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N. C. Esty

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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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ON THE CONTRACTIBILITY OF CERTAIN HYPERSPACES

N. C. ESTY

Abstract. In this paper we will show the contractibility of the hyperspace $\mathbf{CL}(M)$ of closed, nonempty subsets of a Borel compact metric space (M, d) under the Vietoris topology given the assumption that in M, the closure of an open ball of a given radius is equal to the closed ball of that radius.

1. Introduction

For hyperspaces of noncompact metric spaces, limited study has been given to those properties which are in some way related to connectedness. When considering hyperspaces, the case of noncompact base space is especially important to the study of *time scales*. Time scales is a relatively new field which is attempting to unify the areas of differential equations and difference equations, resulting in a single cohesive set of methods for approaching both continuous and discrete problems. The approach is interesting, because when considering a single dynamic equation, the solutions one gets when treating it as a differential equation can be quite similar to those produced by treating it as a difference equation – but they can also be wildly different, for example, continuous in the first case and chaotic in the second. The field, created and heavily investigated by Stefan Hilger [6], [7], has generated a large amount of interest. See the book by Martin Bohner and Allan Peterson [2] for an excellent introduction to the calculus of time

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scales. Generally speaking, time scales considers a dynamic equation as separate from its domain and treats the domain as merely a parameter, which can range from $\mathbb{Z}$ in the difference equation case to $\mathbb{R}$ in the differential equation case. The domain itself, called a *time scale*, may be any nonempty closed subset of $\mathbb{R}$. When the domain is $\mathbb{Z}$, the derivative is the normal difference operator, and when the domain is $\mathbb{R}$ it is the usual derivative. Another useful case is when the domain is $q\mathbb{R}$, which corresponds to the study of q-difference equations. Of course, even more complicated domains can be used, from disjoint intervals to Cantor sets, and many of these have interesting applications.

After the time scale analogs to calculus have been developed, the next naturally occurring questions have to do with convergence. If one considers a given dynamic equation, over a sequence of time scales which converges, and if over each time scale we have a solution, do the solutions converge? Some questions along these lines are addressed in [11]. In order to ask this type of question, we must agree on the topology on the space of time scales. This space is exactly $\mathbf{CL}(\mathbb{R})$, and so we are led into the study of hyperspaces.

Historically, two of the most popular topologies on hyperspaces of metric spaces are the Hausdorff topology and the Vietoris topology. (See Ernest Michael's paper on hyperspaces [13] for a discussion of both topologies, as well as several others.) The two are equal when the space is compact; however, if the space is noncompact, the former results in a metrizable topology and the latter does not. Due to this, the former may initially seem more appealing, but for the purposes of time scales it is less useful. Consider the sequence of subsets $\mathbb{T}_n = [-n, n]$. Under the Hausdorff topology, the sequence does not converge to $\mathbb{R}$. However, we would like it to in order that we might answer questions about dynamic equations on $\mathbb{R}$ by considering only the solutions over compact sets. The Vietoris topology gives this convergence. For this reason, we are more interested in the Vietoris topology.

We would like to know what the hyperspace $\mathbf{CL}(\mathbb{R})$ looks like, and more generally, the space $\mathbf{CL}(M)$, where M is any metric space. Much is known about hyperspaces of compact metric spaces. (See the recent book *Hyperspaces* [8] by Alejandro Illanes and Sam B. Nadler, Jr. or [3], [12], and [15] for properties of hyperspaces of continua.) However, little study has been given to the problem of characterizing the hyperspace of some basic, noncompact metrizable spaces (such as the real line) endowed with the Vietoris topology. It was shown by Michael [13] that $\mathbf{CL}(\mathbb{R})$ is completely regular, separable, and first countable, and shown that $\mathbf{CL}(\mathbb{R})$ is not a normal space by results of V. M. Ivanova [9], James Keesling [10], and N. V. Veličhko [14]. In 2002, Camillo Costantini and Wiesław Kubiś [4] showed that $\mathbf{CL}(\mathbb{R})$ is path connected but not locally connected. In a recent paper [5], I showed that $\mathbf{CL}(\mathbb{R})$ is simply connected. The proof involved an explicit construction of a homotopy. A similar idea is used in this paper to prove the more general statement below.

