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I. Tsereteli and $\boxed{\text{L. Zambakhidze}}$

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METRIZABILITY AND DIMENSION

I. TSERETELI AND L. ZAMBAKHIDZE

ABSTRACT. It is proved that under some natural restrictions any topologically closed subclass of the class of all Tychonoff spaces, where the three classical dimension functions coincide, is a subclass of the class of all metrizable spaces with a countable base providing a complete solution for a problem of A.V.Arhangel'skiı́, next to a partial solution of L.A.Tumarkin's problem.

1. NOTATION

The term *space* is to be understood as a *topological space*.

All spaces under consideration are at least Tychonoff. Below we consider several classes of spaces. All of them are assumed topologically closed, i.e., any considered class of spaces is closed under homeomorphisms.

The class of all separable and metrizable spaces is denoted by T_{SM} and the class of all finite-dimensional separable metrizable spaces - by T_{fSM}.

N' is denoted as the set $\{-1, 0, 1, 2, \dots\} \bigcup \{\infty\}$ together with the natural order relation and addition operation.

A N'-valued function d, defined on a class T of spaces, is said to be *topologically invariant* if for any $X \in T$ and any $Y \in T$, with X homeomorphic to Y, we have: $d(X) = d(Y)$.

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For any natural number n, I^n signifies the standard n-dimensional closed cube $[0;1]^n$. Besides, I^0 denotes the one-point set $\{0\}$ and I^{-1} is the empty set.

N stands for the set of all natural numbers.

For any space X and any $A \subset X$, the boundary of A in X is denoted by $Fr_X A$ (or, simply, by FrA).

For any space X and any $A \subset X$, $[A]_X$ (or simply, $[A]$) denotes the closure of A in X.

dim, ind and Ind denote, respectively, the covering, the small inductive and the large inductive dimension functions. This work follows definitions given e.g. in [6], [13] [1].

It is noteworthy, that in the class of all Tychonoff spaces there are alternative definitions for covering and large inductive dimension functions, denoted below by $dim^\star$ and $Ind^\star$, respectively. $dim^\star$ and $Ind^\star$ are defined as follows: for any Tychonoff space X $dim^\star X = dim\beta X$ [2] and $Ind^\star X = Ind\beta X$, where βX is the Čech-Stone compactification of X. For any normal space X we have $dim^\star X = dimX$ and $Ind^\star X = IndX$. However, in the class of all Tychonoff spaces the function $dim^\star$ (respectively, $Ind^\star$), in general, differs from the function dim (respectively, Ind). It must be underscored, also, that for any Tychonoff space X, $dim^\star X = 0$ iff $Ind^\star X = 0$ (see [6]).

Some of the results given below were announced (without proofs) in [19].

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[1]Definition of ind [6]. For any space X, $indX = -1$ iff $X = \emptyset$. $indX \le n$ ($n \ge 0$) iff for any point $x \in X$ and any neighborhood V of x there is an open subset U of X with $x \in U \subset V$ and $indFr_X U \le n-1$.

Definition of Ind [6]. For any space X, $IndX = -1$ iff $X = \emptyset$. $IndX \le n$ ($n \ge 0$) iff for any closed subset $A \subset X$ and for any neighborhood V of A there is an open subset U of X with $A \subset U \subset V$ and $IndFr_X U \le n-1$.

Definition of dim [6]. For any space X and any integer $n \ge -1$, $dimX \le n$ iff any finite open cover of the space X can be refined by an open cover (of X) of order $\le n$ (order of a family $\mathcal{F}$ of subsets of a set X is the largest integer n such that $\mathcal{F}$ contains $n+1$ sets with non-empty intersection; if no such integer exists, then the order of $\mathcal{F}$ is equal to ∞).

[2]The following (internal) characterization of $dim^\star$ is well-known (see e.g. [4]): for any Tychonoff space X and any integer $n \ge -1$, $dim^\star X \le n$ iff any finite functionally open cover of X can be refined by a finite functionally open cover of order $\le n$.

The main results of the article were presented at the 22nd Summer Conference on "Topology and its Applications" (Universidad Jaume I de Castellon, Spain, 24-27 July 2007).

2. Introduction

For any $X \in T_{SM}$, denote: $\widetilde{d}\ (X) = dimX$. It is well-known (see e.g. [4], [9]), that the class T_{SM} and the function $\widetilde{d}$ have the following properties:

P_1. For any $X \in T_{SM}$, $\widetilde{d}\ (X) = dimX = indX = IndX$.

P_2. For any $X \in T_{SM}$ and any $A \subset X$ we have: $A \in T_{SM}$.

P_3. For any $X \in T_{SM}$ and any $A \subset X$ we have: $\widetilde{d}\ (A) \leq \widetilde{d}\ (X)$.

P_4. For any $n = -1, 0, 1, 2, \dots$ we have: $\widetilde{d}\ (I^n) = n$.

P_5. If $X \in T_{SM}$ and $X = \bigcup\limits_{i=1}^{\infty} X_i$, where each X_i is a closed subset of X, then $\widetilde{d}\ (X) = sup\{\widetilde{d}\ (X_i) : 1 \leq i < \infty\}$.

P_6. If $X \in T_{SM}$ and $X = A \bigcup B$, then $\widetilde{d}\ (X) \leq \widetilde{d}\ (A) + \widetilde{d}\ (B) + 1$.

P_7. If $X \in T_{SM}$ and $0 \leq \widetilde{d}\ (X) = n < \infty$, there exist $X_1, \dots, X_{n+1} \subset X$ such, that $X = \bigcup\limits_{i=1}^{n+1} X_i$ and $\widetilde{d}\ (X_i) \leq 0$ for each $i = 1, \dots, n+1$.

P_8. For any $X \in T_{SM}$ there is a compactification bX of X such that $bX \in T_{SM}$ and $\widetilde{d}\ (bX) = \widetilde{d}\ (X)$.

