal{U}_j$. □

**Theorem 3.** *If $(X,d)$ is $\sigma$-totally bounded and has property $\mathsf{S}_c^+(\mathcal{O}_{fin},\mathcal{O})$, then $X$ has the Haver property in $d$.*

*Proof:* Write $X = \bigcup_{n<\infty} X_n$, where each $X_n \subset X$ is $d$-totally bounded and $X_n \subset X_{n+1}$. Let $(\epsilon_n : n < \infty)$ be a sequence of positive reals. By replacing $\epsilon_n$'s if necessary, we may assume that $\epsilon_{n+1} < \frac{1}{2}\cdot\epsilon_n$ always. For each $n$, put $\delta_n = \frac{2^n-1}{2^n}\cdot(\frac{1}{3}\cdot\epsilon_n)$. For each $n$, choose a finite set $F_n \subset X_n$ such that $\{B(x,\delta_n) : x \in F_n\}$ covers $X_n$, and put $\mathcal{U}_n = \{B(x,\frac{1}{3}\cdot\epsilon_n) : x \in F_n\} \bigcup \{X \setminus \bigcup\{\overline{B(x,\delta_n)} : x \in F_n\}$, a finite open cover of $X$. Observe that for each $n$, $\overline{B(x,\delta_n)} \subset B(x,\epsilon_n)$, and $X_n \bigcap (X \setminus \bigcup\{\overline{B(x,\delta_n)} : x \in F_n\}) = \emptyset$.

Apply $\mathsf{S}_c^+(\mathcal{O}_{fin}, \mathcal{O})$ to the sequence $(\mathcal{U}_n : n < \infty)$. For each $n$, find a finite pairwise disjoint refinement $\mathcal{H}'_n$ of $\mathcal{U}_n$ and find a sequence $m_1 < m_2 < ... < m_k < ...$ such that for each $x \in X$, for all but finitely many $k$, there is a $j$ with $m_k \leq j < m_{k+1}$ and $x \in \cup\mathcal{H}'_j$. Now for each $n$, put

$$\mathcal{H}_n = \{V \in \mathcal{H}'_n : (\exists x \in F_n)(V \subseteq B(x, \frac{1}{2} \cdot \epsilon_n))\}.$$

Claim. $\bigcup_{n<\infty} \mathcal{H}_n$ covers $X$.
Consider $x \in X$. Choose $N$ so large so that for all $n \geq N$, $x \in X_n$, and for all $m_k \geq N$, there is $j \in [m_k, m_{k+1})$ with $x \in \cup\mathcal{H}'_j$. Choose $k$ with $m_k \geq N$ and $j$ with $m_k \leq j < m_{k+1}$ with $x \in V$ for some $V \in \mathcal{H}'_j$. We have that $x \in X_j$, so $V$ is not a subset of $X \setminus (\bigcup\{\overline{B(y, \delta_j)} : y \in F_j\})$, which means that $V \in \mathcal{H}_j$.

Since the diameter of any element of an $\mathcal{H}_n$ is less than $\epsilon_n$, the sequence $(\mathcal{H}_n : n < \infty)$ witnesses the Haver property of $X$ for $(\epsilon_n : n < \infty)$. □

Note that the Hurewicz property plus $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ does not imply the Haver property. If this were to imply the Haver property, then by Theorem 1 of [3], it would follow that $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ plus the Hurewicz property implies $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. Compactness implies the Hurewicz property, and [5] shows that $\mathsf{S}_c(\mathcal{O}_2, \mathcal{O})$ plus compact does not imply $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.

**Theorem 4.** *If $X$ is a metrizable space and has the Hurewicz property, then the following are equivalent.*

(1) *$X$ has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*
(2) *$X$ has $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$.*

*Proof:* $1 \Rightarrow 2$: It is clear.