Theorem 1.1. *If (M, d) is a Borel compact metric space, and if in M, the closure of any open ball of a given radius is the closed ball of that radius, then $\boldsymbol{CL}(M)$ endowed with the Vietoris topology is contractible.*

Recall that Borel compact means that the space is complete and regularly bounded, so that all closed balls in the space are sequentially compact (equivalently, compact). It does not mean that M itself is compact.

2. Notation

For ease of reference, here are some of the notations we will use throughout the paper.

M: a Borel compact metric space with a metric d.
$\mathbf{CL}(M)$: the hyperspace of all closed, non-empty subsets of M.
X: a point in $\mathbf{CL}(M)$, namely, a closed non-empty subset of M.
$B(x,r)$: an open ball centered at x with radius r,
i.e., $\{y \in M : d(y,x) < r\}$.
$\overline{B}(x,r)$: a closed ball centered at x with radius r,
i.e., $\{y \in M : d(y,x) \leq r\}$.
U: an open set in M.
z: a fixed arbitrary point in M, chosen to play the part of the origin.

3. The Vietoris topology

We will introduce some notation for the discussion of the topology. For open $U \subset M$, define

$$U^- = \{A \in \mathbf{CL}(M) : A \cap U \neq \emptyset\}$$
$$U^+ = \{A \in \mathbf{CL}(M) : A \subset U\}.$$

The term "hit and miss topology" is occasionally used to describe topologies like the Vietoris topology and the Fell topology, because sets of $\mathbf{CL}(M)$ in U^- "hit" the set U, and sets in U^+ "miss" its complement. For a more extensive discussion of hit and miss topologies, see the paper by Gerald Beer and Robert K. Tamaki [1].

The Vietoris topology has as a subbase all sets of the form U^- and U^+ as U ranges through open sets of M. It is also possible to describe the basis elements of the topology; however, we prefer to think of the Vietoris topology as the supremum of the upper and lower Vietoris topologies, the first being generated by all sets of the form U^+ and the second being generated by all finite intersections $U_1^- \cap \cdots \cap U_n^-$.

4. Contractibility of $\mathbf{CL}(M)$

In the following, we will consider $\mathbf{CL}(M)$ endowed with the Vietoris topology.

Theorem 4.1. *If (M, d) is a Borel compact metric space, and $\overline{B}(x,r) = \overline{B(x,r)}$ for all $x \in M$, $r > 0$, then $\boldsymbol{CL}(M)$ is contractible.*

The idea is as follows: using an arbitrarily chosen point, which plays the role of the origin, and starting at time zero, we will expand a large sphere, such that at time one it encompasses the entire space. The purpose of this sphere is merely to associate to each point of a subset X a time at which it is "inside the bubble." Once a point is inside the bubble, it becomes activated, and we begin expanding it as the center of a closed solid sphere, enlarging at a rate such that the closed sphere will include all of the space by time one. In this way, at any given moment, the only points which have started to expand are within the large bubble, a compact set, and yet, at time one, all points of the space are included.

The condition on the closure of the open ball $B(x,r)$ is needed to make this function continuous. Indeed, if there are balls for which this is not true, then the growth process of the spheres described above may have moments in time where suddenly the sphere includes a point which was not being converged upon by the growing sphere, creating a discontinuity. Borel compactness gives us sequential compactness inside these spheres.

Proof: Fix an arbitrary point $z \in M$ to play the analog of the origin. Let $d_x = d(x, z)$ for all $x \in M$.

We define a radius function $r(t, d)$ to be

$$r(t,d) = \begin{cases} 0 & t \leq \frac{d}{1+d} \\ \frac{t-\frac{d}{1+d}}{1-t} & t > \frac{d}{1+d} \end{cases}.$$

We will use this function to determine the size of the closed solid sphere around points of X at a given time. Note that before a point of X becomes activated, $r(t, d)$ is zero. We define the function $H : \mathbf{CL}(M) \times [0, 1] \to \mathbf{CL}(M)$ by $H(X, 1) = M$ and for $t \in [0, 1)$,

$$H(X,t) = \bigcup_{x \in X} \overline{B}(x, r(t, d_x)).$$

This function creates the process described above. We claim that H is a contraction of the space $\mathbf{CL}(M)$ to the point M. It is clear that $H(X, 0) = X$ and $H(X, 1) = M$. It may be clear that $H(X, t)$ is, in fact, a member of $\mathbf{CL}(M)$ for all other times between 0 and 1, but it is worthwhile to show, since the proof of continuity contains similar techniques.

