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DENDRITES WITH UNIQUE n-FOLD HYPERSPACE

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Abstract. Let X be a metric continuum and n be a positive integer. Let $C_n(X)$ be the hyperspace of the nonempty closed subsets of X with at most n components. In this paper we prove the following result: Let X be a dendrite whose set of end points is closed and Y is any continuum such that $C_n(X)$ is homeomorphic to $C_n(Y)$ for some $n \geq 3$, then X is homeomorphic to Y.

Introduction

A *continuum* is a nonempty compact, connected, metric space. Given a continuum X, we denote

$$2^X = \{A \subset X : A \text{ is nonempty and closed}\}.$$

The topology on this space will be induced by the Hausdorff metric H. All concepts not defined here will be taken as in [31]. We also consider the following subspaces of 2^X :

$$C(X) = \left\{A \in 2^X : A \text{ is connected}\right\},$$

$$C(p, X) = \{A \in C(X) : p \in A\},$$

and if n is a positive integer, the *n-fold hyperspace* of X is

$$C_n(X) = \left\{A \in 2^X : A \text{ has at most } n \text{ components}\right\}.$$

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All these spaces, called hyperspaces, are continua (see [23], [30]).

Let $\mathcal{H}(X)$ be any one of the hyperspaces defined above. We say that a continuum X has *unique hyperspace* $\mathcal{H}(X)$ provided that X is homeomorphic to Y whenever $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$.

A *dendrite* is a locally connected continuum which contains no simple closed curves.

We denote by $\mathfrak{D}$ the set of dendrites X whose set of end points is closed.

It has been proved that every $X \in \mathfrak{D}$, which is not an arc, has unique hyperspace $C(X)$ [12]. Moreover, if the dendrite X is not an element of $\mathfrak{D}$, then X does not have unique hyperspace $C(X)$ [7].

The purpose of this paper is to prove the following result: Let $X \in \mathfrak{D}$ and Y be a continuum. If $C_n(X)$ is homeomorphic to $C_n(Y)$ for some $n \geq 3$, then X is homeomorphic to Y. In the case that $n = 2$, other techniques are needed, and this case is proved in [13] and [21].

In the first section, we give the notation and definitions which we use in this paper. In the second section, we prove a characterization of the set $\mathfrak{D}$. In order to do this, some theorems about dimension will be proved. In the third section, we prove relations between subsets of $C_n(X)$ given in the first section. In the fourth section, we give two results: First, we prove that if X, $Y \in \mathfrak{D}$ and $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y; second, we assume that $X \in \mathfrak{D}$ and Y is any dendrite and we prove that the assertion $C_n(X)$ is homeomorphic to $C_n(Y)$ implies that $Y \in \mathfrak{D}$. In the fifth section, we prove that if $C_n(X)$ is homeomorphic to $C_n(Y)$ for $X \in \mathfrak{D}$ and Y is any continuum, then Y is a dendrite.

Results related to the subject of this paper can be found in [1]–[7], [11]–[14], [16]–[22], [25]–[28]. Some proofs in this paper are similar to those in [20].

1. The notation and definitions

The symbol $\mathbb{N}$ will denote the set of positive integers. Let Z be any metric space. The interior, closure, and boundary of a subset A in Z will be denoted by $\mathrm{Int}_Z(A)$, $\mathrm{Cl}_Z(A)$, and $\mathrm{Bd}_Z(A)$, respectively.

For a subset A of Z, $p \in Z$, and $\varepsilon > 0$, $B_Z(p, \varepsilon)$ denotes the *ε-ball* centered at p and $N(\varepsilon, A) = \cup_{p \in A} B_Z(p, \varepsilon)$.

For $n \in \mathbb{N}$, an *n-cell* is any space homeomorphic to $\Pi_{i=1}^{n} [0,1]_i$ where each $[0,1]_i = [0,1]$. A 1-cell is called an *arc*. A *Hilbert cube* is a space homeomorphic to $I^\infty = \Pi_{i=1}^{\infty} [0,1]_i$ where each $[0,1]_i = [0,1]$. An *n-od* $(3 \leq n < \infty)$ is a continuum Y which contains a continuum Z, such that $Y - Z = \cup_{i=1}^{n} Z_i$, where $Z_i \neq \emptyset$ for each i and $\mathrm{Cl}_Y(Z_i) \cap Z_j = \emptyset$ whenever $i \neq j$. A *simple n-od* $(n \geqslant 3)$ is the union of n arcs emanating from a single point v and otherwise disjoint from one another. Given subsets $U_1, ..., U_m$ of X, let $\langle U_1, ..., U_m \rangle_n = \{A \in C_n(X) \colon A \subset U_1 \cup \cdots \cup U_m$ and $A \cap U_i \neq \emptyset$ for each $i \in \{1, ..., m\}\}$. It is known that the family of sets of the form $\langle U_1, ..., U_m \rangle_n$, where $U_1, ..., U_m$ are open subsets of X, is a basis for the topology of $C_n(X)$ induced by the Hausdorff metric [30, Theorem 0.13].

Let X be a dendrite. Every two points x, y in X are joined by a unique arc contained in X which will be denoted by $[x, y]$. The *order of X at a point p*, *ord*(p, X), is defined as the number of components of $X - \{p\}$. This definition coincides with the definition in [24, p. 274]. Points of order 1, 2, and greater than or equal to 3 are called *end points*, *ordinary points*, and *ramification points* of X, respectively. The set of end points, ordinary points, and ramification points of X are denoted by $E(X)$, $O(X)$, and $R(X)$, respectively. Then $X = O(X) \cup R(X) \cup E(X)$. We write $E(X) = E_I(X) \cup E_a(X)$, where $E_I(X)$ is the set of isolated points of $E(X)$ and $E_a(X) = E(X) - E_I(X)$ is the set of accumulation points of $E(X)$. A point q in X will be called *essential* in either of the following two cases: (I) $\mathrm{ord}(q, X) = \omega$; (II) there exists an arc $\alpha \subset X$ and a sequence $\{q_n\}_{n \in \mathbb{N}}$ of different points in α which converges to q and such that $q_n \in R(X)$ for each $n \in \mathbb{N}$. We call q $I-essential$ and $II-essential$, respectively.

We write $\dim(X)$ to denote the *dimension of the space* X; and $\dim_p(X)$ stands for the *dimension of the space X at the point p* (where $p \in X$) (see [32]).

