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GEOMETRIC CHARACTERISTICS AND COMMON HYPERCYCLIC SUBSPACES

JUAN BÈS AND ÖZGUR MARTIN

ABSTRACT. We consider geometric characteristics of countably many hypercyclic operators on a Banach space and study their effect on the existence of common hypercyclic subspaces.

1. Introduction

The Invariant Subset Problem asks whether every bounded operator on an infinite dimensional Banach space supports a non-trivial closed invariant set, that is, a closed set that is neither $\{0\}$ nor the whole space and which is mapped into itself by the operator. Read [40] showed that the answer is negative in general, providing a remarkable counterexample on ℓ_1. The Invariant Subset Problem remains open for the case in which the Banach space is a Hilbert space.

The notions of hypercyclic operators and hypercyclic vectors arise naturally in this study of invariant subsets. We say that a bounded operator T on a Banach space X is *hypercyclic* provided there is some $x \in X$ (called a *hypercyclic vector for* T) whose orbit

$$\{x, Tx, T^2x, \dots\}$$

is dense in X. In this way, T lacks non-trivial closed invariant subsets if and only if every vector (but the origin) is hypercyclic.

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Many nice things are known about the set $HC(T)$ of hypercyclic vectors of a hypercyclic operator T. Ansari [2] showed the surprising fact that $HC(T) = HC(T^n)$ $(n \in \mathbb{N})$, see also [18]. Indeed, Ansari's result may now be concluded from the remarkable result by Bourdon and Feldman [16] that the orbit of a bounded operator must be either (everywhere) dense or nowhere dense (see also related results by Costakis [23], Peris [38], Costakis and Peris [24], and Feldman [26]). More recently, León-Saavedra and Müller showed that $HC(T) = HC(\lambda T)$ for each scalar λ of modulus one.

But our interest here is on the linear structure of the set $HC(T)$. It is well known that every hypercyclic operator T supports a dense *hypercyclic manifold*, that is, a dense subspace consisting entirely (but the origin) of hypercyclic vectors for T. This was independently proved by Bourdon [15] and Herrero [32], who stated it in the complex Hilbert case setting (see also [8], [27], and [12]), and later by Wengenroth [43] who showed that this fact holds for operators on any topological vector space.

In contrast, a hypercyclic operator T may fail to support a *hypercyclic subspace,* that is, a closed and infinite dimensional subspace contained in $HC(T) \cup \{0\}$. The first such example was found by Montes-Rodríguez [36], in the unilateral weighted backward shift $T = B_\lambda$ on ℓ_2 given by

$$(x_0, x_1, \dots) \overset{B_\lambda}{\mapsto} (\lambda x_1, \lambda x_2, \dots), \tag{1.1}$$

where $|\lambda| > 1$. (The fact that B_λ is hypercyclic is a classical result of Rolewicz [41]). On the same paper, Montes provided a sufficient condition for an operator to support a hypercyclic subspace, used later with León-Saavedra [33] to show that every separable infinite dimensional Banach space supports an operator with a hypercyclic subspace. Other closely related results and/or alternative approaches, including extensions to non-normable Fréchet spaces, are due to Chan [19], Chan and Taylor [21], Montes-Rodríguez and Romero-Moreno [37], Bernal and Montes-Rodríguez [11], Bonet et al [17], Martínez-Giménez and Peris [35], Bernal [9], Petersson [39], and Conejero and Bès [13].

A complete characterization of when $HC(T) \cup \{0\}$ contains a closed and infinite dimensional subspace was provided by González et al [28] for the case when T is hereditarily hypercyclic (Definition 2.1):

Theorem 1.1. (González et al) *A hereditarily hypercyclic operator T on a Banach space X has a hypercyclic subspace if and only if its essential spectrum intersects the closed unit disk, and if and only if some sequence of its iterates $\{T^{n_k}\}_{k\in\mathbb{N}}$ converges pointwise to zero on a closed, infinite dimensional subspace of X.*

