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A BASE, A QUASI-BASE, AND A MONOTONE NORMALITY OPERATOR FOR $C_k(P)$

KENICHI TAMANO

Abstract. The following three constructions are given for the space $C_k(P)$ of all continuous real-valued functions on the space of irrationals with the compact-open topology:

(1) a σ-closure-preserving base,
(2) a σ-closure-preserving quasi-base, and
(3) a monotone normality operator.

We prove more generally that $C_k(X)$ is an M_1-space for any Polish space X, which answers a question of P. M. Gartside and E. A. Reznichenko.

Introduction

A space X is *stratifiable* if for each closed set F of X and $n \in \omega$, we can assign an open set $G_n(F)$ (called *a stratification*) such that $F = \bigcap_{n\in\omega} G_n(F) = \bigcap_{n\in\omega} \operatorname{cl} G_n(F)$, $G_n(F) \subset G_{n+1}(F)$ for any $n \in \omega$, and $G_n(F) \subset G_n(F')$ whenever $F \subset F'$. A space is an M_1-*space* if it has a σ-closure-preserving base. It is an old problem of Jack G. Ceder [1] whether or not every stratifiable space is an M_1-space. The problem is sometimes called the $M_3 \Rightarrow M_1$-question, since a space is stratifiable if and only if it is an M_3-space.

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Let $C_k(X)$ be the space of all continuous real-valued functions on a space X with the compact-open topology. P. M. Gartside and E. A. Reznichenko [2] have shown that $C_k(X)$ is stratifiable whenever X is a complete separable metrizable space, i.e., a Polish space. They asked whether $C_k(X)$ is an M_1-space for any Polish space X or not.

In §1, we provide a positive answer to the question by using an idea of T. Mizokami and N. Shimane [12], [13].

Theorem 1. *$C_k(X)$ is an M_1-space for any Polish space X.*

We don't know the answer to the following question, whose negative answer implies the negative answer to the $M_3 \Rightarrow M_1$-question.

Question 1. Is every subspace of $C_k(X)$ an M_1-space for any Polish space X?

Also, the following question, whose positive answer implies a positive answer to Question 1, remains open. Recall that a space is *a μ-space* if it is embeddable in some $\prod_{n\in\omega} X_n$, where each X_n is paracompact and a countable union of closed metrizable subspaces.

Question 2. Is $C_k(P)$ a μ-space?

More generally, we can ask the following question.

Question 3 ([11], [10], [14])**.** Is every stratifiable space a μ-space?

To show that $C_k(X)$ is stratifiable for any Polish space X, Gartside and Reznichenko proved that it has a σ-cushioned pair base, i.e,. it is an M_3-space. Heikki J. K. Junnila [9] and Gary Gruenhage [3] proved that a space is an M_3-space if and only if it is an M_2-space, i.e., it has a *σ-closure-preserving quasi-base* $\mathcal{B}$, i.e., for each point x in an open set U, there is some $B \in \mathcal{B}$ with $x \in B^o \subset B \subset U$, and $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ where each $\mathcal{B}_n$ is closure-preserving. By Theorem 1, $C_k(X)$ has a σ-closure-preserving base. But there might be simpler or more natural ways to get such a base or a σ-closure-preserving quasi-base. In §2, we show a method to construct a σ-closure-preserving quasi-base for $C_k(P)$, which is different from the way in §1. The construction, which is inspired by Gruenhage's technique in [3], might be interesting in itself.

Construction 1. *We construct a σ-closure-preserving quasi-base for $C_k(P)$.*

In §3, we show the following construction.

Construction 2. *We construct a monotone normality operator for $C_k(P)$.*

Constructions 1 and 2 might be helpful for further investigation of $C_k(P)$ to get answers to the open questions 1 through 3.

We adopt the following notation: Let ω be the set of nonnegative integers. The letters $n, m, k, l, \ldots$ always denote members of ω, so, for example, $n \geq 1$ means $n \in \omega - \{0\}$.

For $f \in C_k(X)$, a compact set $K \subset X$, and $\varepsilon > 0$, define $B(f, K, \varepsilon) = \{g \in C_k(P) : |g(x) - f(x)| < \varepsilon \text{ for any } x \in K\}$.

For the background information on stratifiable spaces, see [4] or [15]. For a current survey, see [5].

1. A σ-closure-preserving base for $C_k(P)$

Theorem 1. *$C_k(X)$ is an M_1-space for any Polish space X.*

Proof: Let Y be a Polish space and put $X = C_k(Y)$. It is shown by Gruenhage and Kenichi Tamano [6] that if Y is a σ-compact Polish space, in particular, if Y is a locally compact Polish space, then $C_k(Y)$ is an M_1-space. So we may assume that Y is not locally compact. Then there is a point $p_0 \in Y$ such that Y is not locally compact at p_0. Take a decreasing neighborhood base $\{U_n : n \in \omega\}$ of p_0. Define $F_n = \{f \in X : |f(p) - f(q)| \leq 1 \text{ for any } p, q \in U_n\}$. Then we can show that

(a) $F_n \subset F_{n+1}$ for each $n \in \omega$,
(b) $\{F_n : n \in \omega\}$ is a closed cover of X, and
(c) for each $f \in F_n$, there is a sequence $\{g(f, i)\}_{i \in \omega}$ in $X \backslash F_n$ converging to f.