If $\mathcal{V}$ is a 2-cell, then $\partial \mathcal{V}$ denotes the *manifold boundary* of $\mathcal{V}$. Suppose that Y is a continuum and let $\Omega(Y) = \{A \in C(Y) :$ there exists a 2-cell $\mathcal{V}$ in $C(Y)$ such that $A \in \mathrm{Int}_{C(Y)}(\mathcal{V}) \cap \partial \mathcal{V}\}$.

Given a dendrite X and $n \in \mathbb{N}$, we define the following subsets of $C_n(X)$.

$\mathcal{L}_n(X) = \{A \in C_n(X)\colon\ A$ has a $2n$-cell neighborhood in $C_n(X)\}$,

$\mathcal{P}_n(X) = \{A \in C_n(X)\colon A \cap R(X) = \emptyset$ and $A \cap E_a(X) = \emptyset\}$,

$\Gamma_n(X)$ the set of elements $A \in C_n(X) - \mathcal{L}_n(X)$ such that there exists a local basis β of open neighborhoods of A such that for each $\mathfrak{U} \in \beta$, $\dim \mathfrak{U} \leq 2n$ and $\mathfrak{U} \cap \mathcal{L}_n(X)$ is arcwise connected.

For $n \geq 2$, $\mathcal{M}_n(X) = \{A \in C_n(X) - C_{n-1}(X) : A \cap R(X) = \emptyset$ and $A \cap E_a(X) = \emptyset\}$.

2. The set $\mathfrak{D}$

In this section we prove a characterization of the set $\mathfrak{D}$ (Theorem 2.6), but first we give a remark and prove some results about dimension.

Remark 2.1. (a) Let X be a continuum, $A \in 2^X$ and U be an open subset of X such that $A \subset U$. Then there is $\varepsilon > 0$ such that $A \subset N(\varepsilon, A) \subset U$.

(b) Let $X \in \mathfrak{D}$ and $T \in C(X)$. If $T \cap E_a(X) = \emptyset$, then T is a tree [8, Theorem 3.1].

Lemma 2.2. *Let $X \in \mathfrak{D}$, $n \in \mathbb{N}$, and $A \in C_n(X)$. Then $A \cap E_a(X) \neq \emptyset$ if and only if $\dim_A(C_n(X)) = \infty$.*

Proof: Let $n \in \mathbb{N}$ and $A = A_1 \cup ... \cup A_k \in C_n(X)$ where $A_1, ..., A_k$ are the components of A and $A_1 \cap E_a(X) \neq \emptyset$. Since X is normal and locally connected, there exists a continuum D_1 such that $A_1 \subset \mathrm{Int}_X(D_1)$ and $A_j \cap D_1 = \emptyset$ for $j \in \{2, ..., k\}$. Let $p \in A_1 \cap E_a(X)$. Since $p \in \mathrm{Int}_X(D_1)$, there exists a sequence $\{e_n\}_{n\in\mathbb{N}}$ of end points of X contained in $\mathrm{Int}_X(D_1) \cap E_a(X)$. Then $D_1 \in \mathfrak{D}$ and $p \in E_a(D_1)$ so that by [12, Proposition 3*(a)*], p is an essential point of D_1. If T is any tree contained in D_1, by [29, Properties 1.2], $p \notin \mathrm{Int}_{D_1}(T)$. Therefore, by [10, Theorem 10], $C(p, D_1)$ is homeomorphic to I^∞. Let $h : C(D_1) \times \{A_2\} \times \cdots \times \{A_k\} \to C_n(X)$ given by $h(Q_1, A_2, ..., A_k) = Q_1 \cup A_2 \cup \cdots \cup A_k$. Then h is a one-to-one continuous function. Since $C(p, D_1) \times \{A_2\} \times \cdots \times \{A_k\}$ is homeomorphic to I^∞, then $\mathcal{T} = h(C(p, D_1) \times \{A_2\} \times \cdots \times \{A_k\})$ is also homeomorphic to I^{∞}. Moreover, $h(A_1,...,A_k)=A\in\mathcal{T}$. Then $\dim_A(\mathcal{T})=\infty$. Since $\mathcal{T}\subset C_n(X)$, we have that $\dim_A(C_n(X))=\infty$.

On the other hand, assume that $A\cap E_a(X)=\emptyset$. Since $X\in\mathfrak{D}$, it follows from Remark 2.1(b) that each component of A is a tree. Therefore, there is a finite graph $G\in C(X)$ such that $A\subset \mathrm{Int}_X(G)$ and $G\cap E_a(X)=\emptyset$. By Remark 2.1(a), there is $\varepsilon>0$ such that $N(\varepsilon,A)\subset \mathrm{Int}_X(G)$. Thus, $B_{C_n(X)}(A,\varepsilon)\subset C_n(G)$. Since $\dim_A(C_n(G))=\dim_A(C_n(X))$, then $\dim_A(C_n(X))$ is finite. □

Lemma 2.3. *Let $X\in\mathfrak{D}$, $n\in\mathbb{N}$, and $A\in C_n(X)$. Then $A\cap E_a(X)\neq\emptyset$ if and only if for each $\varepsilon>0$, we have that*

$$\dim(B_{C_n(X)}(A,\varepsilon))=\infty.$$

Proof: Since $A\cap E_a(X)\neq\emptyset$, by Lemma 2.2, $\dim_A(C_n(X))=\infty$. Then, for each $\varepsilon>0$, we have that $\dim(B_{C_n(X)}(A,\varepsilon))=\infty$.

On the other hand, assume that $A\cap E_a(X)=\emptyset$. By the second part of the proof of Lemma 2.2, there is a finite graph $G\in C(X)$ and $\varepsilon>0$ such that $B_{C_n(X)}(A,\varepsilon)\subset C_n(G)$ and hence, $\dim(B_{C_n(X)}(A,\varepsilon))\leq \dim(C_n(G))$. By [27, Theorem 5.1], we have that $\dim(C_n(G))<\infty$. Therefore, $\dim(B_{C_n(X)}(A,\varepsilon))<\infty$. □

Lemma 2.4. *Let $X\in\mathfrak{D}$, $n\in\mathbb{N}$, and $A\in C_n(X)$ such that $A\cap R(X)\neq\emptyset$. Then $\dim(\mathcal{S})\geq 2n+1$ for each neighborhood S of A in $C_n(X)$.*