More recently, increasing attention has been drawn to the set $\cap_{T\in\mathcal{F}} HC(T)$ of common hypercyclic vectors of a given family $\mathcal{F}$ of hypercyclic operators acting on the same Banach space X. For example, Abakumov and Gordon [1] showed that the weighted shifts B_λ on ℓ_2 defined in (1.1) satisfy that the set $\cap_{|\lambda|>1} HC(B_\lambda)$ contains a dense subspace (but the origin). Other important advances on common hypercyclic subspaces for the case when $\mathcal{F}$ is uncountable were obtained by Bayart ([4],[5]), Bayart and Matheron [6], Costakis and Sambarino [25], León and Müller [34], Chan and Sanders [20], and Conejero et al [22]. When the family $\mathcal{F}$ is uncountable, however, it may fail to have a common hypercyclic vector, even if each $T \in \mathcal{F}$ supports a hypercyclic subspace [3]. In the countable case, Grivaux ([29]; see also [10]) showed that $\cap_{T\in\mathcal{F}} HC(T)$ always contains a dense subspace (but the origin), and Aron et al [3] provided a sufficient condition under which the countable family $\mathcal{F}$ supports a common hypercyclic subspace, see Theorem 2.2.

The purpose of this note, which stems from the paper by Aron et al [3], is to study the effect of certain geometric characteristics on the existence of common hypercyclic subspaces. For an operator T on a Banach space X, consider the geometric characteristics

$$\begin{aligned} G(T) &= \inf\{\|TJ_W\| \ : \ W \in \mathcal{I}\ \} \\ C(T) &= \inf\{\|TJ_W\| \ : \ W \in \mathcal{J}\ \}, \end{aligned} \tag{1.2}$$

where $\mathcal{I}$ and $\mathcal{J}$ denote the collections of closed infinite dimensional subspaces and of closed, finite codimensional subspaces of X, respectively, and where J_W denotes the canonical inclusion of W into X for each $W \in \mathcal{I}$. We show that given a sequence $\{T_\ell\}_{\ell\in\mathbb{N}}$ of hereditarily hypercyclic operators on a Banach space X, the condition

$$\liminf_{n\to\infty} C(T_\ell^n) < \infty \qquad (\ell \in \mathbb{N})$$

is sufficient, but not necessary, for the T_ℓ's to have a common hypercyclic subspace, while in turn for the latter to happen

$$\liminf_{n\to\infty} G(T_\ell^n) < \infty \qquad (\ell \in \mathbb{N})$$

is a necessary condition, but not a sufficient one. We show that these results hold in the more general setting of universality (Definition 2.1). For more on the notions of hypercyclicity and universality, we refer to the article of Godefroy and Shapiro [27], the surveys by Grosse-Erdmann [30], [31], and the forthcoming book by Bayart and Matheron [7].

2. Geometric characteristics and common universal subspaces

In what follows, $B(X,Z)$ denotes the space of bounded operators between the Banach spaces X and Z, and $B(X) = B(X,X)$. Also, $\mathbb{N}$ denotes the set of positive integers.

Definition 2.1. Given a sequence $\mathcal{F} = \{T_j\}_{j\in\mathbb{N}}$ in $B(X,Z)$, we say that $x \in X$ is a *universal vector* for $\mathcal{F}$ provided $\{T_j x : j \in \mathbb{N}\}$ is dense in X; the set of such universal vectors is denoted $HC(\mathcal{F})$. (Hence when $\mathcal{F} = \{T^j\}_{j\in\mathbb{N}}$, the universal vectors for $\mathcal{F}$ are precisely the hypercyclic vectors for T). Also, any closed, infinite dimensional subspace $Y \subset \{0\} \cup HC(\mathcal{F})$ is called a *universal subspace* for $\mathcal{F}$.

The sequence $\mathcal{F}$ is said to be *universal* (respectively, *densely universal*) provided $HC(\mathcal{F})$ is non-empty (respectively, dense in X). $\mathcal{F}$ is called *hereditarily universal* (respectively, *hereditarily densely universal*) provided $\{T_{n_k}\}_{k\in\mathbb{N}}$ is universal (respectively, densely universal) for each increasing sequence (n_k) of positive integers.