Properties (a) and (b) are easy to see. To show (c), fix $n \in \omega$. Since $\operatorname{cl} U_n$ is not compact, we can take a countable family $\{V_i\}_{i \in \omega}$ of nonempty open sets in U_n such that $\{V_i\}_{i \in \omega}$ is discrete in Y. Take a point $q_i \in V_i$ for each $i \in \omega$. For each $i \in \omega$, take a function $g(f, i) \in C_k(Y)$ satisfying that $g(f, i) = f(p)$ for any $p \in Y \backslash V_i$, and

$$g(f, i)(q_i) = f(q_{i+1}) + 2. \tag{1}$$

By (1), we have that $g(f, i) \notin F_n$. To show that $\{g(f, i)\}_{i \in \omega}$ converges to f, let $B(f, K, \varepsilon)$ be a neighborhood of f. Since K is compact, only finitely many V_i's can meet K. Thus, there is $i_0 \in \omega$

such that $K \cap V_i = \emptyset$ for any $i \geq i_0$. Then $g(f,i) \in B(f,K,\varepsilon)$ for any $i \geq i_0$.

Now Theorem 1 follows from (a), (b), and (c) above and the following Lemma 1. □

Lemma 1 is essentially proved in Mizokami and Shimane's papers (see Lemma 2.6 in [12], or Lemma 12 in [13]). We give a short proof here to make this paper self-contained.

Lemma 1 ([12])**.** *Let X be a stratifiable space with an increasing sequence $\{F_n\}_{n\in\omega}$ of closed sets of X such that for any $n \in \omega$ and $x \in F_n$, there is a sequence $\{y(x,i)\}_{i\in\omega}$ in $X\backslash F_n$ converging to x. Then X is an M_1-space.*

To show Lemma 1, we need the following two lemmas. Lemma 2 gives us a method to fatten a closed set to a regular closed set. For each triple $(\mathcal{B}, F, H)$, fix some $\mathcal{B}'$ constructed in Lemma 2, which will be denoted by $\mathcal{B}' = \Phi(\mathcal{B}, F, H)$.

Lemma 2. *Let F and H be closed sets of a stratifiable space X with $H \subset F$. Assume that for each $x \in F\backslash H$, there exists a sequence $\{y(x,i)\}_{i\in\omega}$ in $X\backslash F$ converging to x. Then for each closure-preserving closed family $\mathcal{B}$ of $X\backslash H$, there is a closure-preserving closed family $\mathcal{B}' = \bigcup_{B\in\mathcal{B}} \mathcal{B}'(B)$ of $X\backslash H$ such that*

- (a) *for each $B' \in \mathcal{B}'(B)$, we have $B \cap F = B' \cap F$, and $B \cap F \subset \mathrm{cl}_X \mathrm{int}_X B'$;*
- (b) *for any neighborhood U of $B \cap F$, there exists $B' \in \mathcal{B}'(B)$ with $B \cap F \subset B' \subset U$; and*
- (c) *for any closure-preserving closed family $\mathcal{C} = \bigcup\{\mathcal{C}(B) : B \in \mathcal{B}\}$ of X with $B \subset C$ for any $C \in \mathcal{C}(B)$, we have that $\{C \cup B' : C \in \mathcal{C}(B),\ B' \in \mathcal{B}'(B), B \in \mathcal{B}\}$ is a closure-preserving closed family of X.*

Proof: Since $\{B \cap F : B \in \mathcal{B}\}$ is a closure-preserving closed family of $X\backslash H$ and since $F\backslash H$ is an F_σ-set of X, there exists a subset $D = \cup_{i\in\omega} D_i$ of $F\backslash H$ such that each D_i is a discrete closed set of X, $D_i \cap D_j = \emptyset$ for $i \neq j$, and $B \cap F = \mathrm{cl}_{X\backslash H}(B \cap D)$ for each $B \in \mathcal{B}$ ([7]). For each point $x \in D$, take a sequence $\{y(x,i)\}_{i\in\omega}$ in $X\backslash F$ converging to x satisfying the following: For each $y(x,i)$, there is an open subset $U(x,i)$ of $X\backslash F$ containing $y(x,i)$ such that

$$F \cap \mathrm{cl}\, U(x,i) = \emptyset; \tag{2}$$

(3) for each $x \in D$, we have $U(x,i) \cap U(x,j) = \emptyset$ for $i \neq j$;

(4) for each $n \in \omega$, $\{\{x\} \cup \bigcup_{i \in \omega} \operatorname{cl} U(x,i) : x \in D_n\}$

is a discrete closed family in X; and

(5) $\{U(x,i) : x \in D,\ i \in \omega\}$ is locally finite in $X \backslash F$.

