TOPOLOGY PROCEEDINGS
Volume 33, 2009
Pages 361–380
http://topology.auburn.edu/tp/

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Electronically published on December 27, 2008

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 33 (2009)
Pages 361-380
E-Published on December 27, 2008

THEOREMS WITH UNIFORM CONDITIONS ON SETS NOT BELONGING TO ALGEBRAS

L.Š. GRINBLAT

Abstract. Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of σ-algebras on a set X, where $|\Lambda| = \aleph_0$, with $\mathcal{A}_\lambda \neq \mathcal{P}(X)$ for any $\lambda \in \Lambda$, and for any finite set $J \subset \Lambda$, where $|J| = h \geq 3$, there exist $2h-2$ pairwise disjoint sets belonging to $\mathcal{P}(X) \setminus \bigcap_{\lambda\in J}\mathcal{A}_\lambda$; then $\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda \neq \mathcal{P}(X)$. If we substitute the estimate $2h-3$ for $2h-2$, this theorem does not hold.

1. Introduction

1.1. The object of the present investigation is sets not belonging to algebras of sets. The present article is a further development of the theory formulated in [Gr1], [Gr2],[Gr3].

Definition 1.1. By an *algebra* $\mathcal{A}$ on a set X we mean a non-empty system of subsets X possessing the following properties: (1) if $M \in \mathcal{A}$, then $X \setminus M \in \mathcal{A}$; (2) if $M_1, M_2 \in \mathcal{A}$, then $M_1 \cup M_2 \in \mathcal{A}$.

1.2. *Some notations and names.* All algebras and measures are considered on some abstract set $X \neq \emptyset$. As usual, $\mathcal{P}(M)$ denotes the set of all subsets of the set M. When it is clear from the context, we will not state explicitly that a set belongs to $\mathcal{P}(X)$. The set M is called *countable* if $|M| = \aleph_0$. By $\mathbb{N}^+$ we denote the set of natural numbers. If $n_1, n_2 \in \mathbb{N}^+$ and $n_1 \leq n_2$, then

$$[n_1, n_2] = \{k \in \mathbb{N}^+ \mid n_1 \leq k \leq n_2\}.$$

2000 *Mathematics Subject Classification.* Primary 03E05; Secondary 54D35.
Key words and phrases. Algebra on a set, ultrafilter, a set not belonging to algebras, pairwise disjoint sets, systems of distinct representatives.

As usual, an algebra $\mathcal{A}$ is called a *σ-algebra*, if for any countable sequence $M_1, ..., M_k, ... \in \mathcal{A}$, we have $\mathcal{A} \ni \bigcup_{k=1}^{\infty} M_k$.

1.3. We now give a useful definition.
Definition 1.3. An algebra $\mathcal{A}$ has a κ lacunae, where κ is a cardinal number, if there exist κ pairwise disjoint sets not belonging to $\mathcal{A}$.

1.4. Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of algebras and $\mathcal{A}_\lambda \neq \mathcal{P}(X)$ for each λ. The question we would like to answer in a general form is as follows: Under what conditions $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \neq \mathcal{P}(X)$? It is very simple to prove that if $|\Lambda| = 2$, then $\mathcal{A}_1 \cup \mathcal{A}_2 \neq \mathcal{P}(X)$. In the case $|\Lambda| = 3$, it is not necessarily true that $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \neq \mathcal{P}(X)$, (see Section 4.3). In our theorems $0 < |\Lambda| \leq \aleph_0$, and, as was demonstrated in [Gr1], these theorems are related to combinatorial set theory. To all appearances Ulam was the first to be engaged in such problems, and the first publication connected with the topic was that of Erdös [E] (this paper contains the well-known theorem of Alouglu-Erdös). Some information about the history of the subject of our investigation after the publication of [E] is set forth in [Gr1]. In fact, Ulam, Alouglu and Erdös investigated sets non-measurable with respect to families of measures. Let $|X| = 2^{\aleph_0}$. Let a σ-additive measure μ be defined on X. Here $\mu(X) = 1$, the measure of a one-point set equals 0, and the measure of each μ-measurable set equals 0 or 1. The measure μ is called *σ-two-valued*. Clearly, there exist μ-non-measurable sets. The Alouglu-Erdös theorem states that, in the assumption of the continuum-hypothesis, for any countable sequence of σ-two-valued measures $\mu_1, ... , \mu_k, ...$ specified on X, there exists a set which is non-measurable with respect to all these measures. The proof of the Alouglu-Erdös theorem is very simple and based on the possibility of constructing the well-known Ulam matrix. The Gitik-Shelah theorem asserts the validity of the Alouglu-Erdös theorem beyond the assumption of the continuum-hypothesis. The first proof of the Gitik-Shelah theorem in [G-S] is metamathematical and uses the forcing method. Purely mathematical proofs were suggested by Fremlin [F] and Kamburelis [K]. It is noteworthy that the Gitik-Shelah theorem is deep and non-trivial. The Gitik-Shelah theorem can be reinstated in our language. Let us consider the σ-two-valued measures $\mu_1, ... , \mu_k, ...$ on X again. For each measure μ_k we examine the algebra $\mathcal{A}_k$ of all μ_k-measurable sets. The Gitik-Shelah theorem asserts that $\bigcup_{k=1}^{\infty} \mathcal{A}_k \neq \mathcal{P}(X)$. We note that here each algebra $\mathcal{A}_k$ has $\aleph_0$ lacunae. If $|X| = \aleph_1$, then the situation is much simpler: each algebra $\mathcal{A}_k$ has $\aleph_1$ lacunae. We will use the Gitik-Shelah theorem in the proofs of our theorems for countability of many σ-algebras.

1.5. This section and the next section deal with the Main Idea that formed the basis of the investigation described in [Gr1], [Gr2], [Gr3]. The essence of this idea is as follows: Sets not belonging to algebras represent global objects, and their study is very complicated. However, the study of sets not belonging to algebras can be reduced to the study of ultrafilters. These ultrafilters represent the most local objects. They represent points of a respective compact extension. Let βX be the Stone-Čech compactification of X in discrete topology; βX is the family of all ultrafilters on X.

Consider an algebra $\mathcal{A}$. We define a notion of $\mathcal{A}$ - equivalent on βX. Two ultrafilters $a \neq b$ are said to be $\mathcal{A}$ - *equivalent* if for any $M \in \mathcal{A}$ we have $M \in a, b$ or $M \notin a, b$.[1] The $\mathcal{A}$ - equivalent relation is symmetric and transitive. Two ultrafilters are $\mathcal{A}$ - equivalent if and only if whenever M belongs to one of these ultrafilters but not to another one, we have $M \notin \mathcal{A}$. If a, b are $\mathcal{A}$ - equivalent ultrafilters, then we say that *a has an $\mathcal{A}$ - equivalent ultrafilter b*, or a is $\mathcal{A}$ - *equivalent to b*. Clearly, there may exist ultrafilters which have no $\mathcal{A}$ - equivalent ultrafilters.

Statement. Consider an algebra $\mathcal{A}$. A set U does not belong to $\mathcal{A}$ if and only if there exist $\mathcal{A}$ - equivalent ultrafilters a, b such that $U \in a,\ U \notin b$.