Proof: Let $\mathcal{S}$ be a neighborhood of A in $C_n(X)$. If $A\cap E_a(X)\neq\emptyset$, by Lemma 2.3, $\dim(\mathcal{S})=\infty$. Suppose that $A\cap E_a(X)=\emptyset$ and let $p\in R(X)\cap A$. Clearly, there exists $B\in\mathcal{S}$ with exactly n components. Let $B=A_1\cup\cdots\cup A_n$ and assume that $p\in A_1$ and that $A_i\cap E_a(X)=\emptyset$ for each $i\in\{1,...,n\}$. Let $D_1,...,D_n$ be pairwise disjoint trees of X such that $A_i\subset \mathrm{Int}_X(D_i)$ for each $i\in\{1,...,n\}$. Let $\mathcal{V}=\langle D_1,...,D_n\rangle_n$. Then the function $h:C(D_1)\times\cdots\times C(D_n)\to\mathcal{V}$ given by $h(Q_1,...,Q_n)=Q_1\cup\cdots\cup Q_n$ is a homeomorphism. By [9, 5.2], for each $i\in\{1,...,n\}$, there exists an m_i-cell $\mathcal{G}_i$ ($m_i\geq 2$), contained in $C(D_i)$, such that $A_i\in\mathcal{G}_i$. Since $p\in R(X)\cap A_1$, by [9, 5.2], $\mathcal{G}_1$ is an m_1-cell, with $m_1\geq 3$. Thus, $h(\mathcal{G}_1\times\cdots\times\mathcal{G}_n)=\mathcal{G}$ is an m-cell for some $m\geq 2n+1$. Moreover, $B\in\mathcal{G}$. Therefore, $B\in\mathcal{S}\cap\mathcal{G}$. Hence, $\mathcal{S}\cap\mathcal{G}$ contains an m-cell, for some $m\geq 2n+1$. Therefore, $\dim(\mathcal{S})\geq 2n+1$. □

Corollary 2.5. *Let $X \in \mathfrak{D}$ such that X is not an arc and let $n \in \mathbb{N}$. Then $C_n(X)$ is not homeomorphic to $C_n([0,1])$.*

Proof: By [27, Corollary 5.3], $\dim(C_n([0,1])) = 2n$. Let $A \in C_n(X)$ such that $A \cap R(X) \neq \emptyset$ and let $\mathcal{S}$ be a neighborhood of A in $C_n(X)$. By Lemma 2.4, $\dim(\mathcal{S}) \geq 2n+1$. Therefore, $C_n(X)$ is not homeomorphic to $C_n([0,1])$. □

We will need two special dendrites. One of them is the dendrite $F_\omega = \bigcup_{n\in\mathbb{N}} [p, a_n]$, where $p = (0,0)$, $a_n = \left(\frac{1}{n}, \frac{1}{n^2}\right)$, and $[p, a_n]$ denotes the straight line segment joining p and a_n in the plane. In other words, F_ω is the unique dendrite for which $R(F_\omega)$ consists of a single point p and $\operatorname{ord}(p, F_\omega) = \omega$.

We define the other dendrite $W = [c, b_1] \cup [\bigcup \{[a_n, b_n] : n \in \mathbb{N}\}]$, where $a_n = \left(\frac{1}{n}, \frac{1}{n}\right)$, $b_n = \left(\frac{1}{n}, 0\right)$, for each $n \in \mathbb{N}$, and $c = (-1, 0)$.

Notice that if $Z = [c, b_1]$, the sequence $\{A_s\}_{s\in\mathbb{N}}$, where $A_s = [c, -\frac{1}{s}] \cup [\frac{1}{s}, 1]$, is such that $\lim A_s = Z$ and $\dim_{A_s}(C_n(X)) < \infty$, for each $s \in \mathbb{N}$.

The class $\mathfrak{D}$ is described in [8]. We obtain the following characterization.

Theorem 2.6. *Let X be a dendrite and $n \in \mathbb{N}$. Then $X \in \mathfrak{D}$ if and only if for each $Z \in C_n(X)$ there is a sequence $\{A_s\}_{s\in\mathbb{N}}$ in $C_n(X)$ such that $\lim A_s = Z$ and $\dim_{A_s}(C_n(X)) < \infty$ for each $s \in \mathbb{N}$.*

Proof: For $n = 1$, this result was proved in [12, Theorem 8]. Thus, we can assume that $n \geq 2$. Let $X \in \mathfrak{D}$. When X is homeomorphic to $[0,1]$, the result follows by [27, Corollary 5.3]. Suppose that X is not an arc and let $Z \in C_n(X)$, $Z = Z_1 \cup \cdots \cup Z_k$ where $Z_1, ..., Z_k$ are the components of Z. By [12, Theorem 8], for each Z_i, with $1 \leq i \leq k$, there is a sequence $\left\{A_s^i\right\}_{s\in\mathbb{N}}$ in $C(X)$ that converges to Z_i and $\dim_{A_s^i}(C(X)) < \infty$ for each $s \in \mathbb{N}$. Therefore, by [12, Theorem 1 and Proposition 3*(a)*], $A_s^i \cap E_a(X) = \emptyset$ for each $s \in \mathbb{N}$. Let $A_s = A_s^1 \cup \cdots \cup A_s^k$, then $\{A_s\}_{s\in\mathbb{N}}$ is a sequence in $C_n(X)$ that converges to Z and $A_s \cap E_a(X) = \emptyset$. By Corollary 2.2, we have that $\dim_{A_s}(C_n(X)) < \infty$ for each $s \in \mathbb{N}$.

Conversely, suppose that $X \notin \mathfrak{D}$. Then $E(X)$ is not a closed set. With the notation used above, by [8, Theorem 3.3], X contains a subspace Y which is homeomorphic to either W or F_ω. Assume first that $W \subset X$ and let $Z = Z_1 \cup \cdots \cup Z_n$, where $Z_1, ..., Z_n$ are nonempty pairwise disjoint subcontinua of X and $Z_1 = [c, b_1]$. By [12, Theorem 8], there is not a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $C(X)$ that converges to Z_1 and $\dim_{T_j}(C(X)) < \infty$ for each $j \in \mathbb{N}$. Therefore, there is not a sequence $\{A_s\}_{s\in\mathbb{N}}$ in $C_n(X)$ that converges to Z and $\dim_{A_s}(C_n(X)) < \infty$ for each $s \in \mathbb{N}$. The proof of the case on which $F_\omega \subset X$ is similar. □

3. Subsets of $C_n(X)$

Previously, we defined subsets of $C_n(X)$ as $\mathcal{L}_n(X)$, $\mathcal{P}_n(X)$, etc., for a dendrite X and $n \in \mathbb{N}$; in this section, we will prove relationships among them. These results will be used to prove Theorem 4.1.

The following remark is a consequence of the definition of $\mathcal{L}_n(X)$.

