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HEREDITARILY NON-TOPOLOGIZABLE GROUPS

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Abstract. A group G is *non-topologizable* if the only Hausdorff group topology that G admits is the discrete one. Is there an infinite group G such that H/N is non-topologizable for every subgroup $H \leq G$ and every normal subgroup $N \lhd H$? We show that an answer to this essentially group theoretic question provides a solution to the problem of c-compactness.

Following Ol'shanskiĭ, we say that a group G is *non-topologizable* if the only Hausdorff group topology that G admits is the discrete one. In 1944, Markov asked whether infinite non-topologizable groups exist ([12]). Markov's problem was solved in 1979 by Hesse, whose result, however, seems to have remained unpublished ([6]). Without being aware of Hesse's solution, in 1980, Ol'shanskiĭ and Shelah independently constructed infinite non-topologizable groups ([13] and [16]). A noteworthy difference between Shelah's construction and the other two is that the earlier requires CH, while the later are ZFC results. Although Ol'shanskiĭ's example is a countable torsion group, Klyachko and Trofimov showed that a non-topologizable group need not satisfy either of these properties.

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Theorem 1. ([8]) *There exists a torsion-free finitely generated non-topologizable group. Thus, there exists a torsion-free non-topologizable group of any cardinality.*

(The second statement is obtained from the first one using the Löwenheim-Skolem theorem.) In a subsequent paper, Trofimov proved that every group embeds into a non-topologizable group of the same cardinality ([21, Thm. 3]). This latter result of Trofimov also shows how strongly non-hereditary the property of non-topologizability is.

Markov himself obtained a criterion of non-topologizability for countable groups with a strong algebraic geometric flavour (Theorem 2 below), whose most elegant proof was given by Zelenyuk and Protasov, more than half a century later ([12] and [15, 3.2.4]). Given a monomial

$$f(x) = g_0 x^{k_1} g_2 x^{k_2} g_3 \ldots g_{n-1} x^{k_n} g_n$$

in a single variable x, with $g_i \in G$ and $k_i \in \mathbb{Z}$, the set

$$V(f) = \{g \in G \mid f(g) = e\}$$

is closed in any Hausdorff group topology on G, because multiplication must be continuous. Thus, if $G \backslash \{e\}$ can be represented as $V(f_1) \cup \ldots \cup V(f_n)$, where each f_i is a monomial, then $\{e\}$ is open in any Hausdorff group topology on G, and therefore G is non-topologizable. In this case, one says that e is *algebraically isolated* in G. The reverse implication also holds if G is countable.

Theorem 2. ([12], [15, 3.2.4]) *A countable group G is non-topologizable if and only if e is algebraically isolated in G.*

Recently, a generalization of Theorem 2 for products of countable groups was obtained by Dikranjan and Shakhmatov, and by Sipachëva ([2, 5.8] and [17]). Shelah's solution, on the other hand, is uncountable and simple. Thus, his result can be rephrased as follows:

Theorem 3. ([16]) *Under the Continuum Hypothesis, there is a group G such that G/N is non-topologizable for every $N \lhd G$.*

We say that G is *hereditarily non-topologizable* if H/N is non-topologizable for every subgroup $H \leq G$ and every normal subgroup $N \lhd H$. Motivated by Shelah's result, we pose the following problem, and show that it is intimately related to the decade-old problem of c-compactness of topological groups, outlined below.

Problem. *Is there an infinite hereditarily non-topologizable group?*

By the well-known Kuratowski-Mrówka Theorem, a (Hausdorff) topological space X is compact if and only if for every (Hausdorff) topological space Y, the projection $p_Y\colon X \times Y \to Y$ is closed. Inspired by this theorem, Dikranjan and Uspenskij called a Hausdorff topological group G *categorically compact* (or briefly, *c-compact*) if for every Hausdorff group H, the image of every closed subgroup of $G \times H$ under the projection $\pi_H\colon G \times H \to H$ is closed in H ([3, 1.1]); they asked whether every c-compact topological group is compact. This question has been an open problem for more than ten years. The most extensive study of c-compact topological groups was done by Dikranjan and Uspenskij in [3], which was a source of inspiration for part of the author's PhD dissertation, as well as his subsequent work ([9], [11], and [10]).

A Hausdorff topological group G is *minimal* if there is no coarser Hausdorff group topology on G ([18] and [4]). So, a discrete group G is non-topologizable if and only if it is minimal. One says that a Hausdorff topological group G is *totally minimal* if every quotient of G by a closed normal subgroup is minimal ([1]), or equivalently, if every continuous surjective homomorphism $f : G \to H$ is open. The following two results of Dikranjan and Uspenskij provide a link between c-compactness and total minimality.

