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ON PARTITIONS OF UNITY IN THE DEDEKIND COMPLETION OF CERTAIN SUBSETS OF CONTINUOUS FUNCTIONS

B. BROSOWSKI AND A.R. DA SILVA

Abstract. In this short paper we prove the existence of a continuous partition $\sum_{\nu=1}^{n} \rho_\nu = 1$ in the Dedekind-completion of a subspace Z of $C(T, \mathbb{R})$, where the functions ρ_ν are constant on certain X-antisymmetric sets, where $Z = \operatorname{span}(X \cup X^2)$. Further, we present some applications of our technique.

1. Basic concepts

It is well-known that the original Dedekind construction of the real numbers, through cuts of rational numbers, holds in a much general framework. Here we follow this line in the specific case of subsets of continuous functions on a compact space endowed with the pointwise order. This allows us to construct partitions of unity in the corresponding Dedekind-completion (Theorem 3.1) and to derive a uniform density result (Theorem 3.2), in the flavor of the classical Stone-Weierstrass theorem.

Let us first recall some basic facts about the Dedekind-completion of partially ordered real vector spaces; for a thorough presentation of the subject see Aliprantis & Tourky or Luxemburg & Zaanen ([AT], [LZ]).

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A partially ordered real vector space Z is called *Dedekind-complete* if every non-empty subset which is bounded from above (below) has a supremum (resp. infimum) in Z.

Theorem 1.1. *Completion Theorem. If Z is an Archimedean [which is defined by*

$$\underset{u,v\in Z}{\forall}\ \underset{n\in\mathbb{N}}{\forall} n\cdot v \leq u \Rightarrow v \leq 0\,]$$

partially ordered real vector space, then there exists a uniquely determined Dedekind-complete partially ordered real vector space $\delta Z \supset Z$ such that every $z \in \delta Z$ satisfies

$$\begin{aligned} z &= \sup\{\ell \in Z \mid \ell \leq z\} \\ &= \inf\{u \in Z \mid u \geq z\}, \end{aligned}$$

δZ is called the Dedekind-completion of Z.

As in the case of the rational numbers the elements of δZ can be considered as *cuts* i.e. as pairs (L, U) of subsets of Z with the following properties:

(α) $\underset{\ell\in L}{\forall}\ \underset{u\in U}{\forall} \ell \leq u$;

(β) If $c \in Z$ and $\underset{\ell\in L}{\forall} \ell \leq c$ then $c \in U$;

(γ) If $c \in Z$ and $\underset{u\in U}{\forall} c \leq u$ then $c \in L$.

The vector space Z is embedded into δZ by the mapping

$$z \longrightarrow (L_z, U_z),$$

where

$$L_z := \{\ell \in Z \mid \ell \leq z\}$$

and

$$U_z := \{u \in Z \mid z \leq u\}.$$

Now let T be a compact Hausdorff space. We denote by $C(T, \mathbb{R})$ (resp. $(C(T, \mathbb{C}))$ the linear space of all continuous mappings $f\colon T \to \mathbb{R}$ (resp. $f\colon T \to \mathbb{C}$) endowed with maximum norm

$$||f|| := \max_{t\in T} |f(t)|.$$

Let X be a unital linear subspace of $C(T, \mathbb{R})$ or $C(T, \mathbb{C})$. A non-empty subset a of T is said to be X-*antisymmetric*, if whenever $h \in X$ and h is real-valued on a then h is constant on a.

For real linear spaces X an X-antisymmetric set is a set of constancy for this unital linear subspace. We denote by $\mathcal{A}(X)$ the collection of all maximal X-antisymmetric subsets of T.

2. Some Lemmas

Lemma 2.1. *Let X be a unital linear subspace of $C(T,\mathbb{R})$. Then for each maximal X-antisymmetric set a_o and for each open set V_o containing a_o there exists a function p_o in $Z :=$ span$(X \cup X^2)$ such that*

(i) $p_o(a_o) = \{0\}$;

(ii) $\underset{t\in V_o}{\forall}\ p_o(t) \geq 0$;

(iii) $\underset{t\in T\setminus V_o}{\forall}\ p_o(t) > 0$.