Remark 3.1. If $X \in \mathfrak{D}$, $n \in \mathbb{N}$, and $A \in \mathcal{L}_n(X)$, then $\dim_A(C_n(X)) \leq 2n$.

By Lemma 2.2, Lemma 2.4, and Remark 3.1, we obtain the following corollary.

Corollary 3.2. *If $X \in \mathfrak{D}$, $n \in \mathbb{N}$, and $A \in \mathcal{L}_n(X)$, then $A \cap E_a(X) = \emptyset$ and $A \cap R(X) = \emptyset$.*

Lemma 3.3. *If $X \in \mathfrak{D}$ and $n \in \mathbb{N}$, then $\mathcal{M}_n(X) \subset \mathcal{L}_n(X)$.*

Proof: Let $A \in \mathcal{M}_n(X)$ and $A_1, ..., A_n$ be the different components of A. Then for each $i \in \{1, ..., n\}$, A_i is an arc or a single point and there exist disjoint arcs J_i in X such that $A_i \subset \mathrm{Int}_X(J_i)$, $J_i \cap R(X) = \emptyset$, and $J_i \cap E_a(X) = \emptyset$. Then $\langle J_1, ..., J_n\rangle_n$ is a neighborhood of A in $C_n(X)$. Moreover, $C(J_1) \times \cdots \times C(J_n)$ is homeomorphic to $\langle J_1, ..., J_n\rangle_n$. Since each $C(J_i)$ is a 2-cell (by [9, p. 267]), we have that $\langle J_1, ..., J_n\rangle_n$ is a $2n$-cell. Therefore, $A \in \mathcal{L}_n(X)$. □

Lemma 3.4. *If $X \in \mathfrak{D}$ and $n \in \mathbb{N}$, then $\dim(\mathcal{P}_n(X)) \leq 2n$.*

Proof: We prove this lemma by induction. Suppose first that $n = 1$. Given $A \in \mathcal{P}_1(X)$, there exists an arc J of X such that $A \subset \mathrm{Int}_X(J)$. Hence, $C(J)$ is a neighborhood of A in $C(X)$. Since $C(J)$ is a 2-cell (by [9, p. 267]), then $\dim_A(C(X)) = 2$. Therefore, $\dim(\mathcal{P}_1(X)) \leq 2$.

Now suppose that $n \geq 2$ and that the lemma is true for $n-1$. We notice that the set $\mathcal{P}_{n-1}(X) = \mathcal{P}_n(X) \cap C_{n-1}(X)$ is closed in $\mathcal{P}_n(X)$ and $\dim(\mathcal{P}_{n-1}(X)) \leq 2(n-1)$. By Remark 3.1 and Lemma 3.3, $\dim(\mathcal{M}_n(X)) \leq 2n$; therefore, since $\mathcal{P}_n(X) = \mathcal{P}_{n-1}(X) \cup \mathcal{M}_n(X)$, by [15, Corollary 1, p. 32], $\dim(\mathcal{P}_n(X)) \leq 2n$. □

Lemma 3.5. *Let $X \in \mathfrak{D}$, $n \in \mathbb{N} - \{1,2\}$, and $A \in C_{n-1}(X) - C(X)$. Assume that $A \cap R(X) = \emptyset$ and $A \cap E_a(X) = \emptyset$. Then there exists a neighborhood $\mathcal{V}$ of A in $C_n(X)$ such that each closed neighborhood $\mathcal{N}$ of A in $C_n(X)$ satisfying $\mathcal{N} \subset \mathcal{V}$ can be separated by a closed set $\mathcal{S} \subset C_{n-1}(X)$ with $\dim(\mathcal{S}) \leq 2(n-1)$ and such that for each $T \in \mathcal{S}$, $T \cap E_a(X) = \emptyset$ and $T \cap R(X) = \emptyset$.*

Proof: Let $A_1, ..., A_m$ be the components of A. By the hypothesis, each A_i is an arc. Choose pairwise disjoint subarcs $J_1, ..., J_m$ of X such that for each $i \in \{1, ..., m\}$, $A_i \subset \text{Int}_X(J_i)$ and $(J_1 \cup \cdots \cup J_m) \cap R(X) = \emptyset$. (Thus, $(J_1 \cup \cdots \cup J_m) \cap E_a(X) = \emptyset$). Put $\mathcal{V} = \langle J_1, ..., J_m \rangle_n$ and let $\mathcal{N}$ be a closed neighborhood of A in $C_n(X)$ such that $\mathcal{N} \subset \mathcal{V}$. We define $\mathcal{R} = \{M \in \mathcal{N} : M \cap J_1$ has at least two components$\}$ and $\mathcal{Q} = \{M \in \mathcal{N} \colon M \cap (J_2 \cup \cdots \cup J_m)$ has at least n-1 components$\}$. It is easy to see that $\mathcal{R}$ and $\mathcal{Q}$ are open in $\mathcal{N}$.

If we choose a point $p \in J_1 - A_1$ such that p is close enough to A_1, then $A_1 \cup \{p\} \in \mathcal{N}$, and $(A_1 \cup \{p\}) \cap J_1$ has two components. So, $\mathcal{R} \neq \emptyset$. Now, choose different points $q_{m+1}, ..., q_n$ in $J_2 - A_2$ such that the points q_i are close enough to A_2. Then $A \cup \{q_{m+1}, ..., q_n\} \in \mathcal{N}$. (Here is the step where we use the assumption $n \geq 3$.) Thus, $\mathcal{Q} \neq \emptyset$. Notice that an element $M \in \mathcal{R} \cap \mathcal{Q}$ contains at least $n+1$ components, which is impossible since $\mathcal{R} \cap \mathcal{Q} \subset C_n(X)$. So, we have proved that $\mathcal{R} \cap \mathcal{Q} = \emptyset$. Let $\mathcal{S} = \mathcal{N} - (\mathcal{R} \cup \mathcal{Q})$. Then $M \in \mathcal{S}$ if and only if $M \in \mathcal{N}$ and M has exactly one component contained in J_1 and at most $n-2$ components contained in $J_2 \cup ... \cup J_m$. Moreover, since $M \in \mathcal{N} \subset \mathcal{V}$, then $\mathcal{S} \subset \{M \in C_{n-1}(X) : M \cap R(X) = \emptyset$ and $M \cap E_a(X) = \emptyset\}$. By Lemma 3.4, $\dim(\mathcal{S}) \leq 2(n-1)$. Therefore, $\mathcal{N}$ can be separated by a closed subset $\mathcal{S}$ of $C_{n-1}(X)$ such that $\dim(\mathcal{S}) \leq 2(n-1)$. □