This can be done because, for example, every stratifiable space is submetrizable. X has a metrizable subtopology τ such that F is a closed set, and each D_i is a closed discrete set in (X,τ).

Let $\mathcal{E}(x,i)$ be a closure-preserving closed quasi-neighborhood base of $y(x,i)$ in X contained in $U(x,i)$. Let $\{G_n(\cdot)\}$ be a stratification of X. For each $B \in \mathcal{B}$, let $\mathcal{B}'(B)$ be the set of all B' which can be expressed as

(6) $B' = (B \cap F) \cup \bigcup\{E(x,i) : x \in B \cap D, i \geq s(x)\}$, where $s : B \cap D \to \omega$ is a function, $E(x,i) \in \mathcal{E}(x,i)$ and $E(x,i) \subset G_j(B)$ for each $i \geq s(x)$ and the unique $j \in \omega$ with $x \in D_j$.

Now it is not difficult to check (a), (b), and (c). We only show (c). Suppose that for each $\lambda \in \Lambda$, $B_\lambda \in \mathcal{B}$, $C_\lambda \in \mathcal{C}(B_\lambda)$, and $B'_\lambda \in \mathcal{B}'(B_\lambda)$. We show that $\bigcup_{\lambda\in\Lambda} C_\lambda \cup B'_\lambda$ is closed in X. Suppose that $p \notin \bigcup_{\lambda\in\Lambda} C_\lambda \cup B'_\lambda$. Since $\mathcal{C}$ is a closure-preserving closed family of X and $p \notin \bigcup_{\lambda\in\Lambda} C_\lambda$, there is $m \in \omega$ such that $p \notin \operatorname{cl}_X G_m(\bigcup_{\lambda\in\Lambda} C_\lambda)$. For each λ, let B_λ be expressed as in (6), i.e., $B'_\lambda = (B_\lambda \cap F) \cup \bigcup\{E_\lambda(x,i) : x \in B_\lambda \cap D,\ i \geq s_\lambda(x)\}$, where $s_\lambda : B_\lambda \cap D \to \omega$ is a function, $E_\lambda(x,i) \in \mathcal{E}(x,i)$ and $E_\lambda(x,i) \subset G_j(B)$ for each $i \geq s(x)$ and the unique $j \in \omega$ with $x \in D_j$.

Then we have

$$E_\lambda(x,i) \subset G_j(B_\lambda) \subset G_m(\bigcup_{\lambda\in\Lambda} B_\lambda) \subset G_m(\bigcup_{\lambda\in\Lambda} C_\lambda)$$

for any $i \geq s_\lambda(x)$ and $j \geq m$ with $x \in B_\lambda \cap D_j$. Put

$$U = X \backslash \operatorname{cl} G_m(\bigcup_{\lambda\in\Lambda} C_\lambda)).$$

Then $\{B'_\lambda : \lambda \in \Lambda\} \cap U$ is the intersection of U and the union of a subfamily of the closure-preserving closed family $\bigcup\{E(x,i) \cup \{x\} : E(x,i) \in \mathcal{E}(x,i), x \in D_0 \cup D_1 \cup \cdots \cup D_{m-1},\ i \in \omega\}$. Thus, $p \notin \operatorname{cl}(\bigcup_{\lambda\in\Lambda} B'_\lambda)$. □

Lemma 3 takes care of the preservation of the closure-preserving property at lower levels in the proof of Lemma 1.

Two subsets A and B of a space X are called *separated* if $(\mathrm{cl}\, A) \cap B = A \cap (\mathrm{cl}\, B) = \emptyset$. Recall that a space X is *monotonically normal* ([8]) if and only if for each pair (A, B) of separated subsets of X, one can assign an open set $D(A, B)$ such that

(a) $A \subset D(A, B)$ (hence, $B \subset D(B, A)$);
(b) $D(A, B) \cap D(B, A) = \emptyset$; and
(c) if (A', B') is a pair of separated sets with $A \subset A'$ and $B \supset B'$, then $D(A, B) \subset D(A', B')$.

The function D is called a *monotone normality operator* for X.

Lemma 3. *Let X be a monotonically normal space with a monotone normality operator $D(H, K)$. Let F be a closed set of X and $\mathcal{B}$ a closure-preserving closed family of X. Suppose that for each $B \in \mathcal{B}$, we have B' with $B' \subset D(B \backslash F, F \backslash B)$. Then $\{B \cup B' : B \in \mathcal{B}\}$ is closure-preserving at any point of F.*

Proof: Suppose that $x \in F$, $\mathcal{C} \subset \mathcal{B}$, and $x \notin \cup\{B \cup B' : B \in \mathcal{C}\}$. Since $\mathcal{B}$ is closure-preserving, $\cup\mathcal{C}$ is a closed set. Let $U = D(F \backslash (\cup\mathcal{C}), (\cup\mathcal{C}) \backslash F) \setminus (\cup\mathcal{C})$. Then U is a neighborhood of x missing $\cup\{B \cup B' : B \in \mathcal{C}\}$. Indeed, let $B \in \mathcal{C}$. Then $B' \subset D(B \backslash F, F \backslash B) \subset D((\cup\mathcal{C}) \backslash F), F \backslash (\cup\mathcal{C}))$. Hence, $B' \cap U = \emptyset$. □

Now we are in a position to prove Lemma 1.