P_9. If $X_1 \in T_{SM}$ and $X_2 \in T_{SM}$, then $X_1 \times X_2 \in T_{SM}$.

P_{10}. If $X_1 \in T_{SM}$, $X_2 \in T_{SM}$ and $X_1 \bigcup X_2 \neq \emptyset$, then $\widetilde{d}\ (X_1 \times X_2) \leq \widetilde{d}\ (X_1) + \widetilde{d}\ (X_2)$.

P_{11}. For any $X \in T_{SM}$ and any $A \subset X$, there is a G_δ-subset H of X with $A \subset H$ and $\widetilde{d}\ (H) = \widetilde{d}\ (A)$.

The following questions are raised naturally.

1. Is $\widetilde{d}$ the only topologically invariant N'-valued function on the class T_{SM}, having the properties $P_2, \dots, P_{11}$ simultaneously?

2. Do there exist topologically invariant N'-valued functions, defined on the classes of spaces wider than the class T_{SM} (containing all standard cubes $[0, 1]^n$, $n = -1, 0, 1, \dots$) and having the properties $P_2, \ldots, P_{11}$ simultaneously? [3]

3. Let T be a (topologically closed) class of spaces, in which the functions dim, ind and Ind coincide and the common value of these functions has the properties $P_2, \ldots, P_{11}$ simultaneously. Is then T a subclass of T_{SM} class? (Exact formulation follows below).

Questions **1** and **2** were studied in the previous papers. Namely, in [16] together with some other results the positive answer to the question **1** is given.

As regards the question **2**, it is particularly proved (see [17]) that in the Tychonoff spaces class there exists no topologically invariant $N^{'}$-valued function, having the properties $P_2, \ldots, P_{11}$ simultaneously. In other words, according to the terminology adopted [17] in the class mentioned, the system $\{P_2, \ldots, P_{11}\}$ is non-realizable.

On the other hand, the present paper studies question **3** (see Question QA below), posed by A.Arhangel'skií on the 47^{th} Conference at Latvia State University in 1988. The study at hand shows that an answer to the given question is positive. Moreover, it is proved, that even when class T and a $N^{'}$-valued function d have but properties P_1, P_2, P_8, P_9 - T is a subclass of the class T_{SM} (Theorem 1).

In this connection, one would interestingly try to single out the combinations of the properties $P_1, \ldots P_{11}$ (i.e., all nonempty subsets of the set $\{P_1, \ldots P_{11}\}$), ensuring the inclusion $T \subset T_{SM}$ (clearly meaning, that the quantity of all combinations of the properties $P_1, \ldots P_{11}$ is equal to $2^{11} - 1 = 2047$). As it turned out (Theorem 1), there are exactly 128 combinations, providing the inclusion $T \subset T_{SM}$. These are exactly the combinations simultaneously containing the properties P_1, P_2, P_8, P_9 .

It is also proved that if given T and $N^{'}$-valued function d, defined on T, having the properties P_2, P_8, P_9 together with $d(X) = dimX = indX = IndX < \infty$ for any $X \in T$, then T is a subclass of the class T_{fSM}. More than that, if T contains the usual unit closed interval [0,1], then T coincides with T_{fSM} (Theorem 2).

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[3] A $N^{'}$-valued function d, defined on such class T of spaces, has the property P_i, $i = 2, \ldots 11$, means that T and d satisfy the condition, formulation of which is obtained from the formulation of P_i by replacing of $\widetilde{d}$ and T_{SM} with d and T, respectively.

When referring to the next question it is known ([6]), that the function $\widetilde{d}$ has the following property:

P_{12}. If $X_1, X_2 \in T_{SM}$, $X_1 \subset X_2$ and $x \in X_2 \setminus X_1$, then $\widetilde{d}$ $(X_1 \bigcup \{x\}) = \widetilde{d}$ (X_1).

L.A.Tumarkin [18] sets up a goal to characterize class T and N'-valued function d defined on T, having simultaneously properties $P_1, P_2, P_3, P_9, P_{10}$ and P_{12} .

A partial solution of this problem (Question QT) is provided below. Namely, we show that if class T and function d together with the properties $P_1, P_2, P_3, P_9, P_{10}, P_{12}$ also have the property P_8, T is a subclass of the class T_{SM} (Theorem 3).

3. Question Formulations

Consider a pair (T, d), where T is a class of spaces and d is a topologically invariant N'-valued function, defined on the class T. Denote the collection of all such pairs by $\mathcal{T}$, i.e. $\mathcal{T} = \{(T, d) |\ T$ is a class of spaces and d is a topologically invariant N'-valued function$\}$. Clearly, $(T_1, d_1) = (T_2, d_2)$ iff $T_1 = T_2$ and $d_1 = d_2$. We introduce the following order relation on $\mathcal{T}$: for any $(T_1, d_1) \in \mathcal{T}$ and $(T_2, d_2) \in \mathcal{T}$, let $(T_1, d_1) \leq (T_2, d_2)$ if and only if $T_1 \subset T_2$ and d_1 is the restriction of d_2 over T_1, i.e., for any $X \in T_1$ we have $X \in T_2$ and $d_1(X) = d_2(X)$. In the case when $(T_1, d_1) \leq (T_2, d_2)$, we say that the pair (T_1, d_1) is a subpair of the pair (T_2, d_2). Clearly, if $(T_1, d_1) \leq (T_2, d_2)$ and $(T_2, d_2) \leq (T_1, d_1)$, then $(T_1, d_1) = (T_2, d_2)$.

Consider the following eleven subcollections of the collection $\mathcal{T}$:

$\mathcal{P}_1 = \{(T, d) \in \mathcal{T} |$ for all $X \in T$, $indX = IndX = dimX = d(X)\}$.

$\mathcal{P}_2 = \{(T, d) \in \mathcal{T} |$ if $X \in T$ and $A \subset X$, then $A \in T\}$.