Lemma 3.6. *If $X \in \mathfrak{D}$ and $n \in \mathbb{N} - \{1,2\}$, then $\mathcal{M}_n(X) = \mathcal{L}_n(X)$.*

Proof: By Lemma 3.3, we need only to prove that $\mathcal{L}_n(X) \subset \mathcal{M}_n(X)$. If $A \in \mathcal{L}_n(X)$, there exists a neighborhood $\mathcal{U}$ of A in

$C_n(X)$ such that $\mathcal{U}$ is a $2n$-cell. By Corollary 3.2, $A \cap E_a(X) = \emptyset$ and $A \cap R(X) = \emptyset$. We need to prove that $A \notin C_{n-1}(X)$. Suppose to the contrary that $A \in C_{n-1}(X)$. Then there exists an element $A_0 \in C_{n-1}(X) - C(X)$ such that $A_0 \in \mathrm{Int}_{C_n(X)}(\mathcal{U})$, $A_0 \cap R(X) = \emptyset$, and $A_0 \cap E_a(X) = \emptyset$. Let $\mathcal{V}$ be a neighborhood of A_0 in $C_n(X)$, as in Lemma 3.5. Then there exists a closed neighborhood $\mathcal{N}$ of A_0 in $C_n(X)$ such that $\mathcal{N} \subset \mathcal{U} \cap \mathcal{V}$ and $\mathcal{N}$ is a $2n$-cell. By the choice of $\mathcal{V}$, $\mathcal{N}$ can be separated by a closed subset $\mathcal{S}$ of $\mathcal{N}$ such that $\dim(\mathcal{S}) \leq 2(n-1)$. This contradicts [15, Corollary 2, p. 48] and completes the proof of the lemma. □

Lemma 3.7. *If* $X \in \mathfrak{D}$ *and* $n \in \mathbb{N} - \{1, 2\}$, *then* $\Gamma_n(X) = \mathcal{P}_1(X)$.

Proof: Let $A \in \Gamma_n(X)$. By Lemma 2.4, $A \cap R(X) = \emptyset$, and by Lemma 2.3, $A \cap E_a(X) = \emptyset$. We need to prove that A is connected. Suppose to the contrary that $A \notin C(X)$. Since $A \notin \mathcal{L}_n(X)$, by Lemma 3.6, $A \notin \mathcal{M}_n(X)$, so $A \in C_{n-1}(X) - C(X)$. Let $\mathcal{V}$ be a neighborhood of A in $C_n(X)$, as in Lemma 3.5. Since $A \in \Gamma_n(X)$, there exists an open neighborhood $\mathcal{N}$ of A in $C_n(X)$ such that $\dim(\mathcal{N}) \leq 2n$, $\mathcal{N} \cap \mathcal{L}_n(X)$ is arcwise connected, and for each element $T \in \mathcal{N}$, $T \cap R(X) = \emptyset$, $T \cap E_a(X) = \emptyset$, and $\mathrm{Cl}_{\,C_n(X)}(\mathcal{N}) \subset \mathcal{V}$. By the choice of $\mathcal{V}$, $\mathrm{Cl}_{C_n(X)}(\mathcal{N})$ can be separated by a closed set $\mathcal{S}$ contained in $C_{n-1}(X)$. Then there exist two nonempty disjoint open subsets $\mathcal{H}$ and $\mathcal{K}$ of $\mathrm{Cl}_{C_n(X)}(\mathcal{N})$ such that $\mathrm{Cl}_{C_n(X)}(\mathcal{N}) - \mathcal{S} = \mathcal{H} \cup \mathcal{K}$. Then $\mathcal{H} \cap \mathcal{N}$ and $\mathcal{K} \cap \mathcal{N}$ are nonempty open subsets of $C_n(X)$. By [28, Theorem 3.3], $C_n(X) - C_{n-1}(X)$ is dense in $C_n(X)$. Then there exist elements $E \in \mathcal{H} \cap \mathcal{N} \cap (C_n(X) - C_{n-1}(X))$ and $F \in \mathcal{K} \cap \mathcal{N} \cap (C_n(X) - C_{n-1}(X))$. Then $E, F \in \mathcal{M}_n(X) \cap \mathcal{N} = \mathcal{L}_n(X) \cap \mathcal{N}$. Since $\mathcal{L}_n(X) \cap \mathcal{N}$ is arcwise connected, there exists a subarc α of $\mathcal{L}_n(X) \cap \mathcal{N}$ joining E and F. By the choice of $\mathcal{S}$, there exists an element $Z \in \alpha \cap \mathcal{S}$. Since $Z \in \alpha \subset \mathcal{L}_n(X) = \mathcal{M}_n(X)$, $Z \notin C_{n-1}(X)$. This is impossible since $Z \in \mathcal{S} \subset C_{n-1}(X)$. This contradiction proves that A is connected.

To prove the opposite inclusion, take $A \in C(X)$ such that $A \cap R(X) = \emptyset$ and $A \cap E_a(X) = \emptyset$. Clearly, $A \notin \mathcal{M}_n(X) = \mathcal{L}_n(X)$ and there exists an open connected subset U of X such that $A \subset U$ and $U \cap R(X) = \emptyset$. Suppose that $U \cap E_a(X) \neq \emptyset$. Since $E_a(X) = \{p \colon p \text{ is a } II\text{-essential point}\}$ ([12, Proposition 3, *(a)*]), we obtain that $U \cap R(X) \neq \emptyset$, a contradiction. Then $U \cap E_a(X) = \emptyset$. Let $\mu = \{B_{C_n(X)}(A, \varepsilon) \subset C_n(X) \colon N(\varepsilon, A) \subset U\}$ and take $\mathcal{G} =$

$B_{C_n(X)}(A, \varepsilon) \in \mu$. Then $\mathcal{G} \subset \mathcal{P}_n$. By Lemma 3.4, $\dim(\mathcal{G}) \leq 2n$. Notice that $\mathcal{G} \cap \mathcal{L}_n(X) = \mathcal{G} \cap \mathcal{M}_n(X) = \mathcal{G} \cap (C_n(X) - C_{n-1}(X))$.