Proof. We must prove the following. Let $U \notin \mathcal{A}$. Then there exist $\mathcal{A}$ - equivalent ultrafilters a, b such that $U \in a,\ U \notin b$. Let us assume the contrary. We fix an ultrafilter q, and $U \in q$. For any ultrafilter $r \ni X \setminus U$, we take a set $V(r) \in r$ such that $V(r) \in \mathcal{A}$ and $V(r) \notin q$. Since the set of all ultrafilters which contain $X \setminus U$ is a compact subset of βX, there exist a finite sequence of sets $V(r_1), \dots, V(r_m)$ with the following properties:

(1) $V(r_k) \in \mathcal{A}$ for any $k \in [1, m]$;
(2) $V(r_k) \notin q$ for any $k \in [1, m]$;
(3) $X \setminus U \subset \bigcup_{k=1}^{m} V(r_k)$.

[1]In our previous publications we called these ultrafilters a, b $\mathcal{A}$-similar.

Let

$$W(q) = X \setminus \bigcup_{k=1}^{m} V(r_k) \ .$$

It is clear that $W(q) \in q$, $W(q) \in \mathcal{A}$, and $W(q) \subset U$. Since the set of all ultrafilters which contain U is a compact subset of βX, there exists a finite sequence of sets $W(q_1), \dots , W(q_n)$ such that $W(q_k) \in \mathcal{A}$ for any $k \in [1, n]$, and $\bigcup_{k=1}^{n} W(q_k) = U$. We have $U \in \mathcal{A}$, a contradiction. □

1.6. *Notation:* If $M \subset \beta X$ (in particular, if $M \subset X$), we denote by $\overline{M}$ the closure of M in βX. Now let us formulate and prove the Main Statement from [Gr1]. In the Main Statement we examine a family of algebras of arbitrary cardinality.

Main Statement. Let $\{\mathcal{A}_\lambda\}$ be a family of algebras. Then $\bigcup \mathcal{A}_\lambda \neq \mathcal{P}(X)$ if and only if there exist closed sets $S, T \subset \beta X$ such that $S \cap T = \emptyset$, and for each λ the following condition holds: there exist $\mathcal{A}_\lambda$ - equivalent ultrafilters s_λ, t_λ such that $s_\lambda \in S, t_\lambda \in T$.

Proof. Let $\bigcup \mathcal{A}_\lambda \neq \mathcal{P}(X)$, ie., there exists $Q \notin \mathcal{A}_\lambda$ for all λ. By virtue of Statement of 1.5, we can put $S \subset \overline{Q}$, $T = \overline{X \setminus Q}$. On the other hand, if the corresponding S and T exist, then there exists $Q \subset X$ such that $S \subset \overline{Q}$ and $T \bigcap \overline{Q} = \emptyset$. By virtue of Statement 1.5, one has $Q \notin \mathcal{A}_\lambda$ for all λ. □

1.7. The following concept was used in [Gr1].

Definition 1.7. A set $M \subset \beta X$ is said to be $\mathcal{A}$ - *equivalent* if $|M| > 1$, any two distinct ultrafilters in M are $\mathcal{A}$-equivalent, and there exist no $\mathcal{A}$-equivalent ultrafilters a, b such that $a \in M, b \notin M$.

An $\mathcal{A}$ - equivalent set is closed. Indeed, let M be an $\mathcal{A}$ - equivalent set. If $L \in \mathcal{A}$, then we have $\overline{M} \subset \overline{L}$ or $\overline{M} \bigcap \overline{L} = \emptyset$. Therefore, any two distinct ultrafilters $a, b \in \overline{M}$ are $\mathcal{A}$-equivalent, and $M = \overline{M}$.

1.8. Let us mention another concept from [Gr1].

Definition 1.8. Consider an algebra $\mathcal{A}$; the set

$$\{a \in \beta X \mid a \text{ has } \mathcal{A} - \text{equivalent ultrafilter}\}$$

is called *the kernel of the algebra* $\mathcal{A}$ and denoted by $\ker \mathcal{A}$.

If an algebra $\mathcal{A} = \mathcal{P}(X)$, then $\ker \mathcal{A} = \emptyset$. If an algebra $\mathcal{A} \neq \mathcal{P}(X)$, then $|\ker \mathcal{A}| \geq 2$ and $\ker \mathcal{A}$ is separated into pairwise disjoint $\mathcal{A}$ -equivalent sets.

1.9. Now we give two very simple propositions.
Proposition 1.9. *An algebra* $\mathcal{A}$ *has* κ *lacunae, where* $2 \leq \kappa \leq \aleph_0$, *if and only if* $|\ker \mathcal{A}| \geq \kappa$.

Proposition 1.10. *Let* $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ *be a finite family of algebras. The algebra* $\bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$ *has* κ *lacunae, where* $2 \leq \kappa \leq \aleph_0$, *if and only if* $\left|\bigcup_{\lambda\in\Lambda} \ker \mathcal{A}_\lambda\right| \geq \kappa$.

1.11. Now we pass to setting forth some results of our previous publications. The proof of the following theorem is presented in [Gr1], Chapter 4.
Theorem 1.11. *Let* $\mathcal{A}_1, ..., \mathcal{A}_n, \mathcal{A}_{n+1}$, *where* $n \in \mathbb{N}^+$, *be a sequence of algebras,* $\bigcup_{k=1}^{n} \mathcal{A}_k \neq \mathcal{P}(X)$, *and assume that* $\mathcal{A}_{n+1}$ *has more than* $\frac{4}{3}n$ *lacunae. Then* $\bigcup_{k=1}^{n+1} \mathcal{A}_k \neq \mathcal{P}(X)$.

1.12. The following theorem is an obvious corollary of Theorem 1.11.
Theorem 1.12. *Let* $\mathcal{A}_1, ..., \mathcal{A}_n$ *be a finite sequence of algebras and assume that each* $\mathcal{A}_k (k \in [1, n])$ *has more than* $\frac{4}{3}(k-1)$ *lacunae. Then* $\bigcup_{k=1}^{n} \mathcal{A}_k \neq \mathcal{P}(X)$.
The estimate $\frac{4}{3}(k-1)$ cannot be improved (see [Gr1], Chapter 4). Note that the condition of this theorem is nonuniform, i.e., the requirement imposed on the algebra $\mathcal{A}_k$ is that the number of pairwise disjoint sets not belonging to it must depend on k.

1.13. The next theorem is a generalization of the theorems of Alouglu-Erdös and Gitik-Shelah. We proved it (see Chapter 5 and 7 [Gr1]), substantially using the Gitik-Shelah theorem.
Theorem 1.13. *Let* $\{\mathcal{A}_k\}_{k\in\mathbb{N}^+}$ *be a family of* σ*-algebras, and each* $\mathcal{A}_k$ *has more than* $\frac{4}{3}(k-1)$ *lacunae. Then* $\bigcup_{k=1}^{\infty} \mathcal{A}_k \neq \mathcal{P}(X)$.