$\mathcal{P}_3 = \{(T, d) \in \mathcal{T} |$ if $X, A \in T$ and $A \subset X$, then $d(A) \leq d(X)\}$.

$\mathcal{P}_4 = \{(T, d) \in \mathcal{T} |$ if $I^n \in T$, where $n \in \{-1, 0, 1, 2, \dots\}$, $I^n \in T$, then $d(I^n) = n\}$.

$\mathcal{P}_5 = \{(T, d) \in \mathcal{T} |$ if $X = \bigcup\limits_{i=1}^{\infty} X_i$, where X as well as each X_i belong to T and each X_i is a closed subset of X, then $d(X) = sup\{d(X_i) \mid i \in \mathbf{N}\}\}$.

$\mathcal{P}_6 = \{(T, d) \in \mathcal{T} |$ if $X, A, B \in T$ and $X = A \cup B$, then $d(X) \leq d(A) + d(B) + 1\}$.

$\mathcal{P}_7 = \{(T,d) \in \mathcal{T} |$ if $X \in T$ and $0 \le d(X) = n < \infty$, then there exist $X_1 \in T, \dots, X_{n+1} \in T$ with $X = \bigcup\limits_{i=1}^{n+1} X_i$ and $d(X_i) \le 0$ for any $i \in \{1, \dots, n+1\}\}$.

$\mathcal{P}_8 = \{(T,d) \in \mathcal{T} |$ if $X \in T$, then there exists a compactification bX of X with $bX \in T$ and $d(bX) = d(X)\}$.

$\mathcal{P}_9 = \{(T,d) \in \mathcal{T} |$ if $X_1 \in T$ and $X_2 \in T$, then $X_1 \times X_2 \in T\}$.

$\mathcal{P}_{10} = \{(T,d) \in \mathcal{T} |$ if $X_1 \in T$, $X_2 \in T$, $X_1 \times X_2 \in T$ and $X_1 \bigcup X_2 \neq \emptyset$, then $d(X_1 \times X_2) \le d(X_1) + d(X_2)\}$.

$\mathcal{P}_{11} = \{(T,d) \in \mathcal{T} |$ if $A \in T$, $X \in T$ and $A \subset X$, then there exists a G_δ-subset H of X such that $H \in T$, $A \subset H$ and $d(H) = d(A)\}$.

For all $k_1, \dots, k_m \in \mathbf{N}$, where $1 \le k_1 < \cdots < k_m \le 11$, denote $[k_1, \dots, k_m] = \bigcap\limits_{i=1}^{m} \mathcal{P}_{k_i}$, i.e., a pair $(T,d) \in \mathcal{T}$ belongs to $[k_1, \dots, k_m]$ if and only if it belongs to $\mathcal{P}_{k_i}$ for all $i = 1, \dots, m$.

Consider the set:

$\mathcal{A} = \{[k_1, \dots, k_m] \mid k_1, \dots, k_m \in \mathbf{N};\ 1 \le k_1 < \cdots < k_m \le 11\}$.

We say that element $[k_1, \dots, k_p] \in \mathcal{A}$ contains element $[l_1, \dots, l_q] \in \mathcal{A}$ (or $[l_1, \dots, l_q]$ is contained in $[k_1, \dots, k_p]$) and write $[k_1, \dots, k_p] \supseteq [l_1, \dots, l_q]$ (or $[l_1, \dots, l_q] \subseteq [k_1, \dots, k_p]$) if $\{l_1, \dots, l_q\}$ is a subset of $\{k_1, \dots, k_p\}$.

As was mentioned above, the three classical dimension functions ind, Ind and dim coincide in the class T_{SM}, i.e. for any $X \in T_{SM}$, we have: $indX = IndX = dimX$. Denote the following function defined on T_{SM} and taking values in N' by $\tilde{d}$: for any $X \in T_{SM}$, $\tilde{d}(X) = indX = IndX = dimX$ (see Introduction). Furthermore, let $\tilde{d}_f$ be the restriction of $\tilde{d}$ over T_{fSM}.

We call $(T_{SM}, \tilde{d})$ and $(T_{fSM}, \tilde{d}_f)$ the *standard pair* and the *finite standard pair* respectively.

An element $[k_1, \dots, k_m] \in \mathcal{A}$ is called a T_{SM}-element of $\mathcal{A}$ if any pair $(T,d) \in [k_1, \dots, k_m]$ is a subpair of the standard pair $(T_{SM}, \tilde{d})$.

The work at hand studies two questions, QA and QT.

Question QA (A. Arhangel'skií (see Introduction)). As it was already noted above, the intersection $\bigcap\limits_{i=1}^{11} \mathcal{P}_i$ appears non-empty:

the standard pair $(T_{SM}, \widetilde{d})$ belongs to $\bigcap\limits_{i=1}^{11} \mathcal{P}_i$. It is of interest whether for every $(T, d) \in \bigcap\limits_{i=1}^{11} \mathcal{P}_i$ there is $(T, d) \leq (T_{SM}, \widetilde{d})$. The terminology provided can also formulate the question otherwise: is $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] \in \mathcal{A}$ the T_{SM}-element of $\mathcal{A}$? The solution of the question is provided in the next section (Theorem 1).

Besides, we establish conditions which provide the coincidence of a pair $(T, d) \in \mathcal{T}$ with the finite standard pair $(T_{fSM}, \widetilde{d}_f)$ (Theorem 2).

To formulate the second question QT, we need the following subcollection $\mathcal{P}_{12}$ of the collection $\mathcal{T}$:

$\mathcal{P}_{12} = \{(T, d) \in \mathcal{T}|$ if $X_1, X_2 \in T$, $X_1 \subset X_2$, $x \in X_2 \setminus X_1$ and $X_1 \cup \{x\} \in T$, $d(X_1 \cup \{x\}) = d(X_1)\}$.