Given $K \in \mathcal{G} \cap \mathcal{L}_n(X) = \mathcal{G} \cap (C_n(X) - C_{n-1}(X))$, there exists $L \in \mathcal{G} \cap \mathcal{L}_n(X)$ with the property that each component of $N(\varepsilon, A) - L$ has the same length and this common length δ is less than $\frac{\varepsilon}{3n}$. Now, we can connect L by an arc γ in $\mathcal{G} \cap (C_n(X) - C_{n-1}(X))$ to an element M such that each component of $N(\varepsilon, A) - M$ has the same length δ and they are evenly distributed in $N(\varepsilon, A)$. Finally, enlarge all the components of $N(\varepsilon, A) - M$ in such a way that M is connected, by an arc σ in $\mathcal{G} \cap (C_n(X) - C_{n-1}(X))$, to an element S with the property that each component of $N(\varepsilon, A) - S$ has the same length $\frac{\varepsilon}{3n}$ and they are evenly distributed in $N(\varepsilon, A) - S$. Since S does not depend on K, we have shown that $\mathcal{G} \cap \mathcal{L}_n(X)$ is arcwise connected. This completes the proof of the lemma. □

Moreover, by [12, Proposition 3], we obtain the following corollary.

Corollary 3.8. *If $X \in \mathfrak{D}$ and $n \in \mathbb{N} - \{1, 2\}$, then $\Omega(X) \subset \Gamma_n(X)$.*

The following equality will be used in the proof of Theorem 4.1.

Remark 3.9. If $X \in \mathfrak{D}$ and $n \in \mathbb{N}$, then we obtain the following equality $\mathrm{Cl}_{C_n(X)}(\Omega(X)) = \mathrm{Cl}_{C(X)}(\Omega(X))$.

4. Previous results

Theorem 4.1. *Let X, $Y \in \mathfrak{D}$ and $n \in \mathbb{N}$. If $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y.*

Proof: For $n = 1, 2$, these results were proved in [12, Theorem 6] and [13, Theorem 12], respectively. Thus, we can assume that $n \geq 3$. Let $h : C_n(X) \to C_n(Y)$ be a homeomorphism. Since $h(\mathcal{L}_n(X)) = \mathcal{L}_n(Y)$, then $h(\Gamma_n(X)) = \Gamma_n(Y)$. By Lemma 3.7, $\Gamma_n(X) = \{A \in C_n(X) \colon A$ is connected, $A \cap R(X) = \emptyset$, and $A \cap E_a(X) = \emptyset\}$. It has been proved that $\Omega(X) = \{\{p\} \in C(X) \colon p \in X - (R(X) \cup E_a(X))\} \bigcap \{\alpha \in C(X) \colon \alpha$ is an arc of $X - R(X)$ containing a point of $E_I(X)\}$ [12, Corollary 1]. Moreover, $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X [12, Theorem 5] (and $\mathrm{Cl}_{C(Y)}(\Omega(Y))$ is homeomorphic to Y). Since $h(\Gamma_n(X)) = \Gamma_n(Y)$, by Corollary 3.8, we obtain that $h(\Omega(X)) = \Omega(Y)$ and $h(\mathrm{Cl}_{C(X)}(\Omega(X))) = \mathrm{Cl}_{C(Y)}(\Omega(Y))$. Hence, X is homeomorphic to Y. □

Theorem 4.2. *Let $X \in \mathfrak{D}$, Y be a dendrite, and $n \in \mathbb{N}$. If $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y.*

Proof: For the cases $n = 1, 2$, these results were proved in [12, Theorem 9] and [13, Theorem 13]. If X is homeomorphic to $[0, 1]$, by [20], the arc has unique hyperspace $C_n([0, 1])$ for $n > 1$. Thus, we can assume that $n \geq 3$ and X is not an arc. Let $h : C_n(X) \to C_n(Y)$ be a homeomorphism and $Z \in C_n(Y)$, then there is $L \in C_n(X)$ such that $h(L) = Z$. By Theorem 2.6, there exists a sequence $\{A_s\}_{s\in\mathbb{N}}$ in $C_n(X)$ that converges to L and $\dim_{A_s}(C_n(X)) < \infty$ for each $s \in \mathbb{N}$. This implies that $\{h(A_s)\}_{s\in\mathbb{N}}$ is a sequence in $C_n(Y)$ that converges to Z and $\dim_{h(A_s)}(C_n(Y)) < \infty$ for each $s \in \mathbb{N}$. Therefore, $Y \in \mathfrak{D}$ (Theorem 2.6). Hence, by Theorem 4.1, we obtain that X is homeomorphic to Y. □

5. Main theorem

Finally, we are able to give the proof of Theorem 5.7, the main theorem. First, we give a remark and five lemmas.

Remark 5.1. Let X be a continuum, $p \in X$, $n \in \mathbb{N}$, $A \in C_n(X)$, and $\varepsilon > 0$. Then $A \in B_{C_n(X)}(\{p\}, \varepsilon)$ if and only if $A \subset B_X(p, \varepsilon)$.

Lemma 5.2. *Let Y be a locally connected continuum, $p \in Y$, and $n \in \mathbb{N}$. Then $\dim_{\{p\}}(C_n(Y)) = \infty$ if and only if p is not in the interior of a finite graph in Y.*

Proof: If $p \in \mathrm{Int}(G)$, where G is a finite graph in Y, then $p \in B_Y(p, \varepsilon) \subset G$ for some $\varepsilon > 0$. By Remark 5.1, we have that $B_{C_n(Y)}(\{p\}, \varepsilon) \subset C_n(G)$. Thus, $\dim_{\{p\}}(C_n(G)) = \dim_{\{p\}}(C_n(Y))$. Since $\dim_{\{p\}}(C_n(G)) \leq \dim(C_n(G)) < \infty$, we obtain $\dim_{\{p\}}(C_n(Y)) < \infty$.

Conversely, suppose that p does not lie in the interior of any finite graph in Y. Hence, by [10, Theorem 4], $C(\{p\}, Y)$ is homeomorphic to I^∞. Then $\dim_{\{p\}}(C(Y)) = \infty$. Therefore, $\dim_{\{p\}}(C_n(Y)) = \infty$. □

The following result is the converse of [12, Lemma 5].

Lemma 5.3. *Let Y be a locally connected continuum and $F \in C(Y)$. If $\dim_{\{p\}}(C(Y)) < \infty$ for each $p \in F$, then F is contained in the interior of a finite graph in Y.*

Proof: As an intermediate step, we prove two claims.

Claim 1. The continuum F is a finite graph.