1.14. We can consider finite families of algebras $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ and require uniformity in the literal sense, i.e., require only one number of pairwise disjoint sets not belonging to each algebra $\mathcal{A}_\lambda$. It is clear that this number depends on $|\Lambda|$. For this purpose we present the following natural definition given in [Gr1], Chapter 2.
Definition 1.14. For each $n \in \mathbb{N}^+$ denote by $\mathfrak{g}(n)$ the minimal natural number possessing the following property: If $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is a family of algebras, $|\Lambda| = n$, and each $\mathcal{A}_\lambda$ has $\mathfrak{g}(n)$ lacunae, then $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \neq \mathcal{P}(X)$.

Let us list some results from [Gr1], Chapter 2, concerning the function $\mathfrak{g}(n)$:

$$\mathfrak{g}(n) = 1 \quad \text{if} \quad n \in [1, 2];$$

$$\mathfrak{g}(n) = 3 \quad \text{if} \quad n \in [3, 4];$$

$$\mathfrak{g}(n) = 5 \quad \text{if} \quad n \in [5, 10].$$

And, finally, a complicated result: $\mathfrak{g}(n) \leq 2 \cdot \lceil \log_2(n-2) \rceil - 1$ if $n > 10$ (by $\lceil \rho \rceil$ we denote the minimum integer $\geq \rho$). Apparently, an exact description of $\mathfrak{g}(n)$ for all n values is a very difficult task. Although Theorem 1.12 has its natural infinite analog – Theorem 1.13, the function $\mathfrak{g}(n)$ is directly related to finite families of algebras only.

1.15. A few words about the content of the following chapters. In the second chapter the main results are presented. In the third chapter we present familiar simple results of experts on combinatorics, pertinent to Hall's theorem on a system of distinct representatives. The forth chapter deals with finite families of algebras, and we prove the main result – Theorem 2.1. The fifth chapter is devoted to countable families of $\sigma-$ algebras, and we prove the main result – Theorem 2.2. One problem is discussed in Section 2.5.

2. Main Results

2.1. Here we formulate and later on we prove two theorems with uniform conditions imposed on algebras.

Theorem 2.1. *Let $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ be a finite not empty family of algebras, with $\mathcal{A}_\lambda \neq \mathcal{P}(X)$ for any $\lambda \in \Lambda$, and for each $J \subset \Lambda$, where $|J| = h \geq 3$, the algebra $\bigcap_{\lambda \in J} \mathcal{A}_\lambda$ has $2h-2$ lacunae. Then $\bigcup_{\lambda \in \Lambda} \mathcal{A}_k \neq \mathcal{P}(X)$.*

2.2. The next theorem is a generalization of the theorems of Alouglu-Erdös and Gitik-Shelah. We prove it substantially using the Gitik-Shelah theorem.

Theorem 2.2. *Let $\{\mathcal{A}_k\}_{k \in \mathbb{N}^+}$ be a family of σ-algebras, with $\mathcal{A}_k \neq \mathcal{P}(X)$ for any $k \in \mathbb{N}^+$, and for any finite set $J \subset \mathbb{N}^+$, where $|J| = h \geq 3$, the algebra $\bigcap_{k \in J} \mathcal{A}_k$ has $2h-2$ lacunae. Then $\bigcup_{k \in \mathbb{N}^+} \mathcal{A}_k \neq \mathcal{P}(X)$.*

Remark 2.3. In the proof of Theorem 2.1 one can assume that $|X| < \aleph_0$ (see [Gr1], Chapter 3). This implies that the algebras considered in Theorem 2.1 can be assumed to be σ-algebras. Therefore Theorem 2.2 is a direct generalization of Theorem 2.1.

2.4. Now we must give the following concept which was used in [Gr1].
Definition 2.4. An algebra $\mathcal{A}$ is said to be an *almost σ-algebra*, if for any countable sequence of sets $M_1, ..., M_k, ...$ such that $\mathcal{P}(M_k) \subset \mathcal{A}$ for each k, we have $\mathcal{A} \ni \bigcup_{k=1}^{\infty} M_k$.

Problem 2.5. The requirement for the algebras $\mathcal{A}_k$ in Theorem 2.2 to be σ-algebras is essential. Using the notion of absolute introduced by Gleason, we can construct a countable sequence of algebras $\mathcal{B}_1, ..., \mathcal{B}_k, ...$ possessing the following property: each $\mathcal{B}_k$ has $\aleph_0$ lacunae, and $\bigcup_{k=1}^{\infty} \mathcal{B}_k = \mathcal{P}(X)$ (see [Gr1], Chapter 5). However, Theorem 1.13 is valid if we substitute σ-algebras with almost σ-algebras. (The proof of this theorem is given in [Gr1], Chapter 7). Naturally, the question arises: Does Theorem 2.2 remain valid if we substitute almost σ-algebras for σ-algebras in its formulation? We explain the difficulty of this problem in Remark 5.8.

3. Finite sequences of sets and the Hall's theorem

3.1. In the present chapter we examine finite sequences of subsets $\sum_1, ..., \sum_n$ of a certain set $\sum$. Let the set $J \subset [1, n]$. We denote

$$b_J = \left| \bigcup_{j \in J} \Sigma_j \right| .$$

The classical Hall's theorem on systems of distinct representatives [H] states the following: If $\sum_1, ... , \sum_n$ is a finite sequence of sets and $b_J \geq |J|$ for each set $J \subset [1, n]$, then there exist pairwise distinct elements $s^1, ... , s^n$ such that $s^k \in \sum_k$. The proof of this theorem is simple. The proof of the following lemma has been reported to the author by N.Alon.

Lemma 3.1. *Let $\sum_1, ..., \sum_n$ be a finite sequence of sets and $b_J \geq 2 \cdot |J|$ for any set $J \subset [1, n]$. Then for any $k \in [1, n]$ we can choose elements $s_1^k, s_2^k \in \sum_k$, since we have chosen $2n$ pairwise distinct elements.*

Proof. Let us consider the sequence of $2n$ sets

$$\sum_1, \sum_1, \sum_2, \sum_2, \dots, \sum_n, \sum_n .$$

This sequence satisfies the condition of the Hall's theorem. Therefore there exists $2n$ pairwise distinct elements $s_1^1, s_2^1, s_1^2, s_2^2, \dots, s_1^n, s_2^n$; and $s_i^k \in \sum_k$. □

Lemma 3.2. *Let $\sum_1, \dots, \sum_n$ be a finite sequence of sets, $n > 1$, and for any set $J \subset [1, n]$, where $|J| = h > 0, b_J \geq 2h - 1$. Then for some $k_0 \in [1, n]$ we can choose an element $s_1^{k_0} \in \sum_{k_0}$; and for each $k \in [1, n] \setminus \{k_0\}$ we can choose two elements $s_1^k, s_2^k \in \sum_k$. As a result we have chosen $2n - 1$ pairwise distinct elements.*

Proof. We can assume that there exists an element

$$z_0 \in \sum \setminus \bigcup_{k=1}^n \sum_k .$$

We examine the set

$$\widehat{\sum}_k = \sum_k \bigcup \{z_0\}$$

for each $k \in [1, n]$. For the sets $\widehat{\sum}_1, \dots, \widehat{\sum}_n$ the condition of Lemma 3.1 is satisfied. Therefore, we can find the corresponding elements s_1^k, s_2^k for all $k \in [1, n]$. Assume that if $z_0 = s_i^k$, then $z_0 = s_2^1$. We have found required elements: $s_1^1 \in \sum_1$, and $s_1^k, s_2^k \in \sum_k$ if $k \in [2, n]$. □