Proof of Lemma 1: Let $\{F_n\}_{n\in\omega}$ be the sequence of closed sets satisfying the assumption of Lemma 1. Put $F_{-1} = \emptyset$. Let S be the set of all finite increasing sequences in ω.

For $s = \langle n_0, n_1, \ldots, n_{k-1}, n_k \rangle \in S$, the number $k+1$ is called the length of s. The length of $\emptyset$ is defined to be 0. The length of s is denoted by $|s|$. For $s = \emptyset$, let $s^\frown n = \langle n \rangle$; and for $s = \langle n_0, n_1, \ldots, n_k \rangle$ and $n \in \omega$ with $n > n_k$, let $s^\frown n = \langle n_0, n_1, \ldots, n_k, n \rangle$.

Let $\bigcup_{n\in\omega} \mathcal{B}_n$ be a σ-closure-preserving closed quasi-base for X. By fattening $\bigcup_{n\in\omega} \mathcal{B}_n$, we construct a σ-closure-preserving regular closed quasi-base $\bigcup_{n\in\omega} \mathcal{C}_n$,

To do that, it suffices to show that for each closure-preserving family $\mathcal{B}$ of closed sets of X, there is a closure-preserving family $\mathcal{C} = \mathcal{C}(\mathcal{B})$ of regular closed sets satisfying

(7) for each $B \in \mathcal{B}$ and an open set U of X there is $C \in \mathcal{C}$ such that $B \subset C \subset U$.

Indeed, put $\mathcal{C}_n = \mathcal{C}(\mathcal{B}_n)$. Then $\bigcup_{n\in\omega}\{\text{int}\, C : C \in \mathcal{C}_n\}$ is the desired σ-closure-preserving base for X.

To complete the proof, let $\mathcal{B}$ be a closure-preserving closed family. We construct $\mathcal{C}$ satisfying the conditions above.

For each $s \in S$, we define a family $\mathcal{B}_s$ of sequences of subsets of X of length $|s| + 1$ by induction on the length of s. Let $\mathcal{B}_\emptyset = \mathcal{B}$. Suppose that $\mathcal{B}_s$ is defined. Let $n \in \omega$. For $s = \emptyset$, let $n \in \omega$ be arbitrary. For $s = \langle n_0, n_1, \ldots, n_k\rangle \neq \emptyset$, we assume that $n > n_k$.

The definition of $\mathcal{B}_{s^\frown n} = \bigcup_{B\in\mathcal{B}_s} \mathcal{B}_{s^\frown n}(B)$ is as follows. We use the operation $\Phi(\cdot, \cdot, \cdot)$ from Lemma 2.

(8) Suppose that $s = \emptyset$. Then for any $B \in \mathcal{B}_s$, define

$$\mathcal{B}_{s^\frown n}(B) = \mathcal{B}_{\langle n\rangle}(B) = \{\langle B, B'\rangle : B' \in \Phi(\mathcal{B}_\emptyset|(X - F_{n-1}), F_n, F_{n-1})(B\backslash F_{n-1}), B' \subset G_n(B) \text{ and } B' \subset \bigcap_{m<n} D(B\backslash F_m, F_m\backslash B)\}.$$

(9) Suppose that $s = \langle n_0, n_1, \ldots, n_k\rangle \neq \emptyset$.
For any $B = \langle B_{-1}, B_0, \ldots, B_k\rangle \in \mathcal{B}_s$, let $B^* = \bigcup_{i=-1}^{k} B_i$.
For any $-1 \le i \le k$, let $B|i = \langle B_{-1}, B_0, \ldots, B_i\rangle$.
We put $n_{k+1} = n$ and define $\mathcal{B}_{s^\frown n}(B) =$

$$\{\langle B, B'\rangle : B' \in \Phi(\{C_k\backslash F_{n-1}\}_{C\in\mathcal{B}_s}, F_n, F_{n-1})(B_k\backslash F_{n-1}), B' \subset G_{n_0}(B_{-1}) \text{ and } B' \subset D((B|i)^*\backslash F_m, F_m\backslash (B|i)^*) \text{ for each } -1 \le i \le k \text{ and } m < n_{i+1}\}.$$

Finally, define $\mathcal{B}_{s^\frown n} = \bigcup\{\mathcal{B}_{s^\frown n}(B) : B \in \mathcal{B}_s\}$. Then, by induction on the length of $s \in S$, and by using Lemma 3, we can show that for each $s \in S$,

(10) $\mathcal{C}_s = \{B^* : B \in \mathcal{B}_s\}$ is a closure-preserving closed family of X.