Proof. Let $a_o \in \mathcal{A}(X)$ and V_o be a neighborhood of a_o. If $a_o = T$ then choose $p_o \equiv 0$. Otherwise, choose a maximal antisymmetric set $a_1 \neq a_o$. Then $a_o \cap a_1 = \phi$ and there exists a function $v_o \in X$ such that

$$v_o\big|_{a_o} = 0 \qquad \text{and} \quad v_o\big|_{a_1} = 1.$$

Now the nonnegative function v_o^2 also vanishes on a_o. Since the case $V_o = T$ is trivial we can assume $V_o \neq T$. Then consider the set

$$\Delta := (T\setminus V_o) \cap \{t \in T \mid v_o^2(t) = 0\}.$$

If $\Delta = \phi$ then the function $p_o := v_o^2$ has the required properties. If $\Delta \neq \phi$ determine for each s in Δ a function v_s such that

$$v_s(s) = 1 \quad \text{and} \quad v_s(a_o) = \{0\} \quad \text{with} \quad \underset{t\in T}{\forall}\ v_s(t) \geq 0.$$

For each point s there exists an open set V_s such that

$$s \in V_s \quad \text{and} \quad \underset{t\in V_s}{\forall}\ v_s(t) > 0.$$

Since Δ is compact there exists a finite number of functions $v_{s_1}, v_{s_2}, \ldots, v_{s_j}$ such that the function

$$p_o := v_o^2 + v_{s_1} + \cdots + v_{s_j}$$

has the required properties. □

Lemma 2.2. *Let X be a unital linear subspace of $C(T,\mathbb{R})$. Then we have*

$$\delta Z \cap C(T,\mathbb{R}) = \{g \in C(T,\mathbb{R}) \mid \underset{a\in\mathcal{A}(X)}{\forall} g|_a = \text{ constant}\},$$

where $Z := \text{ span}(X \cup X^2)$.

Proof. "$\supset$." Choose an element $\bar{g} \neq 0$ in the set

$$\{g \in C(T,\mathbb{R}) \mid \underset{a\in\mathcal{A}(X)}{\forall} g|_a = \text{ constant}\},$$

an X-antisymmetric set a_o in $\mathcal{A}(X)$ and $\varepsilon > 0$ with $0 < \varepsilon < ||\bar{g}||$. There exists an open set V_o containing the compact set a_o such that

$$\underset{t\in V_o}{\forall} \bar{g}(t) \leq c_o + \varepsilon,$$

where c_o denotes the constant value of $\bar{g}$ on a_o. For the set V_o and a_o there exists, by Lemma 2.1, a function p_o with the properties (i), (ii) and (iii). Now, consider the function

$$p := c_o + \varepsilon + \beta p_o\,,$$

which is an element of Z.

If one chooses

$$\beta > \frac{2||\bar{g}|| - (c_o + \varepsilon)}{\inf\limits_{t\in T\setminus V_o} p_o(t)} > 0,$$

then we have

$$\underset{t\in V_o}{\forall} p(t) \geq c_o + \varepsilon \geq \bar{g}(t)$$

and

$$\underset{t\in T\setminus V_o}{\forall} p(t) \geq c_o + \varepsilon + 2||\bar{g}|| - (c_o + \varepsilon) = 2||\bar{g}||.$$

Consequently, $p \in U_{\bar{g}}$ and

$$\underset{t\in a_o}{\forall} \bar{g}(t) = c_o \leq p(t) \leq \bar{g}(t) + \varepsilon.$$

Since $\varepsilon > 0$ and $a_o \in \mathcal{A}(X)$ are arbitrary we conclude that

$$\bar{g}(t) = \inf_{u\in U_{\bar{g}}} u(t).$$

Similarly we can prove that

$$\bar{g}(t) = \sup_{t\in L_{\bar{g}}} \ell(t).$$

Consequently, $\bar{g} \in \delta Z$. Since $\bar{g} \in C(T)$ we have $\bar{g} \in \delta Z \cap C(T)$.

"$\subset$." Take $\bar{g} \in \delta Z \cap C(T, \mathbb{R})$. We have $\bar{g}(t) = \inf\limits_{u \in U_{\bar{g}}} u(t)$. Since every $u \in Z$ must be constant on $a \in \mathcal{A}(X)$, then the same is true for $\bar{g}$ whenever $\bar{g}(t) = \inf\limits_{u \in U_{\bar{g}}} u(t)$, that concludes proof. □

3. The main result

Theorem 3.1. *Let T be a compact Hausdorff space, X be a unital linear subspace of $C(T, \mathbb{R})$ and $Z :=$ span$(X \cup X^2)$. To each a in $\mathcal{A}(X)$ let there be associated a compact subset D_a of $T \backslash a$. Then there exist elements*

$$a_1, a_2, \ldots, a_n \quad in \quad \mathcal{A}(X)$$

and functions

$$b_1, b_2, \ldots, b_n \quad in \quad \delta Z \cap C(T, \mathbb{R})$$

which satisfy

$$b_1 + b_2 + \cdots + b_n = 1,$$
$$b_\nu \geq 0,\ b_\nu(D_{a_\nu}) = \{0\},\ \nu = 1, 2, \ldots, n.$$

Proof. For each $a \in \mathcal{A}(X)$ choose an open set V_a containing a and $\overline{V}_a \cap D_a = \phi$. By Lemma 2.1 there exists a function p_a with the following properties:

(i) $p_a(a) = \{0\}$;
(ii) $\forall_{t \in V_a}\ p_a(t) \geq 0$;
(iii) $\forall_{t \in D_a}\ p_a(t) > 0$.