Let $(X, t) \in \mathbf{CL}(M) \times (0, 1)$. We wish to show that $H(X, t)$ is a closed, non-empty subset of M. It is clearly non-empty because it contains X, itself a non-empty subset. Let $y_n \in H(X, t)$ be a sequence of points converging to a point $y \in M$. We have two possibilities. First, it may be that a subsequence of the y_n is in X. In this case, because X is closed, y must also be in X, which is contained in $H(X, t)$. The other possibility can be reduced to the case where all the $y_n \in H(X, t) \setminus X$, by removing the finite number of $y_n \in X$.

If $y_n \in H(X, t)$, but $y_n \notin X$, then to each y_n, we can associate an $x_n \in X$ such that $y_n \in \overline{B}(x_n, r(t, d_{x_n}))$. Because $y_n \neq x_n$, it must be that $r(x_n, d_{x_n}) > 0$ and so x_n must be one of the activated points, i.e., $x_n \in X \cap \overline{B}(z, \frac{t}{1-t})$. That set is sequentially compact because it is the intersection of a closed set and a closed bounded sphere inside our space M; therefore, every sequence has a convergent subsequence. By abuse of notation, we say $x_n \to x$. Since X is closed, $x \in X$. The corresponding subsequence y_n still converges to the point y.

We claim that $y \in \overline{B}(x, r(t, d_x))$. Because $x_n \to x$, we know that $d_{x_n} \to d_x$, and continuity of the radius function means that $r(t, d_{x_n}) \to r(t, d_x)$. Let $\delta > 0$. We can choose N large enough so that $d(x, x_n) < \delta/3$, $d(y, y_n) < \delta/3$, and $|r(t, d_x) - r(t, d_{x_n})| < \delta/3$ for $n > N$. Then

$$\begin{aligned} d(y,x) &\leq d(y,y_n) + d(y_n,x_n) + d(x_n,x) \\ &< \delta/3 + r(t,d_{x_n}) + \delta/3 \\ &< r(t,d_x) + \delta. \end{aligned}$$

Because this is true for all $\delta > 0$, we have that $d(y, x) \leq r(t, d_x)$ and so $y \in \overline{B}(x, r(t, d_x)) \subset H(X, t)$. This shows that $H(X, t)$ is, in fact, a closed subset of M.

Therefore, it remains to show that H is continuous with respect to the Vietoris topology. It is enough to show continuity with respect to the upper and lower Vietoris topologies.

We begin with the upper. Consider $H(X_0, t_0)$. Let U^+ be an open set in the upper Vietoris topology containing $H(X_0, t_0)$. If $t_0 = 1$, then U must be M and therefore, U^+ will contain $H(X, t)$ for all $(X, t) \in \mathbf{CL}(M) \times [0, 1]$. So assume $t_0 \neq 1$. Suppose that H is not continuous at (X_0, t_0). Then there exists some open set U^+ containing $H(X_0, t_0)$ such that for all neighborhoods of (X_0, t_0), there is a point in the neighborhood which does not map into U^+ under H.

Define $B^k = B(X_0, 1/k) = \{x \in M :$ there is some $x_0 \in X_0$ s.t. $d(x, x_0) < 1/k\}$. These B^k are open, and so a nested sequence of neighborhoods of X_0 can be given by $V_k = B^k \cap U$. Consider neighborhoods of (X_0, t_0), $N_k = V_k^+ \times (t_0 - 1/k, t_0 + 1/k)$, if $t_0 > 0$, and $N_k = V_k^+ \times [0, 1/k)$, if $t_0 = 0$. Choose K large enough that $1/K < \min\{t_0, 1 - t_0\}$, if $t_0 > 0$, and let $K = 2$, if $t_0 = 0$. We will use only $k \geq K$. Since we assumed H is not continuous at (X_0, t_0), for each $k \geq K$, we get a point $(X_k, t_k) \in N_k$ such that $H(X_k, t_k) \not\subset U$. Note that $t_k \in (t_0 - 1/k, t_0 + 1/k) \subset (0, 1)$ and $t_0 \in (0, 1)$ for the k in question (indeed, t_k cannot be 0, because otherwise, $H(X_k, t_k) = X_k \subset V_k \subset U$).