Indeed, suppose that $s = \langle n_0, \ldots, n_k\rangle \in S\backslash\{\emptyset\}$. Take $s^- \in S$ with $s = s^{-\frown} n$ and assume that $\mathcal{C}_{s^-} = \{B^* : B \in \mathcal{B}_{s^-}\}$ is a closure-preserving closed family. Note that by (8) and (9), for each $B = \langle B_{-1}, B_0, \ldots, B_k\rangle \in \mathcal{B}_s$, we have

$$B_k \subset D((B|k-1)^*\backslash F_{n_k-1}, F_{n_k-1}\backslash (B|k-1)^*).$$

Hence by Lemma 3, $\{B^* : B \in \mathcal{B}_s\}$ is a closure-preserving closed family of X.

Let $\mathcal{C} = \{\bigcup_{s\in S}(B_s)^* :$ the family $\{B_s : s \in S\}$ satisfies that $B_s \in \mathcal{B}_s$ and $B_s \subset B_{s'}$ (i.e., B_s is an initial segment of a sequence

$B_{s'}$) for any $s, s' \in S$ with $s \subset s'$}. By Lemma 2, it is easy to check (7) and that $C \subset \mathrm{cl}\,\mathrm{int}\, C$ for each $C \in \mathcal{C}$.

It remains to show that $\mathcal{C}$ is a closure-preserving closed family. Assume that $\{C^\lambda : \lambda \in \Lambda\} \subset \mathcal{C}$. For each $\lambda \in \Lambda$, let $C^\lambda = \bigcup_{s \in S}(B_s^\lambda)^*$, and let $B_s^\lambda = \langle B_{-1}^\lambda, B_0^\lambda, \ldots, B_{|s|}^\lambda \rangle$. We show that $\bigcup_{\lambda \in \Lambda} C^\lambda$ is closed. To show this, let $p \in X \backslash (\bigcup_{\lambda \in \Lambda} C^\lambda)$.

Note that $B_{-1}^\lambda \in \mathcal{B}_\emptyset = \mathcal{B}$ for each $\lambda \in \Lambda$. Let $E = \bigcup_{\lambda \in \Lambda} B_{-1}^\lambda$. Since $\mathcal{B}$ is a closure-preserving closed family, E is a closed set. It follows from $p \notin E$ that there is $m \in \omega$ such that

$$p \in F_m \backslash \mathrm{cl}\, G(E, m). \tag{11}$$

Define $S(m) = \{s = \langle n_0, \ldots, n_k \rangle \in S \backslash \{\emptyset\} : n_k \leq m\}$. For each $s = \langle n_0, \ldots, n_k \rangle \in S(m)$, let $T(s)$ be the set of all $t \in S$ such that $t \subset s$, or $s \subset t$ with $t = \langle n_0, \ldots, n_l \rangle, l > k$, and $n_{k+1} > m$.

Put $T = \bigcup_{s \in S(m)} T(s)$ and $S' = S \backslash (\{\emptyset\} \cup T) = \{s = \langle n_0, \ldots, n_k \rangle \in S \backslash \{\emptyset\} : n_0 > m\}$. Define $V = X \backslash \mathrm{cl}\, G(E, m)$. Then by (8), (9) and (11), we have $(B_s^\lambda)^* \cap V = \emptyset$ for any $s \in S' \cup \{\emptyset\}$ and $\lambda \in \Lambda$.

For each $s \in S(m)$, define $\mathcal{T}(s) = \{(B_t^\lambda)^* : t \in T(s), \lambda \in \Lambda\}$.

To complete the proof of the closure-preserving property of $\mathcal{C}$, it suffices to show that

$$p \notin \mathrm{cl}(\bigcup \bigcup_{s \in S(m)} \mathcal{T}(s)). \tag{12}$$

Note that $S(m)$ is a finite set. Hence, it suffices to show that for each $s \in S(m)$, we have

$$p \notin \mathrm{cl}(\bigcup \mathcal{T}(s)) = \mathrm{cl}(\bigcup \{(B_t^\lambda)^* : t \in T(s), \lambda \in \Lambda\}). \tag{13}$$

Put $H = \bigcup \{(B_s^\lambda)^* : \lambda \in \Lambda\}$. Then $p \notin H$, and by (10), H is a closed set. Define a neighborhood W of p by

$$W = D(F_m \backslash H, H \backslash F_m) \setminus H.$$

Recall that $C^\lambda = \bigcup_{t \in S}(B_t^\lambda)^*$. Let $s = \langle n_0, \ldots, n_k \rangle$, $s \subset t$, and $s \neq t$. Then t can be written as $t = \langle n_0, \ldots, n_k, n_{k+1}, \ldots, n_l \rangle$ with $n_{k+1} > m$. Then for any $\lambda \in \Lambda$ and for any j with $k + 1 \leq j \leq l$, by (9), we have

$$B_j^\lambda \subset D((B_s^\lambda)^* \backslash F_m,\ F_m \backslash (B_s^\lambda)^*) \subset D(H \backslash F_m, F_m \backslash H).$$