Theorem 3.3. *Let $\sum_1, \dots, \sum_n$ be a finite sequence of non-empty sets, $n > 2$, and for any set $J \subset [1, n]$, where $|J| = h > 1, b_J \geq 2h - 2$. Then for some various $k_*, k_{**} \in [1, n]$ we can take elements $s_1^{k_*} \in \sum_{k_*}, s_1^{k_{**}} \in \sum_{k_{**}}$; and for each $k \in [1, n] \setminus \{k_*, k_{**}\}$ we can take two elements $s_1^k, s_2^k \in \sum_k$. As a result we have chosen $2n - 2$ pairwise distinct elements.*

Proof. We can assume that there exists an element

$$z_0 \in \sum \setminus \bigcup_{k=1}^n \sum_k .$$

We examine the set

$$\widehat{\sum}_k = \sum_k \bigcup \{z_0\}$$

for each $k \in [1, n]$. For the sets $\widehat{\sum}_1, \dots, \widehat{\sum}_n$ the condition of Lemma 3.2 is satisfied. Therefore we can assume that it is possible to take an element $s_1^1 \in \widehat{\sum}_1$ and elements $s_1^k, s_2^k \in \widehat{\sum}_k$, where $k \in [2, n]$. As a result we have chosen $2n-1$ pairwise distinct elements.

Case 1. $z_0 \neq s_1^1$. We assume that if $z_0 = s_i^k$, then $z_0 = s_2^2$. We have found required elements: $s_1^1 \in \sum_1, s_1^2 \in \sum_2$, and $s_1^k, s_2^k \in \sum_k$, where $k \in [3, n]$.

Case 2. $z_0 = s_1^1$. Let $z_* \in \sum_1$. We assume that if $z_* = s_i^k$, then $z_* = s_1^2$. We have found required elements: $z_* \in \sum_1, s_2^2 \in \sum_2$, and $s_1^k, s_2^k \in \sum_k$, where $k \in [3, n]$. □

4. Finite sequences of algebras

4.1. The following lemma has been proved in [Gr1]. Here we present it with a very similar, but more elegant proof.

Lemma 4.1. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a finite sequence of algebras, $n \geq 3$, and assume that there exist $2n-2$ pairwise distinct ultrafilters $s_1^1, \dots, s_1^n, s_2^3, \dots, s_2^n$, and $s_i^k \in \ker \mathcal{A}_k$. Then $\bigcup_{k=1}^n \mathcal{A}_k \neq \mathcal{P}(X)$.*

Proof. Put

$$\mathfrak{S}_n = \{s_1^1, \dots, s_1^n, s_2^3, \dots, s_2^n\}.$$

In the course of our proof each ultrafilter from $\mathfrak{S}_n$ will be marked with the figures 0 or 1. For the sake of convenience we consider a finite set

$$S_0 \subset \beta X \setminus \mathfrak{S}_n \ .$$

Each ultrafilter from S_0 we mark by 1. For each ultrafilter $s_i^k \in \mathfrak{S}_n$ we take $\mathcal{A}_k$ - equivalent ultrafilter t_i^k. Without any loss of generality, we assume that $n \geq 5$.

- Case 1. $t_1^1 \notin \mathfrak{S}_n \cup S_0$. Let us mark s_1^1 by 1 and mark s_2^3 by 0.
- Case 2. $t_1^1 = s_{i_0}^{k_0} \in \mathfrak{S}_n$ *and* $k_0 \in [3, n]$. Let $t_1^1 = s_2^3$. Let us mark s_1^1 by 1 and mark s_2^3 by 0.
- Case 3. $t_1^1 \in S_0$. Let us mark s_1^1 and s_2^3 by 0.
- Case 4. $t_1^1 = s_1^2$. We divide this case into four cases.
  - Case 4-1. $t_1^2 \notin \mathfrak{S}_n \cup S_0$. Let us mark s_1^1, s_2^3, s_2^4 by 0 and mark s_1^2 by 1.
  - Case 4-2. $t_1^2 = s_{i_0}^{k_0} \in \mathfrak{S}_n$ *and* $k_0 \in [3, n]$. Let $t_1^2 = s_2^3$. Let us mark s_1^1, s_2^3, s_2^4 by 0 and mark s_1^2 by 1.

– Case 4-3. $t_1^2 \in S_0$. Let us mark s_1^2, s_2^3, s_2^4 by 0 and mark s_1^1 by 1.
– Case 4-4. $t_1^2 = s_1^1$. Let us mark s_1^2, s_2^3, s_2^4 by 0 and mark s_1^1 by 1.

All ultrafilters marked up to the moment by 1 constitute the set $\widetilde{S}$; clearly, $S_0 \subset \widetilde{S}$. We examine algebras $\mathcal{A}_k$ for which there exist non-marked ultrafilters $s_i^k \in \mathfrak{S}_n$. In Cases 1,2,3 there are algebras $\mathcal{A}_2, \dots, \mathcal{A}_n$, and the set of non-marked ultrafilters is

$$\mathfrak{S}_{n'} = \{s_1^2, \dots, s_1^n, s_2^4, \dots, s_2^n\}.$$

In Case 4 there are algebras $\mathcal{A}_3, \dots, \mathcal{A}_n$, and the set of non-marked ultrafilters is

$$\mathfrak{S}_{n'} = \{s_1^3, \dots, s_1^n, s_2^5, \dots, s_2^n\}.$$

In the next step we consider $\mathfrak{S}_{n'}$ instead of $\mathfrak{S}_n$ and $\widetilde{S}$ instead of S_0. We continue by means of induction. We mark in a clear way all ultrafilters from $\mathfrak{S}_n$, and denote the set of all ultrafilters marked by 1 by S. We would like to note the obvious fact which does not influence the proof: we can achieve that $S \subset \mathfrak{S}_n$. For this we should assume that $S_0 = \emptyset$. By virtue of our constructions for each $k \in [1, n]$ there exist $\mathcal{A}_k$ - equivalent ultrafilters s_k, t_k, and $s_k \in S, t_k \notin S$. By virtue of the Main Statement from Section 1.6, $\bigcup_{k=1}^n \mathcal{A}_k \neq \mathcal{P}(X)$. □

4.2. *The Proof of Theorem 2.1.* The Theorem is obviously true when $|\Lambda| = 1$. As it has been said above it is easy to demonstrate that the Theorem is true when $|\Lambda| = 2$. Let $|\Lambda| = n > 2$. We will number algebras from 1 to n: $\mathcal{A}_1, \dots, \mathcal{A}_n$. We have:

a) $|\ker \mathcal{A}_k| \geq 2$ for any $k \in [1, n]$;

b) for any set $J \subset [1, n]$, where $|J| = h > 1$, $\left|\bigcup_{k \in J} \ker \mathcal{A}_k\right| \geq 2h - 2$.