Question QT (L.Tumarkin [18]). Let $(T, d) \in \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_3 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{10} \bigcap \mathcal{P}_{12}$. Characterize the class T. Partial solution of the problem is to follow (Theorem 3).

4. Results

The results concerning Question QA are given in the first part of this section.

Lemma 1. *Let T be a (topologically closed) class of (Tychonoff) spaces, having the following properties simultaneously:*

p_1) If $X \in T$ and $Y \subset X$, then $Y \in T$.

p_2) For any $X \in T$, there exists a compactification bX of X, such that $bX \in T$.

p_3) For any $X \in T$, such that $indX = 0$, there exists a compactification bX of X, with $bX \in T$ and $ind(bX) = indX$.

p_4) If $X_1 \in T$ and $X_2 \in T$, then $X_1 \times X_2 \in T$.

Then the conditions 1) - 5) below are equivalent:

1) For all $X \in T$, $dimX \leq 0$ iff $IndX \leq 0$ iff $indX \leq 0$.

2) For all $X \in T$, $dimX \leq 0$ iff $indX \leq 0$.

3) For all $X \in T$, $IndX \leq 0$ iff $indX \leq 0$.

4) If $X \in T$, where X is a locally σ-compact space (i.e., any point of X has a neighborhood the closure of which is σ-compact), $IndX \leq 0$ iff $indX \leq 0$.

5) $T \subset T_{SM}$.

Proof. 1) $\Longrightarrow$ 2) is obvious.

2) $\Longrightarrow$ 3). It is common knowledge that even in the class of T_1 spaces, $dimX{=}0 \Longleftrightarrow IndX{=}0$ (see [6], Appendix), which implies $indX{=}0 \Longrightarrow IndX{=}0$ and it is evident that $IndX{=}0 \Longrightarrow indX{=}0$.

3) $\Longrightarrow$ 4) is obvious.

4) $\Longrightarrow$ 5). Let $X \in T$ and suppose first that $indX \leq 0$. By condition p_3, there exists a compactification bX of X with $bX \in T$ and $ind(bX) = indX$.

Consider the space $Z = bX \times bX \times bX$. Note, that by p_4, $Z \in T$ and clearly, $indZ \leq 0$. Now it is meant to show that the space Z is hereditarily normal. To this end, it is sufficient to prove that any open subspace G of Z is normal (see e.g. [3, 2.1.7]).

Indeed, let G be any open subspace of Z. Since Z is compact, G is locally compact and therefore G is clearly locally σ-compact. But $indZ \leq 0$. Consequently, by monotonicity of ind, $indG \leq 0$. According to the condition of the lemma, we have: $IndG \leq 0$. Hence, G is normal (see [6]). Thus Z is hereditarily normal.

Based on Katetov's theorem, claiming that a compact space K is metrizable if and only if the product $K \times K \times K$ is hereditarily normal (see e.g. [7]), the above mentioned implies that the compact space bX is metrizable. Hence, bX has a countable base. Consequently, X being topologically embeddable in bX, $X \in T_{SM}$.

Suppose now that X were an arbitrary space belonging to T. It has to be shown that $X \in T_{SM}$. Assume on the contrary, that $X \notin T_{SM}$. Then any compactification of the space X is non-metrizable. In particular, the compactification bX of the space X, which appears in p_2, must be non-metrizable as well. Then, by Shnejder's theorem (which states that a compact space K is metrizable if and only if the diagonal $\Delta_K = \{(x,x)|x \in K\}$ is the G_δ-subset of the product $K \times K$ (see e.g. [3, 4.2.B])), one can infer that the diagonal $\Delta_{bX} = \{(x,x)|x \in bX\}$ is not a G_δ-subset of the product $bX \times bX$. As one can clearly see, the complement $A = (bX \times bX) \setminus \Delta_{bX}$ is not Lindelöf. Otherwise, as A is an open subset of the compact space $bX \times bX$ and therefore locally compact, A must be a F_σ-subset of $bX \times bX$. This is impossible given the fact that its complement Δ_{bX} is not a G_δ-subset of $bX \times bX$. But as $bX \in T$, $bX \times bX \in T$ by p_4, and since $A \subset bX \times bX$, $A \in T$ by p_1. Proceeding from the fact that space A is not Lindelöf, a theorem by W. Pfeffer [14] admits the existence of a locally countable [4] non-Lindelöf subspace Y of the space A with $|Y| = \omega_1$ (where $|Y|$ denotes cardinality of Y). At the same time Y is locally σ-compact.

With $A \in T$, $Y \in T$ by p_1 and since the space Y is locally countable and regular, it is not difficult to show that $locindY = 0$ [5]. As it was noted by C. Dowker [2], $indY = locindY$. Thus, $indY = 0$. But being already shown above, the latter implies that $Y \in T_{SM}$. In other words, Y particularly has a countable base and therefore is a Lindelöf space, entailing a contradiction. Consequently, $X \in T_{SM}$.

5) $\Longrightarrow$ 1) is obvious. □

Theorem 1 (see Question QA). *Element* $[k_1, \ldots, k_p] \in \mathcal{A}$ *is a* T_{SM}*-element of* $\mathcal{A}$ *if and only if* $[k_1, \ldots, k_p] \supseteq [1, 2, 8, 9]$. *(In particular, it follows that* $[1, 2, \ldots, 11]$ *is the* T_{SM}*-element of* $\mathcal{A}$*)* [6].

Proof. (Sufficiency). Let $[k_1, ..., k_p] \in\mathcal{A}$ and $[k_1, ..., k_p] \supseteq [1, 2, 8, 9]$. It has to be shown that $[k_1, ..., k_p]$ is a T_{SM}-element of $\mathcal{A}$.