Hence, $(B_t^\lambda)^* \cap W = \emptyset$. Thus, $\bigcup_{\lambda \in \Lambda} C^\lambda \cap W = \emptyset$, which completes the proof of the closure-preserving property of $\mathcal{C}$. $\square$

2. A σ-closure-preserving quasi-base for $C_k(P)$

Construction 1. *We construct a σ-closure-preserving quasi-base for $C_k(P)$.*

Let P be the set of nondecreasing functions of ω^ω. Note that P is homeomorphic to both ω^ω and the set of irrational numbers with the usual topology. For each $x, y \in P$, $x \leq y$ means that $x(n) \leq y(n)$ for any $n \in \omega$. For each $p \in P$, define $K_p = \{x \in P : x \leq p\}$, and let $\mathcal{K} = \{K_p : p \in \mathcal{K}\}$. For $A \subset P$, define $m(A) = \{a \in A : a' = a$ for any $a' \in A$ with $a' \leq a\}$. Let $Q = \{q \in P : q$ is bounded, i.e., there is $n \in \omega$ such that $q(i) = q(i+1)$ for any $i \geq n\}$.

Lemma 4 ([2]). *If U is a clopen set of P, then $m(U)$ is a finite set in Q. For any $K \in \mathcal{K}$ and any clopen set U of P, if $K \cap U \neq \emptyset$, then $K \cap m(U) \neq \emptyset$.*

For $f \in C_k(P)$ and $n \geq 1$, define $V(f, n) = B(f, n^\omega, \frac{1}{2^n})$. Let $W(f) = \bigcup\{B(\mathbf{0}, K_p, \frac{1}{4}) : p \in P,\ |f(p)| > \frac{3}{4}\}$. Note that since $K_p \cap Q$ is dense in K_p for any $p \in P$, we have $W(f) = \bigcup\{B(\mathbf{0}, K_q, \frac{1}{4}) : q \in Q,\ |f(q)| > \frac{3}{4}\}$.

Lemma 5. *For any $f \in C_k(P)$, there is $n \geq 1$ such that $V(f, n) \cap W(f) = \emptyset$.*

Proof: If $|f|^{-1}[\frac{3}{4}, \infty) = \emptyset$, then $W(f) = \emptyset$, and any $n \geq 1$ satisfies the condition.

Assume that $|f|^{-1}[\frac{3}{4}, \infty) \neq \emptyset$. Let U be clopen with $|f|^{-1}[\frac{3}{4}, \infty) \subset U \subset |f|^{-1}(\frac{1}{2}, \infty)$. We show that

$$B(f, m(U), \frac{1}{4}) \cap W(f) = \emptyset, \tag{14}$$

which implies that $V(f, n) \cap W(f) = \emptyset$ for any $n \geq 2$ satisfying $m(U) \subset n^\omega$.

To show (14), assume that $p \in P$ satisfies $|f(p)| > \frac{3}{4}$. We show that

$$B(f, m(U), \frac{1}{4}) \cap B(\mathbf{0}, K_p, \frac{1}{4}) = \emptyset. \tag{15}$$

Since $|f|^{-1}[\frac{3}{4}, \infty) \subset U$, we have $p \in U \cap K_p$. Hence, there exists $q \in m(U) \cap K_p$. Since $q \in U \subset |f|^{-1}(\frac{1}{2}, \infty)$, we have $f(q) > \frac{1}{2}$, which implies (15). □

Define a closed set M_0 of $C_k(P)$ by $M_0 = \{f \in C_k(P) : |f(p)| \leq \frac{3}{4}$ for any $p \in P\}$.

For each $f \in C_k(P) - M_0$, define $n(f) = \min\{n \geq 1 : V(f,n) \cap W(f) = \emptyset\}$, and for each $n \geq 1$, define $M_n = M_0 \cup \{f \in C_k(P) \setminus M_0 : n(f) \leq n\}$. Note that $M_n \subset M_{n+1}$, for each $n \in \omega$, and $C_k(P) = \bigcup\{M_n : n \in \omega\}$.

Lemma 6. *Each M_n, $n \in \omega$, is a closed set of $C_k(P)$.*

Proof: Let $n \geq 1$. We show that M_n is closed in $C_k(P)$. Suppose that $f \in C_k(P) \setminus M_n$. Then $V(f,n) \cap W(f) \neq \emptyset$. Hence, there exists $q \in Q$ such that $|f(q)| > \frac{3}{4}$ and $V(f,n) \cap B(\mathbf{0}, K_q, \frac{1}{4}) \neq \emptyset$. Take

$$h \in V(f,n) \cap B(\mathbf{0}, K_q, \frac{1}{4}). \tag{16}$$

Take $m \in \omega$, satisfying

$$h \in V(g,n) \text{ for any } g \in V(f,m), \tag{17}$$

$$q \in m^\omega, \text{ and} \tag{18}$$

$$\frac{1}{2^m} < |f(q)| - \frac{3}{4}. \tag{19}$$

To get (17), define $\varepsilon = \sup\{|f(p) - h(p)| : p \in n^\omega\}$. Then $0 \leq \varepsilon < \frac{1}{2^n}$. Take $m \in \omega$ with $m \geq n$ and $\frac{1}{2^m} < \frac{1}{2^n} - \varepsilon$. Then m satisfies (17).