We use the following theorem, a minor modification of [3, Theorem 3.1] which is stated with $Z = X$.

Theorem 2.2. (Aron et al) *For $\ell \in \mathbb{N}$, let $\mathcal{F}_\ell := \{T_{\ell,j}\}_{j\in\mathbb{N}}$ be a hereditarily densely universal sequence in $B(X,Z)$. Suppose there exists a closed, infinite dimensional subspace Y of X so that*

$$T_{\ell,j} x \underset{j\to\infty}{\to} 0 \qquad \text{for each } x \in Y \text{ and } \ell \in \mathbb{N}. \tag{2.1}$$

Then the sequences $\mathcal{F}_\ell$ ($\ell \in \mathbb{N}$) have a common universal subspace. That is, there exists a closed, infinite dimensional subspace X_1 of X so that $\{T_{\ell,j} x\}_{j\in\mathbb{N}}$ is dense in Z for each $0 \neq x \in X_1$ and $\ell \in \mathbb{N}$.

We observe the following sufficient condition in terms of geometric characteristics for the existence of common universal subspaces.

Theorem 2.3. *For $\ell \in \mathbb{N}$, let $\mathcal{F}_\ell := \{T_{\ell,j}\}_{j\in\mathbb{N}}$ be a hereditarily densely universal sequence in $B(X,Z)$ so that*

$$sup\{C(T_{\ell,j}) :\ j \geq 1\} \ < \ \infty. \tag{2.2}$$

Then the sequences $\mathcal{F}_\ell$ ($\ell \in \mathbb{N}$) have a common universal subspace.

Proof. It suffices to show that condition (2.1) of Theorem 2.2 holds. Without loss of generality, we may assume that there is a dense subset X_0 of X so that for each $\ell \in \mathbb{N}$,

$$T_{\ell,j} \underset{j\to\infty}{\to} 0$$

pointwise on X_0. For each $\ell \in \mathbb{N}$, let $c_\ell > \sup\{C(T_{\ell,j}) :\ j \geq 1\}$ and pick closed subspaces $W_{\ell,n}$ of finite codimension in X so that $\|T_{\ell,j}J_{W_{\ell,j}}\| < c_\ell$ ($j \in \mathbb{N}$). Hence the closed finite codimensional subspaces $M_r := \cap_{1\leq \ell,q\leq r} W_{\ell,q}$ satisfy $M_1 \supset M_2 \supset \ldots$ and

$$\|T_{\ell,j}J_{M_r}\| < c_\ell \tag{2.3}$$

for every $1 \leq l, j \leq r$ and $r \in \mathbb{N}$. The rest of the proof (which we outline for completeness' sake) continues as in [3, Theorem 4.1]. Let (e_n) be a normalized basic sequence so that $e_n \in M_n$ ($n \in \mathbb{N}$), and let (e_n^*) in X^* be the corresponding sequence of coordinate functionals. Also, let (ϵ_n) be a decreasing sequence of positive scalars so that $\sum \epsilon_n < \frac{1}{2K}$, where K is the basis constant of (e_n). Next, get (z_n) in X_0 so that

$$\|z_r - e_r\| < \min\{\frac{\epsilon_r}{1 + \|T_{\ell,j}\|} :\ \ell, j \leq r\} \tag{2.4}$$

for every $r \in \mathbb{N}$. Notice that $\sum \|e_r^*\|\,\|z_r - e_r\| < 1$, so any subsequence (z_{n_k}) of (z_n) is equivalent to the corresponding basic sequence (e_{n_k}). It remains to find a subsequence (z_{n_k}) of (z_n) and increasing sequences $(j_{\ell,n})_{n\geq 1}$ ($\ell \in \mathbb{N}$) of positive integers so that