To prove Claim 1, suppose that $p \in F$. As $\dim_{\{p\}}(C(Y)) < \infty$, by Lemma 5.2, there exists a finite graph G_p in Y such that $p \in \mathrm{Int}(G_p)$. Since $F \subset \cup_{p \in F}\mathrm{Int}(G_p)$, there exist $p_1, ..., p_s \in F$ such that $F \subset \cup_{j=1}^{s} G_{p_j}$. Thus, $F = \cup_{j=1}^{s}(F \cap G_{p_j})$. Notice that $F \cap G_{p_j}$ is a finite graph for each $j \in \{1, ..., s\}$.

We will show that if $(F \cap G_{p_j}) \cap (F \cap G_{p_k}) \neq \emptyset$ for $j \neq k$, then $(F \cap G_{p_j}) \cup (F \cap G_{p_k})$ is a finite graph. Fix $q \in (F \cap G_{p_j}) \cup (F \cap G_{p_k})$. Since $\dim_{\{q\}}(C(Y)) < \infty$, we have that $\mathrm{ord}(q, (F \cap G_{p_j}) \cup (F \cap G_{p_k}))$ is finite. Suppose that $(F \cap G_{p_j}) \cup (F \cap G_{p_k})$ is not a finite graph. By [31, Theorem 9.10], there exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ such that $\mathrm{ord}(x_i, (F \cap G_{p_j}) \cup (F \cap G_{p_k})) \geq 3$ for each i and $x_i \neq x_j$ when $i \neq j$. We can assume that this sequence converges to a point $p \in F$. By an argument similar to that in the proof of [31, Lemma 9.11], there exists a dendrite L such that L is homeomorphic to W_0 or L is homeomorphic to F_ω such that p is an essential point. By [12, Theorem 1], we have that $\dim_{\{p\}}(C(Y)) = \infty$. This contradiction proves that $(F \cap G_{p_j}) \cup (F \cap G_{p_k})$ is a finite graph.

Claim 2. The $\mathrm{Bd}_Y(F)$ contains a finite number of points.

Suppose to the contrary that $\mathrm{Bd}_Y(F)$ has an infinite number of points. Since F has a finite number of end points, we can assume there exist points $x_i \in \mathrm{Bd}_Y(F)$, $i \in \mathbb{N}$, satisfying $\mathrm{ord}(x_i, Y) \geq 3$ for each i and $x_i \neq x_j$ when $i \neq j$. By an argument similar to that in the proof of [31, Lemma 9.11], we obtain a contradiction. Thus, we prove Claim 2.

Now we return to the proof of lemma. Let $\mathrm{Bd}_Y(F) = \{x_1, ..., x_r\}$. Notice that if α is an arc in Y such that $F \cap \alpha \neq \emptyset$ and $\alpha - F \neq \emptyset$, then there exists $i \in \{1, ..., r\}$ such that $x_i \in F \cap \alpha$.

We will see that x_i is not an accumulation point of $R(Y)$ for each $i \in \{1, ..., r\}$. Suppose to the contrary that there exists a sequence $\{z_m\}_{m \in \mathbb{N}}$ in $R(Y) - \{x_i\}$ such that this sequence converges to x_i and $z_m \neq z_l$ if $m \neq l$. By an argument similar to that in the proof of [31, Lemma 9.11], we obtain a contradiction. This completes the proof that x_i is not an accumulation point of $R(Y)$.

Fix $x_i \in \mathrm{Bd}_Y(F)$ and let $k_i = sup\,\{s \geq 1 :$ there exist arcs $\alpha_1, ...,$ α_s in Y such that x_i is an end point of α_j and $F \cap \alpha_j = \{x_i\}$ for each $j \in \{1, ..., s\}$, and $\alpha_j \cap \alpha_l = \{x_i\}$, if $j \neq l\}$.

Given $i \in \{1, ..., r\}$, let $\alpha_1, ..., \alpha_{k_i}$ be arcs in Y satisfying the condition described in the definition of k_i. We may assume that $\alpha_j \cap R(Y) = \{x_i\}$ for each $j \in \{1, ..., k_i\}$. Given $j \in \{1, ..., k_i\}$, let z_j be the end point of α_j such that $z_j \neq x_i$. Let W_i be a closed arcwise connected neighborhood of $x_i \in Y$ such that $W_i \cap \{x_1, ..., x_r\} = \{x_i\}$ and $W_i \cap (\{z_1, ..., z_{k_i}\} \cup (R(Y) - \{x_i\})) = \emptyset$. We claim that $W_i \subset F \cup \alpha_1 \cup ... \cup \alpha_{k_i}$. Suppose to the contrary that there exists a point $z \in W_i - (F \cup \alpha_1 \cup ... \cup \alpha_{k_i})$. Let $\gamma : [0, 1] \to W_i$ be a one-to-one function such that $\gamma(0) = z$ and $\gamma(1) = x_i$ and let $t_0 = min \gamma^{-1}(F \cup \alpha_1 \cup ... \cup \alpha_{k_i})$. If $\gamma(t_0) \in F$, then $\gamma(t_0) = x_i$; this contradicts the choice of $\alpha_1, ..., \alpha_{k_i}$, and if $\gamma(t_0) \notin F$, then $\gamma(t_0) \in R(Y) \cap W_i - \{x_i\}$, which contradicts the choice of W_i. Therefore, $W_i \subset F \cup \alpha_1 \cup ... \cup \alpha_{k_i}$. Let $G = F \cup W_1 \cup ... \cup W_r$. Thus, G is a finite graph and $F \subset \mathrm{Int}_Y(G)$. □

The following result extends [12, Lemma 5] and Lemma 5.3 to $C_n(X)$ for each $n \in \mathbb{N}$.

Lemma 5.4. *Let Y be a locally connected continuum, $n \in \mathbb{N}$, and $A \in C_n(Y)$. Then $\dim_A(C_n(Y)) < \infty$ if and only if $\dim_{\{p\}}(C_n(Y)) < \infty$, for each $p \in A$.*

Proof: Suppose that A is not contained in the interior of any finite graph. By [10, Theorem 4], we have $\dim_A(C(Y)) = \infty$. Hence, $\dim_A(C_n(Y)) = \infty$, a contradiction to the hypothesis. Thus, there exists a finite graph G in Y such that $A \subset \mathrm{Int}(G)$. Let $p \in A$; by Lemma 5.2, $\dim_{\{p\}}(C_n(Y)) < \infty$.

Conversely, let $A \in C_n(Y)$ and $\dim_{\{p\}}(C_n(Y)) < \infty$ for each $p \in A$. Suppose that A is connected. Since $\dim_A(C(Y)) \leq \dim_A(C_n(Y))$, by Lemma 5.3, there exists a finite graph G contained in Y such that $A \subset \mathrm{Int}_Y(G)$. Hence, $A \in \mathrm{Int}_{C_n(Y)}(C_n(G))$.