For that purpose, take any $(T, d) \in [k_1, ..., k_p]$ and show that $(T, d) \leq (T_{SM}, \widetilde{d})$. Since $(T, d) \in\mathcal{P}_1$, it suffices to prove, that $T \subset T_{SM}$.

$(T, d) \in [k_1, ..., k_p] = \bigcap\limits_{i=1}^{m} \mathcal{P}_{k_i} \subset\mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9$. Thus, in particular, $(T, d) \in\mathcal{P}_2$, $(T, d) \in\mathcal{P}_8$ and $(T, d) \in\mathcal{P}_9$. This implies that class T has the properties p_1, p_2, p_3 and p_4 from Lemma 1. At the same time, with $(T, d) \in\mathcal{P}_1$, the condition 1) from Lemma 1 is fulfilled. Hence, by Lemma 1, $T \subset T_{SM}$.

(Necessity). Let $[k_1, ..., k_p] \in\mathcal{A}$ be T_{SM}-element of $\mathcal{A}$. It is designed to show that $[k_1, ..., k_p] \supseteq [1, 2, 8, 9]$. Clearly, suffice it to prove that when $[k_1, ..., k_p] \in\mathcal{A}$ and $[k_1, ..., k_p] \not\supseteq [1, 2, 8, 9]$, $[k_1, ..., k_p]$ is not T_{SM}-element of $\mathcal{A}$.

For that purpose, a pair $(T_i, d_i) \in\mathcal{T}$ will be constructed for any $i = 1, 2, 8, 9$ in such a way that $(T_i, d_i) \in\mathcal{P}_j$ for every $j \in \{1, ..., 11\} \setminus \{i\}$ and $T_i \not\subset T_{SM}$.

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[4] A space is said to be locally countable if any point has an open countable neighborhood.

[5] Recall (see [2]) that $locindX \leq n$ $(n \geq -1)$ means that any point $x \in X$ has an open neighborhood O_x with $ind[O_x] \leq n$.

[6] Here we not only answer (positive) the Question QA but point out all T_{SM}-elements of $\mathcal{A}$.

Construction of (T_1, d_1). Let $T_1 = \{X \mid X$ is a Tychonoff space and $indX \leq 0\}$. Let d_1 be also the following function (defined on T_1): for any $X \in T_1$, $d_1(X) = indX$. Then for every $i = 2, \ldots, 11$ the pair (T_1, d_1) belongs to $\mathcal{P}_i$ but, obviously, $T_1 \not\subset T_{SM}$ (note, that $(T_1, d_1) \notin \mathcal{P}_1$).

In fact, it is obvious that for any $i \in \{2 \ldots 11\} \setminus \{8\}$, $(T_1, d_1) \in \mathcal{P}_i$. It is only to be verified that $(T_1, d_1) \in \mathcal{P}_8$.

Let $indX = 0$ and suppose $\mathcal{CO}(X)$ were the system of all clopen subsets of X. Obviously, for any $U_1, U_2 \in \mathcal{CO}(X)$, there is: $U_1 \bigcup U_2 \in \mathcal{CO}(X)$ and for every $U \in \mathcal{CO}(X)$, there is: $[X \setminus U] = X \setminus U$. Therefore, $\mathcal{CO}(X)$ is a π-base in the sense of E.Skljarenko [15] (or, equivalently, $\mathcal{CO}(X)$ is algebraically closed in sense of A.Arhangel'skií and V.Ponomorev [1]). Let bX be the π-compactification of X associated with that π-base (see [15]). It is known that the system $\{O_{bX} < U >\}_{U \in \mathcal{CO}}$ (where $\{O_{bX} < U >\}$ is the maximal open subset of bX with $\{O_{bX} < U >\} \bigcap X = U$) forms an open base of bX [15]. Since the compactification bX is perfect with respect to any $U \in \mathcal{CO}(X)$ [7], $\{O_{bX} < U >\}_{U \in \mathcal{CO}}$ constitutes bX base, consisting of clopen subsets of bX. Hence, $indbX = 0$ and therefore $bX \in T_1$.

Construction of (T_2, d_2). Let T_2 be the following class of spaces: $T_2 = \{X | X$ is a compact space and $indX \leq 0\}$. Clearly, for every $X \in T_2$ there is $dimX = IndX = indX$. Let d_2 be the function on T_2 defined as follows: for any $X \in T_2$, $d_2(X) = indX$. Then it is evident that $(T_2, d_2) \notin \mathcal{P}_2$ and for any $i \in \{1, \ldots, 11\} \setminus \{2\}$, $(T_2, d_2) \in \mathcal{P}_i$. Note, also, that $T_2 \not\subset T_{SM}$.

Construction of (T_8, d_8). Provided $T_8 = \{X | X$ is a Tychonoff space with cardinality not greater than $\aleph_0\}$ and d_8 is the function, defined on T_8: for any $X \in T_8$, $d_8(X) = dimX$, pair (T_8, d_8) belongs to $\mathcal{P}_i$ for any $i \in \{1, \ldots, 11\} \setminus \{8\}$, but, evidently, $T_8 \not\subset T_{SM}$.

Construction of (T_9, d_9). Let C be a discrete uncountable space and suppose $\alpha C = C \cup \{a\}$ were the Alexsandroff one-point compactification of the space C with $\{a\}$ being the one-point remainder. Let T_9 be the following class of spaces: $T_9 = \{X | X$ is a space topologically embeddable in $\alpha C\}$. Also, let d_9 be the following function,

[7] A compactification cX of a space X is said to be perfect with respect to an open subset V of X if the equality $[Fr_X V]_{cX} = Fr_{cX}(O_{cX} < V >)$ takes place (see [15]).

defined on T_9: for any $X \in T_9$, $d_9(X) = indX$. It is scheduled to show that for any $i \in \{1, \ldots, 11\} \setminus \{9\}$, pair (T_9, d_9) belongs to $\mathcal{P}_i$, but $T_9 \not\subset T_{SM}$.