$$T_{l,j_{l,n}} \underset{n\to\infty}{\to} 0$$

pointwise on $Y := \overline{\text{span}\{z_{n_k} :\ k \in \mathbb{N}\}}$. To do this, let $n_0 := 1$, and $j_{\ell,1} > 1$ so that $\|T_\ell^{j_{\ell,1}} z_{n_0}\| < \frac{\epsilon_{n_0}}{2}$ ($\ell \in \mathbb{N}$), and let $n_1 := \max\{n_0+1, j_{1,1}\}$. Next, for $\ell \in \mathbb{N}$ get $j_{\ell,2} > \max\{n_1, j_{\ell,1}, 2\}$ so that

$$\|T_\ell^{j_{\ell,2}} z_{n_i}\| < \frac{\epsilon_{n_i}}{2^2}$$

for $i = 0, 1$, and let $n_2 := \max\left(\{n_1+1\} \cup \{j_{\ell,j} :\ 1 \le \ell, j \le 2\}\right)$. Continuing this process inductively, we obtain sequences (n_k) and $(j_{\ell,k})$ so that for each $k \in \mathbb{N}$

$$\begin{aligned} &j_{\ell,k} > \max\{n_{k-1}, j_{\ell,k-1}, k\} \\ &\|T_{\ell,j_{\ell,k}} z_{n_i}\| < \frac{\epsilon_{n_i}}{2^k} \quad (0 \le i \le k-1). \end{aligned} \tag{2.5}$$

Now, let $0 \ne z = \sum \alpha_i z_{n_i} \in Y := \overline{\text{span}\{z_{n_k} :\ k \in \mathbb{N}\}}$. Let R be the basis constant for (z_{n_k}). So $|\alpha_i| \le 2R\|z\|$ for each $i \in \mathbb{N}$. Given $\ell \in \mathbb{N}$ fixed, for each $k > \ell$ we have

$$T_{\ell,j_{\ell,k}} z = \sum_{i=1}^{k-1} \alpha_i T_{\ell,j_{\ell,k}} z_{n_i} + T_{\ell,j_{\ell,k}}\Big(\sum_{i=k}^{\infty} \alpha_i (z_{n_i} - e_{n_i})\Big) + T_{\ell,j_{\ell,k}}\Big(\sum_{i=k}^{\infty} \alpha_i e_{n_i}\Big).$$

By (2.5),

$$\|\sum_{i=1}^{k-1} \alpha_i T_{\ell,j_{\ell,k}} z_{n_i}\| \le 2R\|z\| \sum_{i=1}^{k-1} \frac{\epsilon_{n_i}}{2^k} \underset{k\to\infty}{\longrightarrow} 0.$$

Also, since $j_{\ell,k} \le n_i$ whenever $i \ge k$, by (2.4) we have

$$\|T_{\ell,j_{\ell,k}}\Big(\sum_{i=k}^{\infty} \alpha_i (z_{n_i} - e_{n_i})\Big)\| \le 2R\|z\| \sum_{i=k}^{\infty} \epsilon_{n_i} \underset{k\to\infty}{\longrightarrow} 0.$$

Finally, for $k > \ell$ we have $\sum_{i=k}^{\infty} \alpha_i e_{n_i} \in M_{n_k} \subset M_{j_{\ell,k}}$, and by (2.3)

$$\|T_{\ell,j_{\ell,k}}\Big(\sum_{i=k}^{\infty} \alpha_i e_{n_i}\Big)\| \le c_\ell\ \|\sum_{i=k}^{\infty} \alpha_i e_{n_i}\| \underset{k\to\infty}{\longrightarrow} 0.$$

So

$$T_{\ell,j_{\ell,k}} z \underset{k\to\infty}{\longrightarrow} 0,$$

and the proof of Theorem 2.3 is now complete. □

Corollary 2.4. *For each $\ell \in \mathbb{N}$, let $T_\ell \in B(X)$ be hereditarily hypercyclic with respect to an increasing sequence of positive integers $(n_{\ell,q})$, and so that*

$$\liminf_{q\to\infty} C(T_\ell^{n_{\ell,q}}) \ < \ \infty. \tag{2.6}$$

Then the T_ℓ's have a common hypercyclic subspace.