Now we consider the case that A is disconnected. Let $A_1, ..., A_m$ be the components of A. Let $Z_1, ..., Z_m$ be disjoint subcontinua of Y such that $A_i \subset \mathrm{Int}_Y(Z_i)$ for each $i \in \{1, ..., m\}$. Since the function $\varphi : C(Z_1) \times ... \times C(Z_m) \to \langle Z_1, ..., Z_m \rangle$, given by $\varphi(Z_1, ..., Z_m) = Z_1 \cup ... \cup Z_m$, is a homeomorphism, we obtain that for each $i \in \{1, ..., m\}$, $\dim_{A_i}(C(Z_i)) \leq \dim_{(A_1,...,A_m)}(C(Z_1) \times ... \times C(Z_m)) = \dim_A(\langle Z_1, ..., Z_m \rangle) = \dim_A(C_n(Y)) < \infty$. Since $C(Z_i)$ is a neighborhood of A_i in $C(Y)$, $\dim_{A_i}(C(Y)) = \dim_{A_i}(C(Z_i))$. By Lemma

5.3, there exists a finite graph G_i contained in Y such that $A_i \subset \mathrm{Int}(G_i)$. We may assume that $G_i \subset Z_i$.

Without loss of generality, let α_1 be an arc in Y such that $G_1 \cap \alpha_1$ and $G_2 \cap \alpha_1$ are one-point sets. Clearly, the set $T_1 = G_1 \cup \alpha_1 \cup G_2$ is a finite graph. Let α_2 be an arc in Y such that $T_1 \cap \alpha_2$ and $G_3 \cap \alpha_2$ are one-point sets. The set $T_2 = T_1 \cup \alpha_2 \cup G_3$ is a finite graph. Continuing in this fashion, $m-1$ applications of [31, Proposition 9.2] give us that $T = G_1 \cup ... \cup G_m \cup \alpha_1 \cup ... \cup \alpha_{m-1}$ is a finite graph. Thus, $A \in \langle \mathrm{Int}_Y(G_i), ..., \mathrm{Int}_Y(G_m) \rangle \subset \mathrm{Int}_{C_n(Y)}(C_n(T))$. Since $\dim_A(C_n(Y)) = \dim_A(C_n(T))$, then $\dim_A(C_n(Y)) < \infty$. □

Lemma 5.5. *Let Y be a locally connected continuum, $p \in Y$, and $n \in \mathbb{N}$. If $\dim_{\{p\}}(C_n(Y)) < \infty$, then $\{p\} \in \mathrm{Cl}_{C(Y)}(\Omega(Y))$.*

Proof: If $\dim_{\{p\}}(C_n(Y)) < \infty$, then $\dim_{\{p\}}(C(Y)) < \infty$. By [12, Lemma 6], we have that $\{p\} \in \mathrm{Cl}_{C(Y)}(\Omega(Y))$. □

Lemma 5.6. *Let $X \in \mathfrak{D}$, Y be a continuum, and $n \in \mathbb{N}$. If $C_n(X)$ is homeomorphic to $C_n(Y)$, then $F_1(Y) \subset \mathrm{Cl}_{C(Y)}(\Omega(Y))$.*

Proof: Let $h : C_n(X) \to C_n(Y)$ be a homeomorphism. Let $\{p\} \in F_1(Y)$. Thus, there exists $A \in C_n(X)$ such that $h(A) = \{p\}$. By Theorem 2.6, there exists a sequence $\{A_s\}_{s \in \mathbb{N}}$ in $C_n(X)$ such that $lim A_s = A$ and $\dim_{A_s}(C_n(X)) < \infty$ for each $s \in \mathbb{N}$. Thus, lim $h(A_s) = h(A) = \{p\}$ and $\dim_{h(A_s)}(C_n(Y)) < \infty$. Therefore, there exists another sequence $\{p_s\}_{s \in \mathbb{N}}$ in $C_n(Y)$ such that $p_s \in h(A_s)$ and $lim\{p_s\} = \{p\}$ for each $s \in \mathbb{N}$. By Lemma 5.4, for each $s \in \mathbb{N}$, $\dim_{\{p_s\}}(C_n(Y)) < \infty$. By Lemma 5.5, $\{p_s\} \in \mathrm{Cl}_{C(Y)}(\Omega(Y))$ for each $s \in \mathbb{N}$. Therefore, $\{p\} \in \mathrm{Cl}_{C(Y)}(\Omega(Y))$. □

Theorem 5.7. *Let $X \in \mathfrak{D}$ and $n \in \mathbb{N}$. Then X has unique hyperspace $C_n(X)$ unless $n = 1$ and X is an arc.*

Proof: For $n = 1, 2$, these results were proved in [12, Theorem 10] and [21, Theorem 3], respectively. Thus, we can assume that $n \geq 3$. Let Y be a continuum and let $h : C_n(X) \to C_n(Y)$ be a homeomorphism. Since X is locally connected, by [30, Theorem 1.92], Y is locally connected. We will prove that Y is a dendrite. For this, we suppose, to the contrary, that there exists a simple closed curve S contained in Y. We know that S is homeomorphic to $F_1(S)(\subset F_1(Y))$. By Lemma 5.6, $F_1(Y) \subset \mathrm{Cl}_{C(Y)}(\Omega(Y))$. Hence, $\mathrm{Cl}_{C(Y)}(\Omega(Y))$ contains a simple closed curve.

Since $C_n(X)$ is homeomorphic to $C_n(Y)$, $\Gamma_n(X)$ is homeomorphic to $\Gamma_n(Y)$; thus, $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to $\mathrm{Cl}_{C(Y)}(\Omega(Y))$. By [12, Theorem 5], $\mathrm{Cl}_{C(X)}(\Omega(X))$ is homeomorphic to X. Id est, X contains a simple closed curve, a contradiction. Therefore, Y does not contain simple closed curves. Thus, Y is a dendrite. By Theorem 4.2, we have that X is homeomorphic to Y. □

Question 5.8. Let $X \in \mathfrak{D}$, Y be a continuum, and $n, m \in \mathbb{N}$. If $C_n(X)$ is homeomorphic to $C_m(Y)$, is X homeomorphic to Y?

A *dendroid* is defined as an arcwise connected and hereditarily unicoherent continuum. It seems natural to ask the next question.

Question 5.9. Given $n \in \mathbb{N}$, does each dendroid with a closed set of end points have unique hyperspace $C_n(X)$?

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