It means that sets $\ker \mathcal{A}_1, \dots, \ker \mathcal{A}_n$ satisfy the condition of the Theorem 3.3. By virtue of Lemma 4.1, $\bigcup_{k=1}^n \mathcal{A}_k \neq \mathcal{P}(X)$. □

4.3. A few words regarding the evaluation $2h - 2$ from Theorem 2.1. For each natural $n \geq 3$ it is possible to construct a sequence of algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ possessing the following properties:

1) $\ker \mathcal{A}_1$ consists of two $\mathcal{A}_1$ - equivalent ultrafilters s_1, s_2;

2) $\ker \mathcal{A}_2$ consists of two $\mathcal{A}_2$ - equivalent ultrafilters s_2, s_3;

2) $\ker \mathcal{A}_3$ consists of two $\mathcal{A}_3$ - equivalent ultrafilters s_1, s_3;

4) if a set $J \subset [1, n], |J| = h \geq 3$ and $J \setminus \{1, 2, 3\} \neq \emptyset$, then
$$\left|\bigcup_{k\in J} \ker \mathcal{A}_k\right| \geq 2h - 2\ .$$
Clearly, $\mathcal{A}_k \neq \mathcal{P}(X)$ if $k > 3$. Indeed, let $n > 3$. We shall show that $\mathcal{A}_4 \neq \mathcal{P}(X)$. We have
$$|\ker \mathcal{A}_1 \cup \ker \mathcal{A}_2 \cup \ker \mathcal{A}_4| \geq 4.$$
Since $|\ker \mathcal{A}_1 \cup \ker \mathcal{A}_2| = 3$, then $\mathcal{A}_4 \neq \mathcal{P}(X)$. It is clear that $\bigcup_{k=1}^{3} \mathcal{A}_k = \mathcal{P}(X)$. Indeed, it is easy to demonstrate that any subset of X belongs to at least one of the algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. Therefore $\bigcup_{k=1}^{n} \mathcal{A}_k = \mathcal{P}(X)$. If $J = \{1, 2, 3\}$, then $|J| = h = 3$ and $\left|\bigcup_{k\in J} \ker \mathcal{A}_k\right| = 2h - 3 = 3$. *Thus, it is impossible to substitute the estimate of $2h - 3$ for $2h - 2$ in Theorem 2.1.*

5. Infinite sequences of σ-algebras

5.1. In the first 11 sections we present the information from [Gr1]. Here we encounter with almost σ- algebras. This is connected to the Problem 2.5.

Definition 5.1. A point $a \in \beta X$ is said to be *irregular* if for any countable sequence of sets $M_1, \dots, M_k, \dots \subset \beta X$ such that $a \notin \overline{M}_k$ for all k, we have $a \notin \overline{\cup M_k}$.

Since a point of βX is an ultrafilter on X and, vice versa, an ultrafilter on X is a point of βX, we will use the notion of the irregular ultrafilter along with the notion of the irregular point. All points of X are irregular. Just a few words about irregular points in $\beta X \setminus X$. The superposition of the absence of irregular points in $\beta X \setminus X$ at an arbitrary cardinality of X is consistent, as is known, with ordinary axioms of the set theory. However, it is possible to construct models with irregular points in $\beta X \setminus X$ when the cardinality of X is "very large".

Definition 5.2. An algebra $\mathcal{A}$ is said to be *simple*, if there exists $Z \subset \beta X$ such that:

(1) $|Z| \leq \aleph_0$;
(2) if $Z \neq \emptyset$, all points of Z are irregular;
(3) $\ker \mathcal{A} \subset \overline{Z}$. [2]

5.3. The proof of the following theorem is seen in [Gr1], Chapter 5.

[2] By virtue of this definition the algebra $\mathcal{P}(X)$ is simple.

Theorem 5.3. *Let* $\{\mathcal{A}_k\}_{k\in\mathbb{N}^+}$ *be a family of almost* σ*-algebras, and let*

$$\bigcup\{\mathcal{A}_k \mid \mathcal{A}_k \textit{ is a simple algebra}\} \neq \mathcal{P}(X).$$

Then $\bigcup_{k=1}^{\infty} \mathcal{A}_k \neq \mathcal{P}(X)$.

Remark 5.4. The Gitik-Shelah theorem is essentially used in the proof of Theorem 5.3, and in the proof of a simpler variant of Theorem 5.3, where instead of almost σ-algebras we can put σ-algebras. This simpler variant of Theorem 5.3 is essentially used in the proof of Theorem 2.2, as well as Theorem 1.13. Theorem 5.3 is essentially used in the proof of generalizing Theorem 1.13 appearing in Section 2.5 (in this generalization we substitute in Theorem 1.13 σ-algebras with almost σ-algebras).

Definition 5.5. The set

$$\{a \in \ker \mathcal{A} \mid a \text{ is an irregular point}\}$$

is called the *spectrum* of an algebra $\mathcal{A}$ and is denoted $sp\mathcal{A}$.

It is clear that if $\mathcal{A}$ is a simpler algebra, then $|sp\mathcal{A}| \leq \aleph_0$.

5.6. The proof of the lemma below is seen in [Gr1], Chapter 7.
Lemma 5.6. *If* $\mathcal{A}$ *is a simple almost* σ*-algebra, then* $\ker \mathcal{A} \subset \overline{sp\mathcal{A}}$.

5.7. The proof of the lemma below is seen in [Gr1], Chapter 7.
Lemma 5.7. *If* $\mathcal{A}$ *is a simple* σ*-algebra and* $a \in sp\mathcal{A}$*, then*

$$\{b \in sp\mathcal{A} \mid a \textit{ is } \mathcal{A}\textit{- equivalent to } b\} \neq \emptyset \ .$$

Remark 5.8. Lemma 5.7 is essentially used in the proof of Theorem 2.2. However, Lemma 5.7, in general, is not valid for almost σ-algebras (see [Gr1], Chapter 7). Is it possible to prove Theorem 2.2 substituting σ-algebras with almost σ-algebras, using Theorem 5.3, but without the need of Lemma 5.7? This is the core of Problem 2.5.

5.9. The following definition was introduced by other authors long before the publication of [Gr1].
Definition 5.9. An algebra is said to be ω-saturated if it does not have $\aleph_0$ lacunae.

An algebra $\mathcal{A}$ is ω-saturated if and only if $|\ker \mathcal{A}| < \aleph_0$. An ω-saturated almost σ-algebra $\mathcal{A}$ is a σ-algebra, and $\ker \mathcal{A} = sp\mathcal{A}$. Obviously, any ω-saturated σ-algebra is simple.

Definition 5.10. A simple but not ω-saturated algebra is called *strictly simple.*

5.11. Let $\mathcal{A}$ be an algebra. Let $b_1, \dots , b_n$ be a finite sequence of pairwise distinct ultrafilters. Let $\mathfrak{Z}$ be a set of ultrafilters, $|\mathfrak{Z}| < n$, and for each $k \in [1, n]$ there exist $\mathcal{A}$-equivalent ultrafilters b_k, b'_k with $b'_k \in \mathfrak{Z}$. Then there exist ultrafilters b_{k_1}, b_{k_2}, b' such that $k_1 \neq k_2, b' \in \mathfrak{Z}$ and $(b_{k_1}, b'), (b_{k_2}, b')$ are two pairs of $\mathcal{A}$-equivalent ultrafilters. Since the relation of $\mathcal{A}$-equivalent is transitive, b_{k_1} and b_{k_2} are $\mathcal{A}$-equivalent ultrafilters. These arguments will be used below.

