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EFIMOV SPACES, CH, AND SIMPLE EXTENSIONS

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Abstract. We give a construction under CH of an inverse system of simple extensions so that its limit is an Efimov space. This example shows that CH alone implies that a conjecture of Mercourakis about measures [8] is false.

1. Introduction

An Efimov space is an infinite compact Hausdorff space with no non-trivial converging sequences and no copies of $\beta\omega$, the Stone-Čech compactification of the integers.

One of the well-known examples of this kind of space was constructed by Fedorčuk [5] with the aid of $\diamondsuit$. His space is the limit of a special type of inverse system: continuous and based on simple extensions. The construction we present here is an improvement in the sense that we are just assuming CH.

Džamonja and Plebanek [3] show that any Efimov space constructed from the Cantor space in an inverse limit of length ω_1 using simple extensions refutes a conjecture of Mercourakis concerning measures on compact spaces. Thus Fedorčuk's example does the job under $\diamondsuit$. It is asked in [3] if CH suffices to refute Mercourakis' conjecture, and our construction answers this affirmatively.

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Efimov spaces have been constructed from CH before (see, for example, [3] and [6]) but ours is the first example which is an inverse limit of simple extensions.

2. The Construction

Recall that $\langle f_{\alpha\beta}, X_\alpha : \alpha < \beta < \varepsilon\rangle$ is an inverse system if $f_{\alpha\beta}$ is a map from X_β to X_α whenever $\alpha < \beta < \varepsilon$ and the equation $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ holds for all $\alpha < \beta < \gamma < \varepsilon$. The inverse limit of the system is the appropriate subspace of the topological product of the family $\{X_\alpha : \alpha < \varepsilon\}$ as described in [4].

Definition 2.1. An inverse system $\langle f_{\alpha\beta}, X_\alpha : \alpha < \beta < \varepsilon\rangle$ is

(1) *continuous* if X_α is the inverse limit of the system $\langle f_{\beta\gamma}, X_\beta : \beta < \gamma < \alpha\rangle$, for any limit ordinal $\alpha < \varepsilon$.
(2) *based on simple extensions* if $X_{\alpha+1}$ is always a simple extension of X_α, i.e. there exists a single point $x_\alpha \in X_\alpha$ so that $f^{-1}_{\alpha,\alpha+1}[x_\alpha]$ contains exactly two points and $f^{-1}_{\alpha,\alpha+1}[x]$ is a singleton for all $x \in X_\alpha \setminus \{x_\alpha\}$.

These kinds of inverse systems are known as *minimal* in the Boolean algebra setting.

The following result, in Boolean algebraic form, is due to Koppelberg [7]; a topological proof can be found in [1].

Proposition 2.2. *If X is the limit of the inverse system of simple extensions $\langle f_{\alpha\beta}, X_\alpha : \alpha < \beta < \varepsilon\rangle$, then X does not map onto $[0,1]^{\omega_1}$, unless X_0 does.*

Observe that in this case, X does not map onto $[0,1]^{\mathfrak{c}}$ either and therefore $\beta\omega$ cannot be embedded in X.

Theorem 2.3. *Under CH, there exists an Efimov space that can be obtained as the limit of an inverse system of simple extensions of length ω_1.*

Proof. The plan is to use induction over β to get the desired inverse system $\langle f_{\alpha\beta}, X_\alpha : \alpha < \beta < \omega_1\rangle$.

From now on, sequence means infinite sequence. Recall that a countable set E converges to a point a if $E \setminus U$ is finite whenever U is a neighborhood of a.

Assume that we have constructed X_α and X_β for some $\alpha < \beta$ and let $x \in X_\alpha$ be arbitrary. To simplify our notation we will write $[x]_\beta$ to denote $f^{-1}_{\alpha\beta}[\{x\}]$.

Fix a partition $\langle P_\alpha : \alpha < \omega_1\rangle$ of ω_1 so that P_α is an uncountable subset of $\omega_1 \setminus \alpha$ for each $\alpha < \omega_1$.

We need some terminology. Let $\gamma < \omega_1$ and $\beta \in P_\gamma$ be arbitrary. If X_γ and X_β have been constructed, then use CH to fix an enumeration $\langle D_\xi : \xi \in P_\gamma\rangle$ of all converging sequences D in X_γ so that $f_{\xi\gamma}[D]$ is finite for each $\xi < \gamma$. Observe that D_β is a convergent sequence in X_γ. A *selector* from D_β is a set $E \subseteq X_\beta$ so that $E \cap [d]_\beta$ is a singleton for each $d \in D_\beta$. Use CH again to get $\langle E_\xi : \xi \in P_\beta\rangle$, an enumeration of all converging selectors from D_β.