To complete the proof, let $V = V(f,m) \setminus M_0$. Then V is a neighborhood of f with $V \cap M_n = \emptyset$. To show this, let $g \in V$. Since $g \in V(f,m)$, by (18), we have $|f(q) - g(q)| < \frac{1}{2^m}$. Hence by (19), we have $|g(q)| \geq |f(q)| - |f(q) - g(q)| > |f(q)| - \frac{1}{2^m} > \frac{3}{4}$. Furthermore, by (16) and (17), we have $h \in V(g,n) \cap B(\mathbf{0}, K_q, \frac{1}{4})$. Hence, $V(g,n) \cap W(g) \neq \emptyset$. Thus, $g \notin M_n$, which completes the proof. □

For each $n \geq 1$, take a locally finite open refinement $\mathcal{G}_n$ of the open cover $\{V(f, n+1) : f \in C_k(P)\}$ of $C_k(P)$. For any $n \in \omega$, let $\mathcal{U}_n = \mathcal{G}_n | (X \setminus M_{n-1})$, where $M_{-1} = \emptyset$. Then $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ is a point finite open cover of $C_k(P)$, hence is interior-preserving. For each $f \in X - M_0$, take an element $U(f)$ of $\mathcal{U}_{n(f)}$ containing f. This can be done because $\mathcal{G}_{n(f)}$ is an open cover of $C_k(P)$ and $f \notin M_{n(f)-1}$.

For each $K = K_p \in \mathcal{K}$, define

$$E(\mathbf{0}, K) = X \setminus \bigcup\{U(f) : f \notin B(\mathbf{0}, K, 1)\}.$$

Then we have $E(\mathbf{0}, K) \subset B(\mathbf{0}, K, 1)$, and $\mathcal{E} = \{E(\mathbf{0}, K) : K \in \mathcal{K}\}$ is a closure-preserving closed family.

Lemma 7. *For each $K \in \mathcal{K}$, $\mathbf{0} \in \operatorname{int} E(\mathbf{0}, K_p)$.*

Proof: Let $K = K_p$. It suffices to show $B(\mathbf{0}, K_p, \frac{1}{4}) \subset E(\mathbf{0}, K_p)$. Suppose that $f \in C_k(P) \backslash B(\mathbf{0}, K_p, 1)$. We show that

$$B(\mathbf{0}, K_p, \frac{1}{4}) \cap U(f) = \emptyset. \tag{20}$$

Note that $f \notin M_0$ because $M_0 \subset B(\mathbf{0}, K_p, 1)$. Since $f \notin B(\mathbf{0}, K_p, 1)$, there exists $p' \leq p$ such that $f(p') \geq 1 > \frac{3}{4}$. Then

$$B(\mathbf{0}, K_p, \frac{1}{4}) \subset B(\mathbf{0}, K_{p'}, \frac{1}{4}) \subset W(f). \tag{21}$$

Note that $U(f) \subset V(f, n(f))$. Indeed, since $U(f) \in \mathcal{U}_{n(f)}$ and $\mathcal{U}_{n(f)}$ is a refinement of $\mathcal{G}_{n(f)}$, there exists $g \in C_k(P)$ with $U(f) \subset V(g, n(f) + 1)$. For any $e \in U(f)$, since $e \in V(g, n(f) + 1)$, we have $|e(x) - g(x)| < \frac{1}{2^{n(f)+1}}$ for any $x \in (n(f) + 1)^\omega$. Note that $f, h \in U(f)$. Hence, $|f(x) - h(x)| < \frac{1}{2^{n(f)}}$ for any $x \in n(f)^\omega$, which implies $U(f) \subset V(f, n(f))$.

By the definition of $n(f)$, we have $V(f, n(f)) \cap W(f) = \emptyset$, which implies that

$$U(f) \cap W(f) = \emptyset. \tag{22}$$

Thus, by (21) and (22), we get (20). □

Now for each $n \geq 1$, let $H_n : C_k(P) \to C_k(P)$ be the homeomorphism defined by $H_n(f) = \frac{1}{n} f$. Define $\mathcal{E}_n = \{H_n(E(\mathbf{0}, K)) : K \in \mathcal{K}\}$. Then we have the following lemma.