5.12. We start preparations for proving Theorem 2.2. In the present section we consider a situation similar to that of Section 4.1, but more complicated. Here we make use of the notion of a *bunch*, which was used in a similar, but simpler form in [Gr1], Chapter 12. We examine a finite sequence of algebras $\mathcal{L}_1, \dots , \mathcal{L}_n$. A set $\mathfrak{S}_n$ of ultrafilters σ_i^k is defined, which can be conveniently represented by a matrix consisting of n rows. The *k-th* row of $\mathfrak{S}_n$ comprises one or two ultrafilters (either σ_1^k, or σ_1^k, σ_2^k and $\sigma_1^k \neq \sigma_2^k$); $\sigma_i^k \in \ker \mathcal{L}_k$. Two ultrafilters from two different rows are not equal. We consider a closed set of ultrafilters S_0 and $\mathfrak{S}_n \cap S_0 = \emptyset$. Each ultrafilter from S_0 marked by 1. We will describe a special process, as a result of which, *under certain assumptions*, each ultrafilter σ_i^k will be marked either by 0 or by 1. Consequently, it will become clear that $\bigcup_{k=1}^n \mathcal{L}_k \neq \mathcal{P}(X)$. (Here the existence of the set S_0 will be taken into account in a certain way). For each ultrafilter σ_i^k we consider an ultrafilter τ_i^k, which is $\mathcal{L}_k$-equivalent to σ_i^k. *It is assumed that if ultrafilter σ_1^k, σ_2^k are considered, they are not $\mathcal{L}_k$-equivalent.* The $k-th$ row of $\mathfrak{S}_n$ is said to be *special of the first kind*, if it involves only one ultrafilter σ_1^k. The $k-th$ row of $\mathfrak{S}_n$ is said to be *special of the second kind*, if it involves two ultrafilters σ_1^k, σ_2^k, and the following is specified: one of them must be marked by 0, and another – 1. The rest of the rows of $\mathfrak{S}_n$ are said to be *ordinary.* We examine pairwise distinct indices $j_1, \dots , j_\ell \in [1, n]$ under the assumption that the rows $j_1, \dots , j_\ell$ of the matrix $\mathfrak{S}_n$ do not involve marked ultrafilters. At first, this assumption does not make sense, since we have not started marking ultrafilters in $\mathfrak{S}_n$ as yet. We say that these indices form the bunch, if the following conditions hold:

1) for each $k \in [1, \ell]$ an ultrafilter $\sigma^{j_k}_{i_k}$ is considered;

2) $i_1 = 1$;

3) $\tau^{j_k}_{i_k} = \sigma^{j_{k+1}}_{i'_{k+1}}$ if $k < \ell$;

4) $i_{k+1} \neq i'_{k+1}$ if $k < \ell$ and the row j_{k+1} is ordinary;

5) $i_{k+1} = i'_{k+1}$ if $k < \ell$ and the row j_{k+1} is special (of the first or second kind);

6) the property of maximality: an index $j_* \in [1, n]$ such that indices $j_1, \dots, j_\ell, j_*$ form the bunch does not exist.

All the ultrafilters from the rows $j_1, \dots, j_\ell$ are called *ultrafilters from the bunch*, and each of them is marked either by 0, or by 1. First we mark them *preliminary*, and afterwards, *if necessary*, they can be marked *finally*. They are marked preliminary as follows:

a) $\sigma^{j_1}_1$ is marked by 1;

b) if there exists the ultrafilter $\sigma^{j_1}_2$, it is marked by 0;

c) if $\sigma^{j_k}_{i_k}$, where $k < \ell$, is marked by 1, then $\sigma^{j_{k+1}}_{i'_{k+1}}$ is marked by 0, and vice versa;

d) if the row j_k is ordinary, then $\sigma^{j_k}_{i_k}$ is marked by 1;

e) if the row j_k is special of the second kind, then $\sigma^{j_k}_1$ and $\sigma^{j_k}_2$ are marked by different figures.

Thus, all ultrafilters from the bunch are preliminary marked. The bunch is said to be *cycled*, if $\tau^{j_\ell}_{i_\ell}$ is an ultrafilter from the bunch; then $\tau^{j_\ell}_{i_\ell} = \sigma^{j_r}_e$ and $r < \ell$. Otherwise, the bunch is said to be *uncycled*; then either (1) $\tau^{j_\ell}_{i_\ell} \notin \mathfrak{S}_n$, or (2) $\tau^{j_\ell}_{i_\ell} = \sigma^u_w \in \mathfrak{S}_n$, the ultrafilter σ^u_w is marked and, it goes without saying, σ^u_w is not an ultrafilter from the bunch. At first, the assumption (2) does not make sense, since we marked the ultrafilters from our bunch only. If the bunch is cycled, four cases are possible.

- Case I. *$\sigma^{j_\ell}_{i_\ell}$ and $\sigma^{j_r}_e$ are marked with different figures.*

Before examining Case II, note that the ultrafilters $\sigma^{j_k}_{i'_k}$ is determined by $k > 1$. If the row j_1 is ordinary, we put $\sigma^{j_1}_{i'_1} = \sigma^{j_1}_2$.

- Case II. *Case I does not hold, the row j_r is ordinary, and $e = i'_r$.* We mark $\sigma^{j_r}_{i'_r}$ with a different figure. If $r > 1$, then we mark each ultrafilter $\sigma^{j_k}_p$, where $k < r$, with a different figure.

- Case III. *Cases I, II do not hold, and among the rows* $j_{r+1}, \dots, j_\ell$ there is an ordinary one. Let a certain $q \in [r+1, \ell]$ and the row j_q be ordinary. We mark $\sigma_{i_q}^{j_q}$ by 0 (previously, $\sigma_{i_q}^{j_q}$ was marked by 1). We mark each ultrafilter $\sigma_p^{j_k}$, where $k > q$, with a different figure.

Thus, in the final version in Cases I, II, III the ultrafilters $\sigma_{i_\ell}^{j_\ell}$ and $\sigma_e^{j_r}$ are marked with different figures. In Cases I, II, III our cycled bunch is called *non-degenerate*.

- Case IV. *Cases I, II, III do not hold.* We call our cycled bunch *degenerate*.

Let us make an *essential* assumption: *degenerate cycled bunches do not exist.* Now let us examine an uncycled bunch. In two cases we should mark ultrafilters from the bunch in a new way.

- Case A. *Ultrafilters* $\sigma_{i_\ell}^{j_\ell}$ *and* $\tau_{i_\ell}^{j_\ell}$ *are marked with the same figures.* Each ultrafilter from the bunch is marked with a different figure. [3]
- Case B. *The ultrafilter* $\sigma_{i_\ell}^{j_\ell}$ *is marked by* 0, *and* $\tau_{i_\ell}^{j_\ell} \notin \mathfrak{S}_n \cup S_0$. Each ultrafilter from the bunch is marked with a different figure.