Let $\beta < \alpha < \varepsilon$ and $x \in X_\alpha$ be arbitrary. Fix $\gamma < \omega_1$ so that $\beta \in P_\gamma$. We define

$$D_\beta(x) := \{d \in D_\beta : x \notin [d]_\alpha\}.$$

In other words, $d \in D_\beta(x)$ if and only if $d \in D_\beta$ and $f_{\gamma\alpha}(x) \neq d$.

Assume that for some $\varepsilon < \omega_1$ we have defined a continuous inverse system of simple extensions $\langle f_{\gamma\beta}, X_\gamma : \gamma < \beta < \varepsilon\rangle$ so that the following holds for each $\beta < \varepsilon$.

(1) $X_0 = 2^\omega$, the Cantor set.
(2) If $\beta + 1 < \varepsilon$ and $i < 2$, then there exist A^i_β, a closed subset of X_β, and H^i_β, an infinite subset of E_β, so that
   (a) $A^0_\beta \cap A^1_\beta = \{x_\beta\}$,
   (b) $X_\beta = A^0_\beta \cup A^1_\beta$,
   (c) $X_{\beta+1} = A^0_\beta \oplus A^1_\beta$ (the topological sum), and
   (d) $f_{\beta,\beta+1}$ is the projection map.
   (e) $[e]_\beta \subseteq A^i_\beta$ for all $e \in H^i_\beta$.
(3) If $x \in U \in \mathcal{T}_\beta$, where $\mathcal{T}_\beta$ is the topology of X_β, then for each finite set $F \subseteq \beta$ there exists W, a clopen subset of X_β, such that $x \in W \subseteq U$ and W *takes care of* (x, F), i.e.

$$[d]_\gamma \cap f_{\gamma\beta}[W] \cap f_{\gamma\beta}[X_\beta \setminus W] = \emptyset,$$

for all $\gamma \in F$ and $d \in D_\gamma(x)$.

Condition (3) is equivalent to saying that $W \cap [d]_\beta$ is a preimage of a clopen subset of $[d]_\gamma$.

Observe that (1) guarantees, according to Proposition 2.2, that the limit will not contain a copy of $\beta\omega$. Hence our main concern is to get rid of all converging sequences.

If ε is a limit ordinal, then X_ε is the limit of this inverse system. To verify (3) let $x \in U_0 \in \mathcal{T}_\varepsilon$ be arbitrary. Fix a finite set $F \subseteq \varepsilon$. Since $\{f_{\xi\varepsilon}^{-1}[V] : \xi < \varepsilon,\ V \in \mathcal{T}_\xi\}$ is a base for $\mathcal{T}_\varepsilon$, there exists $\alpha < \varepsilon$ and $U \in \mathcal{T}_\alpha$ so that $F \subseteq \alpha$ and $x \in f_{\alpha\varepsilon}^{-1}[U] \subseteq U_0$. Apply the inductive hypothesis to α, $f_{\alpha\varepsilon}(x)$, U, and F to get a clopen set W in X_α which takes care of $(f_{\alpha\varepsilon}(x), F)$ and satisfies $f_{\alpha\varepsilon}(x) \in W \subseteq U$. For each $\gamma \in F$ we have $D_\gamma(x) = D_\gamma(f_{\alpha\varepsilon}(x))$, so $f_{\alpha\varepsilon}^{-1}[W]$ is a clopen subset of X_ε which takes care of (x, F).

Now assume that $\varepsilon = \alpha + 1$. Let $\beta < \omega_1$ be such that $\alpha \in P_\beta$. Hence $\beta \leq \alpha$ and therefore E_α and D_β have been defined. Since X_0 is compact metrizable and $\alpha < \omega_1$, X_α is compact metrizable too. In particular the family $\{[e]_\alpha : e \in E_\alpha\}$ must have an accumulation point, i.e. there exists a point x_α in X_α so that the set $\{e \in E_\alpha : [e]_\alpha \cap V \neq \emptyset\}$ is infinite for each neighborhood V of x_α.

The next step is to find an infinite set $H_\alpha \subseteq E_\alpha$ so that $\langle [e]_\alpha : e \in H_\alpha \rangle$ converges to x_α, i.e. for each neighborhood U of x_α all but finitely many $e \in H_\alpha$ satisfy $[e]_\alpha \subseteq U$. Let's start by fixing a local decreasing base $\{B_n : n \in \omega\}$ for x_α.