We can assume that $p_a(t) \geq 1$ for each $t \in D_a$. Then the function

$$q_a(t) := \min(1, p_a(t))$$

is 1 on D_a, 0 on a, and satisfies the condition $0 \leq q_a(t) \leq 1$, and, by Lemma 2.2, $q_a \in \delta Z \cap C(T, \mathbb{R})$. Choose a positive real number $\delta < 1/2$. The open set

$$W_a := \{t \in T \mid q_a(t) < \delta\}$$

contains a and satisfies $W_a \cap D_a = \phi$.

The continuous function

$$\bar{q}_a(t) := \frac{\max(0, q_a(t) - \delta)}{1 - \delta}$$

has the properties

$$\bar{q}_a(t) = \begin{cases} 1, & \text{if} \quad t \in D_a \\ 0, & \text{if} \quad t \in W_a \end{cases}$$

and $0 \leq \bar{q}_a(t) \leq 1$.

By compactness of T there exist

$$a_1, a_2, \ldots, a_n \quad \text{in} \quad \mathcal{A}(X)$$

such that

$$T = W_{a_1} \cup \cdots \cup W_{a_n} .$$

Then the functions

$$\begin{aligned} b_1 &:= 1 - \bar{q}_{a_1} \\ b_2 &:= \bar{q}_{a_1}(1 - \bar{q}_{a_2}) \\ &\vdots \\ b_n &:= \bar{q}_{a_1} \ldots \bar{q}_{a_{n-1}}(1 - \bar{q}_{a_n}) \end{aligned}$$

have the required properties, compare Rudin [Ru p.40]. □

We have then the following density result:

Theorem 3.2. *Let X be a unital subspace of $C(T, \mathbb{R})$. Then*

$$\overline{\mathrm{ALG}}(X) = \delta Z \cap C(T, \mathbb{R}),$$

where $Z := \mathrm{span}(X \cup X^2)$ and $\mathrm{ALG}(X)$ denotes the algebra generated by X.

Proof. By Lemma 2.2 we have the inclusion

$$\overline{\mathrm{ALG}}(X) \subset \delta Z \cap C(T, \mathbb{R}).$$

To prove the reverse inclusion choose a function g in $\delta Z \cap C(T, \mathbb{R})$. For each $\varepsilon > 0$ there exist functions $p_1, \ldots, p_m$ and $q_1, \ldots, q_\ell$ in Z such that for each $t \in T$

$$-\varepsilon < g(t) - \min(p_1(t), \ldots, p_m(t)) \leq 0$$

and

$$0 \leq g(t) - \max(q_1(t), \ldots, q_\ell(t)) < \varepsilon.$$

In fact, choose an X-antisymmetric set $a \in \mathcal{A}(X)$. Then, by Lemma 2.1, there exists a function $\hat{p}_a \in U_g$ such that

$$\underset{t \in a}{\forall}\, g(t) < \hat{p}_a(t) = g(t) + \frac{\varepsilon}{2}.$$

Now there exists an open set V_a containing a such that

$$\underset{t \in V_a}{\forall}\, g(t) < \hat{p}_a(t) < g(t) + \varepsilon.$$

By compactness of T there exist $a_1, \ldots, a_m \in \mathcal{A}(X)$ such that $\bigcup_{\mu=1}^{m} V_{a_\mu} = T$. If we set $p_\mu := \hat{p}_{a_\mu}$, $\mu = 1, 2, \ldots, m$ then we have

$$\underset{t \in T}{\forall}\, g(t) \leq p_\mu(t),$$

and consequently

$$g(t) - \min(p_1(t), \ldots, p_m(t)) \leq 0.$$

To prove the other estimate choose an arbitrary point $t \in T$. Then there exists a μ with $1 \leq \mu \leq m$ such that

$$\min(p_1(t), \ldots, p_m(t)) \leq p_\mu(t) < g(t) + \varepsilon$$

or

$$-\varepsilon < g(t) - \min(p_1(t), \ldots, p_m(t)).$$

Similarly we can prove the existence of functions $q_1, \ldots, q_\ell$ in Z with

$$\underset{t \in T}{\forall}\, 0 \leq g(t) - \max(q_1(t), \ldots, q_\ell(t)) \leq \varepsilon.$$

Since in each unital subalgebra of $C(T, \mathbb{R})$ the maximum and the minimum of a finite number of elements of the algebra belong to the uniform closure of the algebra, the reverse inclusion follows. $\square$

Corollary. *Let $A \subset C(T, \mathbb{R})$ be a unital algebra. Then we have*

$$\overline{A} = \delta A \cap C(T, \mathbb{R}).$$

Final comments. The above results indicate that it might be possible to obtain alternative proofs for classical density results such as Stone-Weierstrass and Bishop theorems through the classical Dedekind-completion construction.

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