Thus, if the bunch is uncycled, in the final version either the ultrafilters $\sigma_{i_\ell}^{j_\ell}, \tau_{i_\ell}^{j_\ell}$ are marked with different figures or $\sigma_{i_\ell}^{i_\ell}$ is marked by 1, and $\tau_{i_\ell}^{j_\ell} \notin \mathfrak{S}_n \cup S_0$.

We have to explain so far what a bunch is and how ultrafilters from a bunch are marked. Now let us pass to the construction of bunches. Let us construct arbitrarily the first bunch and mark ultrafilters from the bunch accordingly. If non-marked ultrafilters are left in $\mathfrak{S}_n$, let us construct arbitrarily the second bunch and mark ultrafilters from the bunch accordingly, etc. Finally, we will mark all the ultrafilters from $\mathfrak{S}_n$. Let S include all ultrafilters marked by 1. Clearly, $S_0 \subset S$ and $\overline{S} = S$. By virtue of our constructions, for each $k \in [1, n]$ it is possible to find a pair $\mathcal{L}_k$ - equivalent ultrafilters s_k, t_k such that $s_k \in S, t_k \notin S$. We denote the set of all ultrafilters t_k by T. It is finite and therefore $\overline{T} = T$. Since $S \cap T = \emptyset$, owing to the Main Statement from Section 1.6, $\bigcup_{k=1}^{n} \mathcal{L}_k \neq \mathcal{P}(X)$.

[3] In this case it is possible that $\tau_{i_\ell}^{j_\ell} \in S_0$.

5.13. *The Proof of Theorem 2.2.* By virtue of Theorem 5.3 (here we can use the simpler version of Theorem 5.3, where instead of almost σ-algebras we put σ-algebras), we assume that all algebras $\mathcal{A}_k$ are simple. Let us divide the sequence of algebras $\mathcal{A}_1, \dots, \mathcal{A}_k, \dots$ into two subsequences. The first subsequence $\mathcal{B}_1, \dots, \mathcal{B}_k, \dots$ includes all ω- saturated algebras. The second subsequence $\mathcal{C}_1, \dots, \mathcal{C}_k, \dots$ includes all strictly simple algebras. We assume, without any loss of generality, that each of these two subsequences includes $\aleph_0$ algebras. Thus,

$$0 < |\ker \mathcal{B}_k| < \aleph_0$$

for each k, all ultrafilters from $\ker \mathcal{B}_k$ are irregular, and for each finite set $J \subset \mathbb{N}^+$, where $|J| = h \geq 2$,

$$\left|\bigcup_{k\in J} \ker \mathcal{B}_k\right| \geq 2h - 2.$$

By virtue of Theorem 3.3, we can construct a set of ultrafilters $\mathfrak{S}$, which can be represented by a matrix with $\aleph_0$ rows: the first row, the second row, ... , the $k-th$ row etc. The $k-th$ row includes either one ultrafilter s_1^k or two various ultrafilters s_1^k, s_2^k. Here $s_i^k \in \ker \mathcal{B}_k$, two ultrafilters from two different rows are not equal, and

$$\left|\{k \in \mathbb{N}^+ \mid \text{ the } k-th \text{ row includes one ultrafilter } s_1^k \text{ alone}\}\right| \leq 2.$$

We assume that ultrafilters s_2^1, s_2^2 do not exist, i.e., in the first row of $\mathfrak{S}$ there is only one ultrafilter s_1^1, in the second row of $\mathfrak{S}$ there is only one ultrafilter s_1^2, and in the $k-th$ row of $\mathfrak{S}$, where $k \geq 3$, there are two ultrafilters s_1^k, s_2^k.

As we know, $|sp\ \mathcal{C}_k| = \aleph_0$. Let

$$|sp\ \mathcal{C}_k \setminus \mathfrak{S}| = \aleph_0.$$

In this case we say that $\mathcal{C}_k$ belongs to a subsequence (D). Otherwise, we say that $\mathcal{C}_k$ belongs to a subsequence (E). Let $\mathcal{D}_1, \dots, \mathcal{D}_k, \dots$ be all algebras of the subsequence (D), and $\mathcal{E}_1, \dots, \mathcal{E}_k, \dots$ – all algebras of the subsequence (E). We assume, without any loss of generality, that the subsequence (E) comprises $\aleph_0$ algebras. We write all the algebras $\mathcal{B}_1, \dots, \mathcal{B}_k, \dots$ and all the algebras of the subsequence (D) in the form of a sequence of algebras $\mathcal{A}'_1, \dots \mathcal{A}'_k, \dots$. This sequence comprises $\aleph_0$ algebras because, according to the assumption, there are $\aleph_0$ algebras $\mathcal{B}_k$. We assume that $\mathcal{A}'_1 = \mathcal{B}_1, \mathcal{A}'_2 = \mathcal{B}_2$. If $\mathcal{A}'_k = \mathcal{B}_p$, the ultrafilter $s_i^p \in \mathfrak{S}$ is denoted by $\tilde{s}_i^k$. If $\mathcal{A}'_k = \mathcal{D}_p$, distinct ultrafilters $\tilde{s}_1^k, \tilde{s}_2^k \in sp\ \mathcal{D}_p$ are considered. Ultrafilters of the sort $\tilde{s}_i^k$ form the matrix $\widetilde{\mathfrak{S}}$ which has $\aleph_0$ rows. There is only one ultrafilter $\tilde{s}_1^1$ in the first row of $\widetilde{\mathfrak{S}}$, and only one ultrafilter $\tilde{s}_1^2$ in the second row of $\widetilde{\mathfrak{S}}$; in the $k-th$ row of $\widetilde{\mathfrak{S}}$, where $k \geq 3$, there are two various ultrafilters $\tilde{s}_1^k, \tilde{s}_2^k$. We can obtain that any two ultrafilters from different rows of $\widetilde{\mathfrak{S}}$ are not equal. For each ultrafilter $\tilde{s}_i^k$ we consider an ultrafilter $\tilde{t}_i^k$, which is $\mathcal{A}'_k$- equivalent to $\tilde{s}_i^k$. All ultrafilters $\tilde{s}_i^k$ are irregular. By virtue of Lemma 5.7 we can assume that all ultrafilters $\tilde{t}_i^k$ are irregular. Two ultrafilters $\tilde{s}_1^k$ and $\tilde{s}_2^k$, where $k \geq 3$, are said to be *neighboring*.

Let us pass to the examination of algebras from the subsequence (E). By virtue of Lemma 5.7 and considerations of Section 5.11, we can assert that for each pair $m, n \in \mathbb{N}^+$ at least one of the following holds:

(1) there exist $\mathcal{E}_m$ - equivalent ultrafilters $\tilde{s}_1^p, \tilde{s}_2^p$ and $p > n$;

(2) there exist $\mathcal{E}_m$ - equivalent ultrafilters $\tilde{s}_i^p, \tilde{s}_{i'}^{p'}$ and $p > p' > n$.