We face two cases. When $\beta < \alpha$ apply (3) to construct a sequence $\langle W_n : n \in \omega \rangle$ of clopen subsets of X_α which satisfies

(i) $x_\alpha \in W_0 \subseteq B_0$,
(ii) $x_\alpha \in W_{n+1} \subseteq W_n \cap B_{n+1}$, and
(iii) W_n takes care of $(x_\alpha, \{\beta\})$,

for each $n \in \omega$.

Define $E_\alpha^n := \{e \in E_\alpha : [e]_\alpha \cap W_n \neq \emptyset\}$. We claim that for all but possibly one $e \in E_\alpha^n$ we get $[e]_\alpha \subseteq W_n$. To prove this assertion let $e \in E_\alpha^n$ and $d \in D_\beta(x_\alpha)$ be so that $e \in [d]_\beta$. Now note that if $[e]_\alpha \setminus W_n \neq \emptyset$, then $e \in f_{\beta\alpha}[W_n] \cap f_{\beta\alpha}[X_\alpha \setminus W_n] \cap [d]_\beta$, a clear contradiction to (iii). Find an infinite set $H_\alpha \subseteq E_\alpha$ so that $H_\alpha \setminus E_\alpha^n$ is finite for all $n \in \omega$ and observe that this H_α works.

For the case $\alpha = \beta$ we have $E_\alpha \subseteq X_\alpha$ and $[e]_\alpha = \{e\}$, for each $e \in E_\alpha$. Clearly any subsequence of E_α which converges to x_α will work as H_α.

Let $\alpha = \{\beta_k : k < \omega\}$ be an enumeration of α. Use (3) to construct $\{e_n : n \in \omega\} \subseteq H_\alpha$, $g : \omega \to \omega$, and $\{U_n : n \in \omega\}$ so that for each $n \in \omega$

(I) $U_0 = X_\alpha$,
(II) U_n is clopen in X_α and takes care of $(x_\alpha, \{\beta_k : k \leq n\})$,
(III) g is an increasing function,
(IV) $x_\alpha \in U_{n+1} \subseteq B_{g(n)} \subseteq U_n \setminus [e_n]_\alpha$, and
(V) $[e_n]_\alpha \subseteq U_n$.

We are going to partition $X_\alpha \setminus \{x_\alpha\}$. Let n be an arbitrary integer. Observe that the set $V_n := U_n \setminus U_{n+1}$ is clopen and takes care of $(x_\alpha, \{\beta_k : k \leq n\})$. Now, given $i < 2$, define

$$b_n^i := \{x_\alpha\} \cup \bigcup_{k=n}^{\infty} V_{2k+i}.$$

The following holds for each $i < 2$.

(a) b_n^i is closed for all $n < \omega$.
(b) If U a neighborhood of x_α, then $b_m^i \subseteq U$, for some $m < \omega$.

We claim that b_n^i takes care of $(x_\alpha, \{\beta_k : k \leq n\})$. Let $k \leq n$ and $d \in D_{\beta_k}(x_\alpha)$ be arbitrary. If $y \in f_{\beta_k\alpha}[b_n^i] \cap f_{\beta_k\alpha}[X_\alpha \setminus b_n^i]$, then we have two possibilities: $y \in f_{\beta_k\alpha}[V_{2\ell+i}]$, for some $\ell \geq n$, or $y = f_{\beta_k\alpha}(x_\alpha)$. In the first case we get $y \in f_{\beta_k\alpha}[V_{2\ell+i}] \cap f_{\beta_k\alpha}[X_\alpha \setminus V_{2\ell+i}]$ and $k \leq n \leq \ell \leq 2\ell + i$, so $y \notin [d]_{\beta_k}$.

Now assume that $y = f_{\beta_k\alpha}(x_\alpha)$. Since $d \in D_{\beta_k}(x_\alpha)$, we get $x_\alpha \notin [d]_\alpha$ and therefore $y \notin [d]_{\beta_k}$.

For each $i < 2$, set $A_\alpha^i := b_0^i$ and $H_\alpha^i := \{e_{2n+i} : n \in \omega\}$. Let $X_{\alpha+1} := (A_\alpha^0 \times \{0\}) \cup (A_\alpha^1 \times \{1\})$ and declare open all the sets of the form $(U^0 \times \{0\}) \cup (U^1 \times \{1\})$, where U^i is open in the subspace topology of $A_\alpha^i \subseteq X_\alpha$. The map $f_{\alpha,\alpha+1} : X_{\alpha+1} \to X_\alpha$ is defined by $f_{\alpha,\alpha+1}(x, i) = x$.