Theorem 4. ([3, 3.6]) *Every closed separable subgroup of a c-compact group is totally minimal.*

Theorem 5. ([3, 5.5]) *A countable discrete group G is c-compact if and only if every subgroup of G is totally minimal.*

A discrete group G is hereditarily non-topologizable if and only if the discrete topology is totally minimal on every subgroup of G. Thus, Theorem 5 yields:

Corollary 6. *A countable discrete group is c-compact if and only if it is hereditarily non-topologizable.* □

Recall that a topological group G has *small invariant neighborhoods* (or briefly, *G is SIN*), if every neighborhood U of $e \in G$ contains an invariant neighborhood V of e, that is, a neighborhood V such that $g^{-1}Vg = V$ for all $g \in G$.Equivalently, G is SIN if its left and right uniformities coincide. In a former paper, the author showed that the problem of c-compactness for locally compact SIN groups can be reduced to the countable discrete case ([10, 4.5]). Therefore, the Problem is equivalent to a special case of the problem of c-compactness.

Theorem 7. *The following statements are equivalent:*

(i) *every locally compact c-compact group admitting small invariant neighborhoods is compact;*
(ii) *every countable hereditarily non-topologizable group is finite.* □

We conclude with an algebraic consequence of hereditary non-topologizability. We denote by $H^{(k)}$ the k-th derived group of a group H, that is, $H^{(1)} = [H, H]$, and $H^{(k)} = [H^{(k-1)}, H^{(k-1)}]$.

Theorem 8. *Let G be a hereditarily non-topologizable group. Then:*

(a) *$G^{(k)}$ has finite index in G for every $k \in \mathbb{N}$;*
(b) *G has a smallest subgroup N of finite index, and $N = [N, N]$;*
(c) *there is $n \in \mathbb{N}$ such that $G^{(n)} = G^{(n+1)}$;*
(d) *if G is soluble, then G is finite;*
(e) *G is a torsion group.*

In light of Corollary 6, Dikranjan and Uspenskij's results imply Theorem 8 (cf. [3, 3.7-3.12]). Nevertheless, for the sake of completeness, we provide here a direct and elementary proof that does not rely on the Prodanov-Stoyanov theorem ([14]).

Proof. (a) Since G is hereditarily non-topologizable, the only Hausdorff group topology on its maximal abelian quotient $A = G/[G,G]$ is the discrete one. Kertész and Szele showed that every infinite abelian group admits a non-discrete metrizable group topology ([7] and [5, I.7.5]). Therefore, A is finite. Hence, the statement follows by an inductive reiteration of this argument for the hereditarily non-topologizable groups $G^{(k)}$.

(b) Since every subgroup of G of finite index contains a *normal* subgroup of G of finite index (namely, the intersection of the conjugates of the given subgroup), it suffices to show that G has a smallest normal subgroup of finite index. To that end, let $\{N_\alpha\}$ be the collection of normal subgroups of finite index in G, and set $N = \bigcap N_\alpha$. The discrete topology is the only Hausdorff group topology on G/N, because G is hereditarily non-topologizable.

On the other hand, G/N embeds into the product $P = \prod G/N_\alpha$, which admits a compact Hausdorff group topology (as each quotient G/N_α is finite). So, the image of G/N in P can be discrete only if it is finite, because every discrete subgroup of a topological group is closed. Thus, N has finite index in G. Since G is hereditarily non-topologizable, so is its subgroup N. Therefore, by (a), $[N, N]$ has finite index in N, and consequently in G. Hence, one has $N \subseteq [N, N]$, by the minimality of N.

(c) Let N be the smallest normal subgroup of finite index of G provided by (b). By (a), $G^{(k)}$ has finite index in G for every $k \in \mathbb{N}$, and thus $N \subseteq G^{(k)}$. Since N has finite index in G, there are only finitely many subgroups of G that contain N. Therefore, the decreasing sequence of subgroups

$$N \subseteq \cdots \lhd G^{(k)} \lhd \cdots \lhd G^{(1)} \lhd G^{(0)} = G$$

must stabilize after finitely many steps.

(d) Let N be the smallest normal subgroup of finite index of G provided by (b). As we have seen in (c), $N \subseteq G^{(k)}$ for every $k \in \mathbb{N}$. Thus, if $G^{(d)} = \{e\}$ for some $d \in \mathbb{N}$, then N is trivial. Therefore, G is finite, because N has finite index in G.

(e) Let $a \in G$. The subgroup $H = \langle a \rangle$ generated by a is hereditarily non-topologizable and abelian. Thus, by (d), H is finite. Therefore, every element of G has finite order. □

It follows from Theorem 8(e) that neither the example of Shelah nor the example of Klyachko and Trofimov are hereditarily non-topologizable, because they are not torsion groups ([16] and [8]). We do not know whether Ol'shanskiĭ's example is hereditarily non-topologizable ([13]).

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