$2 \Rightarrow 1$: By Theorem 3, $X$ has the Haver property. By Theorem 1 from [3], we have that $X$ has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. □

## 4. An extension of the class of finite C-spaces

In §3 of [6], Borst introduces the notion of a finite C-space: A topological space $X$ is a *finite C-space* if there is, for each sequence $(\mathcal{U}_n : n < \infty)$ of finite open covers of $X$, an $n$ and a sequence $(\mathcal{V}_j : j \leq n)$ such that each $\mathcal{V}_j$ is a disjoint refinement of $\mathcal{U}_j$ and $\bigcup_{j\leq n} \mathcal{V}_j$ is an open cover of $X$. And a space $X$ is said to have condition $K$ if it has a compact subset $C$ such that for every open subset $U$ of $X$ with $C \subset U$, the set $X \setminus U$ is finite dimensional. And in Theorem 3.8 of [6], the following equivalence is proved.

**Theorem 5** (Borst)**.** *For separable metric spaces $X$, the following are equivalent.*

(1) *$X$ is a finite C-space.*
(2) *$X$ has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and condition K.*

Thus, also in the class of spaces with condition K, $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ is equivalent to $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$. And there are spaces with condition K and $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ which do not have the Hurewicz property: Let $C$ be the compact metric space from [14]. It has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and is infinite dimensional. Let $P$ be the space of irrational numbers. Then $X$, the topological sum of $C$ and $P$, has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and condition K. It is well known that the closed subset P of $X$ does not have the Hurewicz property, and therefore, $X$ does not have the Hurewicz property.

As pointed out in [6], the space $\mathsf{K}_\omega$, consisting of the elements $x$ of $[0,1]^{\mathbb{N}}$ for which $x(n) > 0$ for only finitely many $n$, is not a finite C-space. For if it were a finite C-space, then by Theorem 1.2 of [6], it has a compactification with property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. But no compactification of $\mathsf{K}_\omega$ has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. (See [7, Example 5.3.6].) But $\mathsf{K}_\omega$ is $\sigma$-compact and so has the Hurewicz property, and it is countable dimensional, and so it has property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.

**Corollary 6.** *Let $X$ be a separable metric space which has an $\mathsf{F}_\sigma$ subset $C$ such that $C$ has the Hurewicz property, and for every open set $U \subset X$ with $C \subset U$, $X \setminus U$ has $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. Then the following are equivalent.*

(1) *$X$ has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*
(2) *$X$ has the property $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$.*

The proof uses the fact that $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ and $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ are preserved by $\mathsf{F}_\sigma$-subsets.

## 5. Remarks

In [10], Hurewicz introduced a property weaker than the Hurewicz property, known as the Menger property: For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of a space $X$, there is a sequence $(\mathcal{V}_n :$

$n < \infty$) of finite sets such that for each $n$, $\mathcal{V}_n \subset \mathcal{U}_n$, and $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of $X$. Theorem 3 shows that if a metrizable space has the Hurewicz property and also $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$, then it has the Haver property. We have the following conjecture.

**Conjecture 1.** *There is a metrizable space $X$ with the Menger property and $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$, which does not have the Haver property in some metric.*

Note that Conjecture 1 implies that the answer to Borst's Question 3.10 [6] is "no."

We also expect that for each $n > 1$, the implication $\mathsf{S}_c(\mathcal{O}_n, \mathcal{O}) \Rightarrow \mathsf{S}_c(\mathcal{O}_{n+1}, \mathcal{O})$ is false.

In Remark D of [12], Elzbieta Pol and Roman Pol showed that a metrizable space has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ if and only if it has the Haver property in all equivalent metrics. This provides another way to conclude Theorem 4: By theorems 1 and 3, we see that the Hurewicz property and $\mathsf{S}_c(\mathcal{O}_{fin}, \mathcal{O})$ imply the Haver property for all equivalent metrics. In addition, by [13, Remark D], Conjecture 1 translates to the statement that Theorem 4 fails if the Hurewicz property is replaced with the Menger property.

**Acknowledgment.** The author thanks Elzbieta and Roman Pol for communicating the inspiring results in [12] and [13].

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