Lemma 8. *$\mathcal{E} = \bigcup_{n \geq 1} \mathcal{E}_n$ is a σ-closure preserving neighborhood base of $\mathbf{0}$ consisting of closed sets.*

Proof of Construction 1: Let D be a countable dense subset of $C_k(P)$. Let $\mathcal{E}$ be as in Lemma 8. Then $\{E + f : E \in \mathcal{E}, f \in D\}$ is the desired σ-closure-preserving quasi-base for $C_k(P)$. □

3. A monotone normality operator for $C_k(P)$

Construction 2. *We construct a monotone normality operator for $C_k(P)$.*

Proof: Let $\{\mathcal{H}_n : n \in \omega\}$ be a family of finite subsets of $C_k(P)$ such that $\bigcup_{n\in\omega} \mathcal{H}_n$ is dense in $C_k(P)$.

For example, let $\{\mathcal{P}_n : n \in \omega\}$ be a sequence of finite partitions of P such that $\mathcal{P}_{n+1} \prec \mathcal{P}_n$ for each $n \in \omega$, and $\bigcup_{n\in\omega} \mathcal{P}_n$ is a base of P. Let $\{Q_n : n \in \omega\}$ be an increasing sequence of finite subsets of rational numbers such that $Q = \cup Q_n$. Define $\mathcal{H}_n$ by $\mathcal{H}_n = \{f \in C_k(P) : f$ has a constant value in Q_n on each $A \in \mathcal{P}_n\}$. Then $\{\mathcal{H}_n : n \in \omega\}$ satisfies the condition stated above.

To get a monotone normality operator, it is sufficient to define $B(f, K, \varepsilon)_f$ for each $f \in C_k(P)$, $K \in \mathcal{K}$, and $\varepsilon > 0$ satisfying:
if

$$B(f, K, \varepsilon)_f \cap B(g, L, \mu)_g \neq \emptyset, \tag{23}$$

then

$$f \in B(g, L, \mu) \text{ or } g \in B(f, K, \varepsilon). \tag{24}$$

To define $B(f, K, \varepsilon)_f$, take $n_f = n(f, K, \varepsilon)$, $m_f = m(f, K, \varepsilon) \in \omega$, and $h_f = h(f, K, \varepsilon) \in \mathcal{H}_{m_f}$ such that

$$m_f \geq n_f, \text{ and} \tag{25}$$

$$|f(p) - h_f(p)| < \frac{1}{2^{n_f}} < \frac{5}{2^{n_f}} < \varepsilon \text{ for any } p \in K. \tag{26}$$

For each $m, n \leq m_f$, and $h \in \mathcal{H}_m$, take a clopen set $U^{f,h,m,n}$ such that

$$|f - h|^{-1}[\frac{4}{2^n}, \infty) \subset U^{f,h,m,n} \subset |f - h|^{-1}(\frac{2}{2^n}, \infty). \tag{27}$$

Define

$$K' = K \cup \bigcup_{h\in\mathcal{H}_m, m,n\leq m_f} m(U^{f,h,m,n}), \text{ and} \tag{28}$$

$$B(f, K, \varepsilon)_f = B(f, K', \frac{1}{2^{m_f+2}}).$$

Now assume (23). To show (24), we may assume, without loss of generality, that

$$m_f \leq m_g. \tag{29}$$

By (25) and (29), we have $n_f \leq m_f \leq m_g$. Hence, the set $U = U^{g,h_f,m_f,n_f}$ appears in the construction of $B(g, L, h)_g$.

Case 1. $U \cap K = \emptyset$.

By the definition of U in the construction of $B(g, L, h)_g$, we have $|g - h_f|^{-1}[\frac{4}{2^{n_f}}, \infty) \cap K = \emptyset$. Hence, we have $|g - h_f| < \frac{4}{2^{n_f}}$ on K. On the other hand, by the definition of h_f in the construction of $B(f, K, \varepsilon)$, we have $|f - h_f| < \frac{1}{2^{n_f}}$ on K. Thus, $|f - g| < \frac{5}{2^{n_f}} < \varepsilon$ on K. Hence, $g \in B(f, K, \varepsilon)$.

Case 2. $U \cap K \neq \emptyset$.

By (23) and (28), we have

$$B(f, K, \frac{1}{2^{m_f+2}}) \cap B(g, m(U), \frac{1}{2^{m_g+2}}) \neq \emptyset.$$

Hence, $|f - g| < \frac{1}{2^{m_f+1}} \leq \frac{1}{2^{n_f+1}}$ on $m(U) \cap K (\neq \emptyset)$. On the other hand, by the definition of U, we have $U \subset |g - h_f|^{-1}(\frac{2}{2^{n_f}}, \infty)$. Hence, $|g - h_f| > \frac{2}{2^{n_f}}$ on U. Hence,

$$|f - h_f| \geq |g - h_f| - |f - g| > \frac{2}{2^{n_f}} - \frac{1}{2^{n_f+1}} = \frac{3}{2^{n_f+1}}$$

on $m(U) \cap K \ (\neq \emptyset)$, contradicting that $|f - h_f| < \frac{1}{2^{n_f}}$ on K. $\square$

Remark 1. In the definition of U, we can replace "$m, n \leq m_f$" by "$n \leq m \leq m_f$."

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