It will be only verified that $(T_9, d_9) \in\mathcal{P}_1$ and $(T_9, d_9) \in\mathcal{P}_8$ (it is obvious that $(T_9, d_9) \in\mathcal{P}_i$ for any $i \in \{1, \ldots, 11\} \setminus \{1, 8\}$ and $T_9 \not\subset T_{SM}$).

For that purpose, note that the following easily verifiable statement holds:

$(\star)$ Let X be any T_1 space and suppose $X_0 \subset X$ were a subspace of X, which is discrete in the induced topology. Assume also, that $X \setminus X_0 = X^{'}$ were a closed subset of X; then: (a) for any point $x \in X_0$ the one point set $\{x\}$ is an open subset of X; (b) for any subset $Y \subset X$, the set $Y \bigcup X^{'}$ is closed in X.

From $(\star)$ it follows that when $A \subset C$ and $B = A \bigcup \{a\}$, B is a closed subset of αC. Hence any such B is compact.

Undoubtably, for any subset $Y \subset \alpha C$ either $Y \subset C$ or $Y = A \bigcup \{a\}$, where $A \subset C$. So, any subspace of αC is either discrete or compact. It is not difficult to show that for any subset $Y \subset \alpha C$ there is: $d_8(X) = dimX = IndX = indX$, i.e, $(T_9, d_9) \in\mathcal{P}_1$.

Finally, show that $(T_9, d_9) \in\mathcal{P}_8$. Take any $Y \subset \alpha C$ and consider the closure $[Y]_{\alpha C}$ of Y in αC. Then $[Y]_{\alpha C}$ is a compactification of Y, with $d_9([Y]_{\alpha C}) = d_9(Y)$, pointing out that $(T_9, d_9) \in\mathcal{P}_8$.

Thus, the construction of pairs (T_1, d_1), (T_2, d_2), (T_8, d_8) and (T_9, d_9) has been fulfilled.

Suppose now that $[k_1, ..., k_p] \in\mathcal{A}$ and $[k_1, ..., k_p] \not\supseteq [1, 2, 8, 9]$; then there exists $i \in \{1, 2, 8, 9\}$ with $i \notin \{k_1, ..., k_p\}$. Clearly, $(T_i, d_i) \in [k_1, ..., k_p]$ and as $T_i \not\subset T_{SM}$, (T_i, d_i) is not a subpair of the standard pair $(T_{SM}, \widetilde{d})$; to put it otherwise, $[k_1, ..., k_p]$ is not a T_{SM}-element of $\mathcal{A}$. □

Remark 1. Particularly, Theorem 1 encompasses the following: suppose we are given a subclass T of the class of all Tychonoff spaces and a $N^{'}$-valued topologically invariant function d defined on T, such that the pair (T, d) satisfies simultaneously the following conditions:

P_1. For any $X \in T$, $d(X) = dimX = IndX = indX$.

P_2. If $X \in T$ and $A \subset X$, $A \in T$.

P_8. If $X \in T$, there exists a compactification bX of X with $bX \in T$ and $d(bX) = d(X)$.

P_9. If $X_1 \in T$ and $X_2 \in T$, $X_1 \times X_2 \in T$.

Then T is a subclass of the class T_{SM} of all separable and metrizable spaces.

As it was noted above (see Notation), there are two different definitions for covering and large inductive dimension functions in the class of all Tychonoff spaces. These are the functions dim and Ind (accepted in the present work) and $dim^\star$ and $Ind^\star$ (see §1).

The following question is aroused naturally.

Question. Suppose we are given a subclass T of the class of all Tychonoff spaces and a N'-valued topologically invariant function d on T, such that pair (T, d) satisfies the conditions P_2, P_8, P_9 (see Remark 1) and also, the following condition:

$P_1^\star$. If $X \in T$, $d(X) = dim^\star X = Ind^\star X = indX$.

Is T then a subclass of the class T_{SM}?

Now it is targeted to show, that in $ZFC + \neg CH$ the answer to this question is negative.

Namely, the following proposition holds.

Proposition $(ZFC + \neg CH)$. *There exists such a class of spaces T' and a topologically invariant N'-valued function d' on T' where T' is not a subclass of T_{SM} but pair (T', d') simultaneously having properties (i), (ii), (iii), (iv) listed below:*

(i) If $X \in T'$, $d'(X) = dim^\star X = Ind^\star X = indX$.

(ii) If $X \in T'$ and $A \subset X$, $A \in T'$.

(iii) If $X \in T'$, there exists a compactification bX of X with $bX \in T'$ and $d'(bX) = d'(X)$.

(iv) If $X_1 \in T'$ and $X_2 \in T'$, $X_1 \times X_2 \in T'$.

Proof. Let D be such a discrete space where $\aleph_0 < |D| < \mathbf{c}$ ($\mathbf{c}$ being the cardinality of continuum) and αD be the Alexandroff one-point compactification of D.

Let T' be the following class of topological spaces:

$T' = \{X|$ there exists a natural number n such that X is topologically embeddable in $(\alpha D)^n\}$.

Now it must be proved that for any $X \in T^{'}$, $dim^{\star}X = Ind^{\star}X = indX$.

Indeed, let $n \in \mathbf{N}$ and X be any subspace of $(\alpha D)^n$.

If $X = \emptyset$, then clearly, $dim^{\star}X = Ind^{\star}X = indX = -1$.

Suppose $X \neq \emptyset$. We will demonstrate that $dim^{\star}X = Ind^{\star}X = indX = 0$.

That $indX = 0$ is easily seen.