To complete the induction we check that property (3) holds. Assume that $(x, i) \in U \in \mathcal{T}_{\alpha+1}$ are arbitrary and fix a finite set $F \subseteq \alpha + 1$. If $x \neq x_\alpha$, find $U_0 \in \mathcal{T}_\alpha$ so that $(x, i) \in U_0 \times \{i\} \subseteq U$ and $x_\alpha \notin U_0$. Let W_0 be a clopen subset of X_α which takes care of $(x, F \setminus \{\alpha\})$ and satisfies $x \in W_0 \subseteq U_0$. Set $W := W_0 \times \{i\}$ and note that for all $\beta \in F$ we have $f_{\beta,\alpha+1}[W] = f_{\beta\alpha}[W_0]$ and $f_{\beta,\alpha+1}[X_{\alpha+1} \setminus W] = f_{\beta\alpha}[X_\alpha \setminus W_0]$ (recall that $f_{\alpha\alpha}$ is the identity map). Since $D_\beta((x, i)) = D_\beta(x)$ we conclude that W takes care of $((x, i), F)$.

Assume now that $x = x_\alpha$. Find $n \in \omega$, so that $b^i_n \times \{i\} \subseteq U$ and $F \setminus \{\alpha\} \subseteq \{\beta_k : k \leq n\}$. Define $W := b^i_n \times \{i\}$. For each $\beta \in F$ and $d \in D_\beta((x_\alpha, i))$ we have that $f_{\beta\alpha}(x_\alpha) \notin [d]_\beta$ and

$$f_{\beta,\alpha+1}[W] \cap f_{\beta,\alpha+1}[X_{\alpha+1} \setminus W] = \{f_{\beta\alpha}(x_\alpha)\} \cup \big(f_{\beta\alpha}[b^i_n] \cap f_{\beta\alpha}[X_\alpha \setminus b^i_n]\big) .$$

Therefore W takes care of $((x_\alpha, i), F)$.

Let X be the limit of our inverse system and let $\pi_\alpha : X \to X_\alpha$ be the bonding map for each $\alpha < \omega_1$. In order to check that X is an Efimov space, assume that S is a converging sequence in X. Let $\gamma < \omega_1$ be the least ordinal so that $\pi_\gamma[S]$ is infinite. Since $\pi_\gamma[S]$ is a convergent sequence in X_γ, there exists $\beta \in P_\gamma$ so that $\pi_\gamma[S] = D_\beta$. Since $f_{\gamma\beta} \circ \pi_\beta = \pi_\gamma$, we can find an infinite set $S_0 \subseteq S$ so that π_β is one-to-one on S_0 and $\pi_\beta[S_0]$ is a selector from D_β, i.e. there exists $\alpha \in P_\beta$ so that $\pi_\beta[S_0] = E_\alpha$. Property (2) provides two infinite subsets of S_0, namely S^0_0 and S^1_0, so that $\pi_\alpha[S^i_0] \subseteq A^i_\alpha$ for each $i < 2$. Therefore $\pi_{\alpha+1}[S^0_0]$ and $\pi_{\alpha+1}[S^1_0]$ cannot converge to the same point. This contradiction ends the proof. □

References

[1] A. Dow, *Efimov spaces and the splitting number*. Spring Topology and Dynamical Systems Conference. Topology Proc. **29** (2005), no. 1, 105–113.

[2] A. Dow, D.H. Fremlin, *Compact sets without converging sequences in the random real model*, Acta Math. Univ. Comenian. (N.S.) 76 (2007), no. 2, 161–171. 54A25

[3] M. Džamonja, G. Plebanek, *On Efimov spaces and Radon measures*, Topology Appl. **154** (2007), no. 10, 2063-2072.

[4] R. Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR **91**c:54001

[5] V. V. Fedorčuk, *Completely closed mappings, and the consistency of certain general topology theorems with the axioms of set theory*, Matematicheskiĭ Sbornik. Novaya Seriya **99** (**141**) (1976), 1-26 (Russian); English transl. Mathematics of the USSR. Sbornik **28** (1976), 3-33, 135. MR 53 #14379, Zbl. 0371.54045

[6] K.P. Hart, *Efimov's problem*, Open Problems in Topology II, (E. Pearl, ed.), Elsevier Science, 2007, pp. 171-177.

[7] S. Koppelberg, *Minimally generated Boolean algebras*, Order **5** (1989), no. 4, 393-406. MR **90g**:06022

[8] S. Mercourakis, *Some remarks on countably determined measures and uniform distribution of sequences*, Monats. Math. **121** (1996), 79–101.

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