Proof. For each $\ell \in \mathbb{N}$, there exists a subsequence $(m_{\ell,q})$ of $(n_{\ell,q})$ so that $(C(T_\ell^{m_{\ell,q}}))_q$ is bounded. So Theorem 2.3 applies to the hereditarily densely universal families $\{T_{\ell,j}\}_{j\in\mathbb{N}} := \{T_\ell^{m_{\ell,j}}\}_{j\in\mathbb{N}}$. □

A simple consequence of Corollary 2.4 is the following particular case of [3, Theorem 4.1].

Example 2.5. Consider the operators on ℓ_2 of the form $T_\ell = \lambda_\ell I + K_\ell$, where λ_ℓ is a scalar of modulus one and where K_ℓ is a compact unilateral backward shift with non-zero weights ($\ell \in \mathbb{N}$). Then $T_1, T_2, \dots$ have a common hypercyclic subspace. To see this, notice that each T_ℓ is hereditarily hypercyclic [33, Proposition 4.3]. Also, $C(T_\ell^m) = C(\lambda_\ell^m I) = 1$ for each $m, \ell \in \mathbb{N}$ since this geometric characteristic is invariant under compact perturbations. Hence the conclusion follows from Corollary 2.4.

Notice in Example 2.5 that $G(T_\ell^m) = 1$ for each $m, \ell \in \mathbb{N}$ as well, since this geometric characteristic is also invariant under compact perturbations. On the other hand, recall that for a hypercyclic operator T its essential spectrum $\sigma_e(T)$ coincides with its essential approximate point spectrum

$$\sigma_{\pi e}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \notin \Phi_+(X)\}$$

($\Phi_+(X)$ denoting the class of operators on X with finite dimensional kernel and closed range), see [33, Proposition 3.1]. Also, Gohberg, Goldenstein, and Markus [44] showed that

$$\lim_{n\to\infty} C(T^n)^{\frac{1}{n}} = \rho_e(T), \tag{2.7}$$

while Zemánek [45, Theorem 8.1] showed that

$$\lim_{n\to\infty} G(T^n)^{\frac{1}{n}} = b_e(T), \tag{2.8}$$

where $\rho_e(T) = \max\{|\lambda| : \lambda \in \sigma_e(T)\}$, and $b_e(T) = \min\{|\lambda| : \lambda \in \sigma_{\pi e}(T)\}$ denote the essential spectral radius of T and the essential injectivity radius of T, respectively. Thus every operator T with a hypercyclic subspace satisfies

$$1 \geq b_e(T) = \lim_{n\to\infty} G(T^n)^{\frac{1}{n}},$$

by Theorem 1.1. Hence it is natural to ask whether we may replace the geometric characteristics $C(T_\ell^{n_{\ell,q}})$ by $G(T_\ell^{n_{\ell,q}})$ in Corollary 2.4. We answer this with Theorem 2.6 and with Example 2.10.

Theorem 2.6. *For $\ell \in \mathbb{N}$, let $\{T_{\ell,j}\}_{j\in\mathbb{N}}$ be a hereditarily densely universal sequence in $B(X, Z)$. Consider the following statements:*

(1) *For each $\ell \in \mathbb{N}$, $\liminf_{j\to\infty} C(T_{\ell,j}) < \infty$.*

(2) *There exists a closed, infinite dimensional subspace Y of X and increasing sequences of positive integers $(n_{\ell,q})$ so that*

$$T_{\ell,n_{\ell,q}} \underset{q\to\infty}{\to} 0 \qquad \text{pointwise on } Y \quad (\ell \in \mathbb{N}).$$

(3) *For each $\ell \in \mathbb{N}$, $\liminf_{j\to\infty} G(T_{\ell,j}) < \infty$.*

Then we have the implications

$$(1) \implies (2) \implies (3).$$

Also, none of the reverse implications hold.