To show that $dim^{\star}X = 0$, it suffices to prove that any functionally closed subset of X can be represented as a countable intersection of clopen subsets of X [10]. Take any functionally closed subset X_0 of X. Let $f : X \to [0,1]$ be a continuous function with $X_0 = f^{-1}(0)$. As $|D| < \mathbf{c}$, clearly, $|X| < \mathbf{c}$ and consequently, $|f(X)| < \mathbf{c}$. By [8, 26, V, Theorem 1, p. 296], $dim^{\star}(f(X)) = 0$. Furthermore, by [8, 26, I, Corollary 1b, p. 286], the one point set $\{0\} \subset f(X)$ is a countable intersection of clopen subsets of $f(X)$, i.e., there are clopen subsets $V_1, \dots V_k, \dots$ of $f(X)$, with $\{0\} = \bigcap\limits_{k=1}^{\infty} V_k$. Obviously, for any $k \in \mathbf{N}$, $f^{-1}(V_k)$ is a clopen subset of X and $X_0 = f^{-1}(0) = f^{-1}(\bigcap\limits_{k=1}^{\infty} V_k) = \bigcap\limits_{k=1}^{\infty} f^{-1}(V_k)$.

Consequently, $dim^{\star}X = 0$. However $dim^{\star}X = 0$ if and only if $Ind^{\star}X = 0$ (see Notation). So, $Ind^{\star}X = 0$ as well.

Thus (for every $X \in T^{'}$), there is: $dim^{\star}X = Ind^{\star}X = indX$.

Denote (for any $X \in T^{'}$): $d^{'}(X) = dim^{\star}X = Ind^{\star}X = indX$.

It is obvious that pair $(T^{'}, d^{'})$ satisfies simultaneously the conditions (i), (ii), (iii) and (iv) and (as αD is non-metrizable), $T^{'}$ is not a subclass of T_{SM}. $\square$

To formulate the next theorem, we have to introduce two subcollections - $\mathcal{P}_1^f$ and $\mathcal{P}_{13}$ of the collection $\mathcal{T}$.

$\mathcal{P}_1^f = \{(T, d) \in \mathcal{T} |$ for any $X \in T, indX = IndX = dimX = d(X) < \infty\}$.

$\mathcal{P}_{13} = \{(T, d) \in \mathcal{T} |\ [0,1] \in T\}$.

Clearly, $\mathcal{P}_1^f \subset \mathcal{P}_1$.

Theorem 2. *$(T_{fSM}, \widetilde{d}_f)$ is the only element of $\mathcal{P}_1^f \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{13}$, i.e. $\mathcal{P} \bigcap_1^f \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{13} = \{(T_{fSM}, \widetilde{d}_f)\}$.*

Proof. Clearly, $\mathcal{P}_1^f \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9 \subset \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9$. Hence, for any $(T,d) \in \mathcal{P}_1^f \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9$ it shall be: $(T,d) \in \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9$. Furthermore, by Theorem 1, $(T,d) \le (T_{SM}, \widetilde{d})$, i.e., $T \subset T_{SM}$ and d is the restriction of $\widetilde{d}$ over T. But $(T,d) \in \mathcal{P}_1^f$. So, for any $X \in T \subset T_{SM}$ there is: $dimX = indX = IndX = d(X) < \infty$. Thus, $X \in T_{fSM}$ and clearly, $d(X) = \widetilde{d}\ (X)$. Consequently, $(T,d) \le (T_{fSM}, \widetilde{d}_f)$.

Now we show that $(T_{fSM}, \widetilde{d}_f) \le (T,d)$. Indeed, take any $X \in T_{fSM}$. By Nöbeling-Pontrjagin theorem (see e.g. [6]), there exists $n \in \mathbf{N}$ such that X is embeddable in $[0,1]^{2n+1}$. But as $(T,d) \in \mathcal{P}_{13}$, $[0,1] \in T$. Furthermore, since $(T,d) \in \mathcal{P}_9$, $[0,1]^{2n+1} \in T$; besides as $(T,d) \in \mathcal{P}_2$, $X \in T$. So, the inclusion $T_{fSM} \subset T$ is shown. Apparently $\widetilde{d}_f$ is the restriction of d over T_{fSM}.

Thus, $(T,d) \le (T_{fSM}, \widetilde{d}_f)$ and $(T_{fSM}, \widetilde{d}_f) \le (T,d)$. Hence, $(T,d) = (T_{fSM}, \widetilde{d}_f)$, i.e. $\mathcal{P}_1^f \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{13} = \{(T_{fSM}, \widetilde{d}_f)\}$. □

Remark 2. There exists $(T,d) \in \mathcal{P}_1^f \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9$, such that $(T,d) \neq (T_{fSM}, \widetilde{d}_f)$ (i.e., $T \subset T_{fSM}$ and $T \neq T_{fSM}$).

Proof. Let T be the class of all singletons. Then for any $X \in T$ there is $dimX = indX = IndX$. For any $X \in T$ let $d(X) = dimX$. Then, clearly, $T \subset T_{fSM}$ and $T \neq T_{fSM}$, i.e., $(T,d) \neq (T_{fSM}, \widetilde{d}_f)$. On the other hand, it is obvious that the pair (T,d) belongs to $\mathcal{P}_1^f \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9$. □

Below the study refers to the consideration of Question QT (s. Introduction). Firstly it is shown that whenever a pair (T,d) belongs simultaneously to $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_9$, (T,d) belongs to $\mathcal{P}_3$, $\mathcal{P}_{10}$ and $\mathcal{P}_{12}$ as well. Namely, there takes place the following assertion.

Assertion. $\mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_9 = \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_3 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{10} \bigcap \mathcal{P}_{12}$.

Proof. It is of no doubt that $\mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_3 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{10} \bigcap \mathcal{P}_{12} \subset \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_9$. Show that $\mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_9 \subset \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_3 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{10} \bigcap \mathcal{P}_{12}$, i.e., if $(T,d) \in \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_9$ - $(T,d) \in \mathcal{P}_3$, $(T,d) \in \mathcal{P}_{10}$ and $(T,d) \in \mathcal{P}_{12}$.

Really, let $(T,d) \in \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_9$.