Take $y_k \in H(X_k, t_k)$ with $y_k \notin U$. Since $X_k \subset B^k \cap U$, $y_k \notin X_k$, and therefore, $y_k \in H(X_k, t_k) \setminus X_k$. This means that $y_k \in \overline{B}(x_k, r(t_k, d_{x_k}))$ for some $x_k \in X_k$ with $r(t_k, d_{x_k}) > 0$. We now have two sequences of points, the y_k and the associated x_k. The following shows that both sequences are completely contained in a closed ball around z of radius $2R$ where $R = (t_0 + 1/K)/(1-(t_0+1/K))$. The choice of K ensures that $R > 0$.

First, we show $x_k \in \overline{B}(z, R)$. We know $r(t_k, d_{x_k}) > 0$, which means $t_k > d_{x_k}/(1+d_{x_k})$. Therefore,

$$d_{x_k} = d(x_k, z) < \frac{t_k}{1-t_k} < \frac{t_0 + 1/k}{1-(t_0+1/k)} < \frac{t_0+1/K}{1-(t_0+1/K)} = R.$$

Next, we show that $y_k \in \overline{B}(z, 2R)$. Because $y_k \in \overline{B}(x_k, r(t_k, d_{x_k}))$, we have

$$d(y_k, x_k) \leq r(t_k, d_{x_k}) = \frac{t_k - \frac{d_{x_k}}{1+d_{x_k}}}{1-t_k} < \frac{t_k}{1-t_k} < R$$

and so $d(y_k, z) \leq d(y_k, x_k) + d(x_k, z) < 2R$.

M is Borel compact, so the set $\overline{B}(z, 2R)$ which contains the x_n and y_n is sequentially compact. In addition, we can associate to each x_k a point $x_0^k \in X_0$ with $d(x_k, x_0^k) < 1/k$. Compactness of $\overline{B}(z, R+1/K)$ gets a convergent subsequence of the x_0^k; by abuse of notation, we say $x_0^k \to x_0$. Because X_0 is closed, $x_0 \in X_0$. Because $1/k \to 0$, it is easy to see the corresponding subsequence of the x_k must also converge to x_0. If we consider now the same subsequence of the y_k, we apply compactness one more time to get another convergent subsequence. Further abuse of notation (to avoid excessive indices) gives $y_k \to y_0$. We claim that $y_0 \in \overline{B}(x_0, r(t_0, d_{x_0}))$. Since we know that each $y_k \in \overline{B}(x_k, r(t_k, d_{x_k}))$, this can be seen by observing that $t_k \to t_0$, $x_k \to x_0$, and the radius function r is continuous. The details follow.

Pick an arbitrary $\epsilon > 0$. It is possible to choose such a large L that for all larger k, we have $d(y_0, y_k) < \epsilon/3$, and $d(x_k, x_0) < \epsilon/3$, and (by continuity of r), $|r(t_k, d_{x_k}) - r(t_0, d_{x_0})| < \epsilon/3$. Therefore,

$$\begin{aligned} d(y_0, x_0) &\leq d(y_0, y_k) + d(y_k, x_k) + d(x_k, x_0) \\ &< \epsilon/3 + r(t_k, d_{x_k}) + \epsilon/3 \\ &< \epsilon/3 + r(t_0, d_{x_0}) + \epsilon/3 + \epsilon/3 \\ &= r(t_0, d_{x_0}) + \epsilon. \end{aligned}$$

Because ϵ was arbitrary, we have that $d(x_0, y_0) \leq r(t_0, d_{x_0})$, or in other words, $y_0 \in \overline{B}(x_0, r(t_0, d_{x_0}))$. But since $x_0 \in X_0$, that means $y_0 \in H(X_0, t_0)$.

So we know that $y_0 \in H(X_0, t_0) \subset U$. Because U is open, there exists some $\epsilon > 0$ such that $B(y_0, \epsilon) \subset U$. By choosing large enough k, $y_k \in B(y_0, \epsilon)$, and thus, $y_k \in U$. This is a contradiction, so H must be continuous with respect to the upper Vietoris topology.