Proof. The implication (1) ⇒ (2) was proved in Theorem 2.3, while (2) ⇒ (3) follows from the uniform boundedness principle. Example 2.7 and Example 2.10 below show that (2) ⇏ (1) and (3) ⇏ (2), respectively. □

In the examples below we use the fact [33, Proposition 4.1] that the essential injectivity radius of a hypercyclic unilateral backward shift B_w on ℓ_2 with weight sequence $w = (w_1, w_2, w_3, \ldots)$ is given by

$$b_e(B_w) = \lim_{n\to\infty} \left(\inf_{k\in\mathbb{N}}\{w_k w_{k+1} \cdots w_{k+n-1}\}\right)^{\frac{1}{n}}.$$

Example 2.7. The implication (2) ⇒ (1) of Theorem 2.6 fails even for a single operator T. Indeed, let $T = B_w$ be the unilateral backward shift on $X = \ell_2$ of weight sequence

$$w = (2, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 1 \ldots).$$

Then $b_e(T) = 1 < 2 = \rho_e(T)$, and T is hereditarily hypercyclic ([42, Theorem 2.8], and [14, Remark 3.2]), so $\sigma_{\pi e}(T) = \sigma_e(T)$. Thus $\sigma_e(T)$ intersects the closed unit disk, and by Theorem 1.1, some sequence of iterates of T must converge pointwise to zero on a closed, infinite dimensional subspace of X. That is, T satisfies statement (2) of Theorem 2.6. However, $\lim_{m\to\infty} C(T^m) = \infty$, since $1 < \rho_e(T) = \lim_{m\to\infty} C(T^m)^{\frac{1}{m}}$, by (2.7).

We use in Example 2.10 the following spectral condition for the existence of common hypercyclic subspaces for finitely many direct sums, which we find of independent interest. Here $\overline{\triangle}^N$ denotes the closed unit polydisk in $\mathbb{C}^N$.

Proposition 2.8. *For* $(1 \leq j \leq r)$*, let* X_j *be a Banach space and let* $T_{\ell,j} \in B(X_j)$ $(1 \leq \ell \leq N)$*. Suppose each direct sum*

$$T_\ell = T_{\ell,1} \oplus \cdots \oplus T_{\ell,r} \in B(X_1 \times \cdots \times X_r) \quad (1 \leq \ell \leq N)$$

is hereditarily hypercyclic and has a hypercyclic subspace. If $T_1, \ldots, T_N$ *have a common hypercyclic subspace, then*

$$\emptyset \neq \overline{\triangle}^N \cap \cup_{j=1}^r \sigma_e(T_{1,j}) \times \cdots \times \sigma_e(T_{N,j}). \tag{2.9}$$

Proof. Let $M \subset X_1 \times \cdots \times X_r$ be a common hypercyclic subspace for $T_1, \ldots, T_N$. Notice that if $P_1, \ldots, P_r$ are the canonical projections, since

$$I\mid_M = P_1\mid_M + \cdots + P_r\mid_M$$

is not strictly singular we have that $P_i\mid_M$ must be non-strictly singular for some $1 \leq i \leq r$, and thus that $P_i(M)$ contains a closed infinite dimensional subspace. But

$$P_i(M) \setminus \{0\} \subset \cap_{1\leq \ell\leq N} HC(T_{\ell,i}).$$

So each $T_{\ell,i}$ $(1 \leq \ell \leq N)$ is hereditarily hypercyclic, and has a hypercyclic subspace. Thus, by Theorem 1.1, $\sigma_e(T_{\ell,i}) \cap \overline{\triangle} \neq \emptyset$ for each $1 \leq \ell \leq N$. □

Theorem 1.1 gives the case $N = 1$ of Proposition 2.8, and that when $N = r = 1$ condition (2.9) is also sufficient. When $N \geq 2$ (or when $N = 1 < r$), however, the converse of Proposition 2.8 is never true, as the following example shows.