1) $(T, d) \in \mathcal{P}_3$. Take any $X \in T$ and any $A \subset X$. Since $(T, d) \in \mathcal{P}_2$, we clearly, have: $A \in T$. Furthermore, since $A \in T$, $X \in T$ and $(T, d) \in \mathcal{P}_1$, then $d(A) = indA$ and $d(X) = indX$. But $indA \leq indX$. Hence, $d(A) \leq d(X)$, i.e., $(T, d) \in \mathcal{P}_3$.

2) $(T, d) \in \mathcal{P}_{10}$. Let $X_1 \in T$ and $X_2 \in T$. Since $(T, d) \in \mathcal{P}_9$, $X_1 \times X_2 \in T$.

As it was shown by P.Ostrand [11], for the dimension function dim the finite sum theorem (see [12]) holds in any space (without any restrictions on separation axioms). In particular, the function dim satisfies the finite sum theorem in X_1 and X_2.

Since $(T, d) \in \mathcal{P}_1$ and $(T, d) \in \mathcal{P}_2$, then for any $A \subset X_1$ and any $B \subset X_2$ there is: $dimA = indA$ and $dimB = indB$. Hence for the function ind the finite sum theorem holds both in X_1 and X_2. But then, as it was established by B.Pasynkov [12, Theorem 2], there takes place inequality $ind(X_1 \times X_2) \leq ind(X_1) + ind(X_2)$. As $X_1 \in T$, $X_2 \in T$, $X_1 \times X_2 \in T$ and $(T, d) \in \mathcal{P}_1$, $d(X_1 \times X_2) = ind(X_1 \times X_2) \leq ind(X_1) + ind(X_2) = d(X_1) + d(X_2)$. So, $(T, d) \in \mathcal{P}_{10}$.

3) $(T, d) \in \mathcal{P}_{12}$. Let $X_1 \in T$, $X_2 \in T$, $X_1 \subset X_2$ and $x \in X_2 \setminus X_1$. Since $(T, d) \in \mathcal{P}_2$, $X_2 \in T$ and $X_1 \bigcup \{x\} \subset X_2$, there is: $X_1 \bigcup \{x\} \in T$.

As $(T, d) \in \mathcal{P}_1$, $d(X_1 \bigcup \{x\}) = dim(X_1 \bigcup \{x\}) = ind(X_1 \bigcup \{x\}) = Ind(X_1 \bigcup \{x\})$ and $d(X_1) = dim(X_1) = ind\ (X_1) = Ind(X_1)$. But $ind(X_1 \bigcup \{x\}) \geq ind(X_1)$. Consequently $d(X_1 \bigcup \{x\}) = ind(X_1 \bigcup \{x\}) \geq ind(X_1) = d(X_1)$.

Furthermore, E.Skljarenko showed in [15], that if B is a regular space and $A \subset B$, then for any point $z \in B \setminus A$, the inequality $Ind(A \bigcup \{x\}) \leq IndA$ holds. Thus, $d(X_1 \bigcup \{x\}) = Ind(X_1 \bigcup \{x\}) \leq IndX_1 = d(X_1)$. The latter with inequality $d(X_1 \bigcup \{x\}) \geq d(X_1)$ (s. above), provides equality $d(X_1 \bigcup \{x\}) = d(X_1)$, i.e., $(T, d) \in \mathcal{P}_{12}$. $\square$

The following theorem provides a partial answer for the Question QT by imposing one additional restriction.

Theorem 3 (A partial answer to the Question QT). *Let* $(T, d) \in \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_3 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{10} \bigcap \mathcal{P}_{12}$ *and suppose, in addition,* $(T, d) \in \mathcal{P}_8$, *i.e.,* $(T, d) \in \mathcal{P}_1 \bigcap \mathcal{P}_2 \bigcap \mathcal{P}_3 \bigcap \mathcal{P}_8 \bigcap \mathcal{P}_9 \bigcap \mathcal{P}_{10} \bigcap \mathcal{P}_{12}$. *In this case* $(T, d) \subset (T_{SM}, \widetilde{d})$. *In particular,* T *is a subclass of the class* T_{SM}.

Proof. As $\mathcal{P}_1\bigcap\mathcal{P}_2\bigcap\mathcal{P}_3\bigcap\mathcal{P}_8\bigcap\mathcal{P}_9\bigcap\mathcal{P}_{10}\bigcap\mathcal{P}_{12}\supset\mathcal{P}_1\bigcap\mathcal{P}_2\bigcap\mathcal{P}_8\bigcap\mathcal{P}_9$, the correctness of the statement follows directly from Theorem 1. $\square$

5. Open questions

Definition. We say that a topological space X is hereditarily strongly zero-dimensional if for any subspace $A\subset X$ equality $dim\beta A=0$ holds, where βA denotes Čech-Stone compactification of A (meaning that, for any $A\subset X$ we have: $dim^\star A=0$ (see §1)).

The following question is posed naturally (s. proposition above).

Question 1. Does there exist a non-metrizable compact space X without any set theoretic assumptions, any finite power of which (i.e., $X^n=\underbrace{X\times\cdots\times X}_{n}$, where $n=1,2,\dots$) is hereditarily strongly zero-dimensional? It is of essence to prove in particular whether any finite power of αC (where C is an uncountable discrete space and αC denotes the Alexandroff one-point compactification of C) is hereditarily strongly zero-dimensional without any set-theoretic assumptions.

It is shown in [5] that any finite power of the "Arrow" (Sorgenfrey Line) is hereditarily strongly zero-dimensional, even though is not a compact space.

Question 2. It is of interest whether any finite power of Alexandroff's "Double Arrow" (s. e.g. [1],[3]) is hereditarily strongly zero-dimensional.

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Department of Mathematics, Tbilisi State University, Tbilisi, Georgia 0128
E-mail address: ivanetsereteli@hotmail.com