Next, consider the lower Vietoris topology. First, we suppose $t_0 = 1$ and $H(X_0, t_0) \in U_1^- \cap \cdots \cap U_n^-$. Choose any $x' \in X_0$ and let $\epsilon > 0$. Let $V = B(x', \epsilon)$ and then V^- is a neighborhood of X_0. Now, for each U_i, pick some $y_i \in U_i$ and let $\Delta_i = d(x', y_i)$. Since n is finite, we can let $\Delta = \max\{\Delta_i\} + \epsilon$. This is chosen this way so that if $x \in B(x', \epsilon)$, then $d(x, y_i) \leq d(x, x') + d(x', y_i) < \Delta$, and so $\overline{B}(x, \Delta) \cap U_i \neq \emptyset$. Now we need only choose $\delta > 0$ sufficiently small such that if $t \in (1 - \delta, 1)$, then $r(t, d_x) > \Delta$ for all $x \in B(x', \epsilon)$. Because the d_x are bounded below by $d_{x'} - \epsilon$ and above by $d_{x'} + \epsilon$, and r is continuous, this can be done. We choose δ sufficiently small that if $t \in (1 - \delta, 1)$, then $r(t, d_{x'} + \epsilon) > \Delta$. Since r is nondecreasing in t, and for all $x \in B(x', \epsilon)$, $d_x < d_{x'} + \epsilon$, we have that $r(t, d_x) > r(t, d_{x'} + \epsilon) > \Delta$. Then we have a neighborhood $V \times (1 - \delta, 1]$ of (X_0, t_0) such that if (X, t) is in that neighborhood, $H(X, t) \in U_1^- \ldots U_n^-$.

Suppose that $t_0 \neq 1$, and consider an open set in the lower Vietoris topology containing $H(X_0, t_0)$ of the form $U_1^- \cap \cdots \cap U_n^-$. The U_i are all open sets in M. Then for every i, there exists some $x_i \in X_0$ with $\overline{B}(x_i, r(t_0, d_{x_i})) \cap U_i \neq \emptyset$. Furthermore, since each U_i is open, and since $\overline{B}(x_i, r(t_o, d_{x_i})) = \overline{B(x_i, r(t_0, d_{x_i}))}$, we can choose some $y_i \in B(x_i, r(t_0, d_{x_i})) \cap U_i$ (the open ball). For each i, let

$$\zeta_i = \frac{1}{2}(r(t_0, d_{x_i}) - d(x_i, y_i)).$$

By the continuity of r and d_x, for every i, there exists some δ_i and $\epsilon_i > 0$ such that for all $x \in B(x_i, \epsilon_i)$ and for all t with $|t - t_0| < \delta_i$, it must be that $|r(t, d_x) - r(t_0, d_{x_i})| < \zeta_i$.

Let $\eta_i = \min\{\zeta_i, \epsilon_i\}$ and let $V_i = B(x_i, \eta_i)$. Let $\delta = \min\{\delta_i\}$ and let $N = (V_1^- \cap \cdots \cap V_n^-) \times ([0, 1) \cap (t_0 - \delta, t_0 + \delta))$. This will be a neighborhood of (X_0, t_0) which maps into $U_1^- \cap \cdots \cap U_n^-$. To show this, suppose (X, t) is an arbitrary point in N. For each i, because $X \in V_i$, there is an $x_i' \in X$ with $d(x_i, x_i') < \eta_i \leq \epsilon_i$. But ϵ_i was chosen such that this means $|r(t_0, d_{x_i}) - r(t, d_{x_i'})| < \zeta_i$, which implies

$$r(t_0, d_{x_i}) - \zeta_i < r(t, d_{x_i'}).$$

We also know that $d(x_i, x_i') < \eta_i \leq \zeta_i$, and so

$$\begin{aligned} d(x_i', y_i) &\leq d(x_i', x_i) + d(x_i, y_i) < \zeta_i + (r(t_0, d_{x_i}) - 2\zeta_i) \\ &= r(t_0, d_{x_i}) - \zeta_i < r(t, d_{x_i'}). \end{aligned}$$

This statement about distance means that y_i is in the growing ball around x_i', part of $H(X,t)$. But y_i is also in U_i, and so we have $H(X,t) \cap U_i \neq \emptyset$. This completes the proof. □

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Department of Mathematics; Stonehill College; 320 Washington Street; Easton, MA 02357

E-mail address: nesty@stonehill.edu