Example 2.9. Let $N \geq 2$, and let $X_1 = \cdots = X_r = \ell_2 \oplus \ell_2$. Consider the operators $T_\ell = T_{\ell,1} \oplus \cdots \oplus T_{\ell,r}$ on $\oplus_{j=1}^r X_j$ $(1 \leq \ell \leq N)$, where

$$T_{i,j} := \begin{cases} U & \text{if } i+j \text{ is even} \\ V & \text{if } i+j \text{ is odd,} \end{cases}$$

where

$$U = U_1 \oplus U_2 = 2B \oplus (I + B_w),$$
$$V = V_1 \oplus V_2 = (I + B_w) \oplus 2B,$$

and where B and B_w are the unilateral backward shifts on ℓ_2 with weighted sequences $(1, 1, \dots)$ and $w = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, respectively. Then the operators $T_1, \dots, T_N$ satisfy (2.8), but have no common hypercyclic subspace. To see this, notice that

$$\sigma_e(U_1) = \sigma_e(V_2) = \{\lambda \in \mathbb{C} : \ |\lambda| = 2 \ \}$$
$$\sigma_e(U_2) = \sigma_e(V_1) = \{1\},$$

and thus Proposition 2.8 gives that U and V have no common hypercyclic subspace. Also, since

$$\sigma_e(U) = \sigma_e(V) = \{1\} \cup \{\lambda \in \mathbb{C} : \ |\lambda| = 2 \ \},$$

we have

$$(1, \dots, 1) \in \sigma_e(T_{1,j}) \times \cdots \times \sigma_e(T_{N,j})$$

for each $j = 1, \dots, r$. Thus, if $M \subset X_1 \times \cdots \times X_r$ is a common hypercyclic subspace for $T_1, \dots, T_N$, arguing as in the proof of Proposition 2.8 some projection $P_i(M)$ must contain a closed and infinite dimensional subspace, giving

$$P_i(M) \setminus \{0\} \subset HC(T_{1,i}) \cap HC(T_{2,i}) = HC(U) \cap HC(V),$$

a contradiction. □

We are ready to complete the proof of Theorem 2.6 with the following.

Example 2.10. Let B be the unweighted unilateral backward shift on ℓ_2, and let B_w be the unilateral backward shift with weight sequence

$$w = (2,\, 2^{-1}, 2, 2,\, 2^{-1},\, 2^{-1}, 2, 2, 2,\, 2^{-1},\, 2^{-1},\, 2^{-1}, \dots).$$

Then $T_1 := 2B \oplus B_w$, $T_2 := B_w \oplus 2B$ on $X_1 = X_2 = \ell_2 \oplus \ell_2$ satisfy statement (3), but do not satisfy statement (2), of Theorem 2.6. To see this, notice that $2B$ is hereditarily hypercyclic with respect to (n), and that B_w is hereditarily hypercyclic with respect to some (n_q) ([42, Theorem 2.8], and [14, Remark 3.2]). So T_1, T_2 are hereditarily hypercyclic with respect to (n_q). Also, $G(T_\ell^m) \leq G(B_w^m)$ for each $m \in \mathbb{N}$ and by (2.8)

$$\lim_{m \to \infty} G(B_w^m)^{\frac{1}{m}} = b_e(B_w) = \frac{1}{2} < 1,$$

so the operators T_1, T_2 satisfy statement (3) of Theorem 2.6:

$$\sup\{G(T_\ell^{n_q}) : q \in \mathbb{N}\} \leq \sup\{G(B_w^{n_q}) : q \in \mathbb{N}\} < \infty \quad (\ell = 1, 2).$$

But notice that

$$T_1 = T_{1,1} \oplus T_{1,2} = 2B \oplus B_w$$
$$T_2 = T_{2,1} \oplus T_{2,2} = B_w \oplus 2B$$

satisfy

$$((\sigma_e(T_{1,1}) \times \sigma_e(T_{2,1})) \cup (\sigma_e(T_{1,2}) \times \sigma_e(T_{2,2}))) \cap \overline{\triangle}^2 = \emptyset,$$

since $\sigma_e(2B) = \{\lambda \in \mathbb{C} : |\lambda| = 2\}$. Thus T_1 and T_2 don't have a common hypercyclic subspace, by Proposition 2.8. Hence, by Theorem 2.2, T_1, T_2 cannot satisfy statement (2) of Theorem 2.6.

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