Let $q \in \mathbb{N}^+$. We put:

$$R_q = \{r \mid \text{ there exists } \tilde{t}_i^k = \tilde{s}_j^r \text{ and } k \leq q \ \},$$

$$q_0 = \begin{cases} 0 & \text{if} \quad R_q = \emptyset, \\ \max R_q & \text{if} \quad R_q \neq \emptyset, \end{cases}$$

$$q' = \max(q, q_0), \ (q')' = q^* \ .$$

Let $n_1 = 2^*$. (I) If it is possible, we take $\mathcal{E}_1$ - equivalent ultrafilters $\tilde{s}_1^{p_1}, \tilde{s}_2^{p_1}$, where $p_1 > n_1$; now the row p_1 of $\widetilde{\mathfrak{S}}$ will be *special of the second kind*. (II) If (I) does not hold, we take $\mathcal{E}_1$- equivalent ultrafilters $\tilde{s}_{i_1}^{p_1}, \tilde{s}_{i'_1}^{p'_1}$, where $p_1 > p'_1 > n_1$; we mark $\tilde{s}_{i_1}^{p_1}$ by 1 and $\tilde{s}_{i'_1}^{p'_1}$ – by 0, and say that the ultrafilters neighbouring with the ultrafilters $\tilde{s}_{i_1}^{p_1}, \tilde{s}_{i'_1}^{p'_1}$ form *a pair of special ultrafilters of the first kind*. In the two cases we put $n_2 = p_1^*$. Let us proceed with our constructions. (I) If, it is possible, we take $\mathcal{E}_2$ - equivalent ultrafilters $\tilde{s}_1^{p_2}, \tilde{s}_2^{p_2}$, where $p_2 > n_2$; now the row p_2 of $\widetilde{\mathfrak{S}}$ will be *special of the second kind*. (II) If (I) does not hold, we take $\mathcal{E}_2$ - equivalent ultrafilters $\tilde{s}_{i_2}^{p_2}, \tilde{s}_{i'_2}^{p'_2}$, where $p_2 > p'_2 > n_2$; we mark $\tilde{s}_{i_2}^{p_2}$ by 1 and $\tilde{s}_{i'_2}^{p'_2}$ by 0, and say that the ultrafilters neighboring the ultrafilters $\tilde{s}_{i_2}^{p_2}, \tilde{s}_{i'_2}^{p'_2}$ form *a pair of special ultrafilters of the first kind*. In the two cases we put $n_3 = p_2^*$, etc. Denote by $\mathfrak{S}'$ the matrix of ultrafilters obtained from $\widetilde{\mathfrak{S}}$ by crossing out all marked ultrafilters. We assume that if there is only one ultrafilter in the $k-th$ row of $\mathfrak{S}'$, it is the ultrafilter $\tilde{s}_1^k$.

The row of $\mathfrak{S}'$ comprising only one ultrafilter is said to be *special of the first kind*. The ultrafilters $\tilde{s}_2^1, \tilde{s}_1^2$ form, by definition, *a pair of special ultrafilters of the first kind*. A row of $\mathfrak{S}'$ that is not special of either the first or the second kind is said to be *ordinary*. Further we consider ultrafilters $\tilde{s}_i^k$ from $\mathfrak{S}'$ only. If $\tilde{s}_1^k, \tilde{s}_2^k$ are $\mathcal{A}'_k$ - equivalent ultrafilters, we mark $\tilde{s}_1^k$ by 1 and $\tilde{s}_2^k$ – by 0. Assume that there exist ultrafilters $\tilde{s}_v^{k_1}, \tilde{s}_1^{k_2}, \tilde{s}_1^{k_3}$ such that the $k_1 - th$ row is ordinary, the $k_2 - th$ and $k_3 - th$ rows are special of the first kind, and

$$\tilde{t}_v^{k_1} = \tilde{s}_1^{k_2}, \tilde{t}_1^{k_2} = \tilde{s}_1^{k_3}, \tilde{t}_1^{k_3} = \tilde{s}_v^{k_3}.$$

It is clear that the ultrafilter neighboring to the ultrafilter $\tilde{s}_v^{k_1}$ belongs to $\mathfrak{S}'$. We call the set of ultrafilters $\{\tilde{s}_v^{k_1}, \tilde{s}_1^{k_2}, \tilde{s}_1^{k_3}\}$ *the ternary cycle*. By virtue of our constructions in the ternary cycle $\tilde{s}_1^{k_2}, \tilde{s}_1^{k_3}$ necessarily form the pair of special ultrafilters of the first kind. Let $\{\tilde{s}_{v'}^{k'_1}, \tilde{s}_1^{k'_2}, \tilde{s}_1^{k'_3}\}$ be another ternary cycle. By virtue of our constructions

$$\{k_1, k_2, k_3\} \bigcap \{k'_1, k'_2, k'_3\} = \emptyset \ .$$

We mark $\tilde{s}_v^{k_1}, \tilde{s}_1^{k_2}$ by 0 and $\tilde{s}_1^{k_3}$ by 1. Let us find all ternary cycles and mark their ultrafilters accordingly. We denote by S_* the set of all ultrafilters until now by 1, and put $S_0 = \overline{S}_*$. Denote by $\mathfrak{S}^*$ the matrix of ultrafilters obtained from $\mathfrak{S}'$ by crossing out all marked ultrafilters. We assume, without any loss of generality, that $\mathfrak{S}^*$ contains $\aleph_0$ rows. Let us write out the numbers of all rows of $\mathfrak{S}^*$:

$$\gamma_1 < \gamma_2 < \dots < \gamma_k < \dots \quad .$$

The $k - th$ row of $\mathfrak{S}^*$ is the $\gamma_k - th$ row of $\mathfrak{S}'$. We assume that if the $k - th$ row of $\mathfrak{S}^*$ contains one ultrafilter only, it must be the ultrafilter $\tilde{s}_1^{\gamma_1}$. A row $\mathfrak{S}^*$ containing only one ultrafilter is said to be *special of the first kind*. Clearly, $\mathfrak{S}^*$ can contain special rows of the second kind and ordinary rows. We put $\mathcal{L}_k = \mathcal{A}'_{\gamma_k}$, and denote the ultrafilters $\tilde{s}_i^{\gamma_k}$ and $\tilde{t}_i^{\gamma_k}$ by σ_i^k and τ_i^k respectively. Thus, $\mathfrak{S}^*$ has become the matrix of ultrafilters σ_i^k. Let us fix $n \in \mathbb{N}^+$. The first n rows of $\mathfrak{S}^*$ form the matrix $\mathfrak{S}_n$. The situation here is similar to that of Section 5.12. If we consider here ultrafilters σ_1^k and σ_2^k, they are not $\mathcal{L}_k$ - equivalent. Here all the ultrafilters σ_i^k, τ_i^k and all the ultrafilters from S_* are irregular, and therefore, if $\tau_i^k \in S_0$, then $\tau_i^k \in S_*$. Clearly, $\mathfrak{S}_n \cap S_0 = \emptyset$. And finally, the most important point is that: when considering here $\mathfrak{S}_n$ *we do not find any degenerate bunches*! Let us explain the reason for this.

Let us examine a cycled bunch $j_1, \dots, j_\ell \in [1,n]$; for each $k \in [1,n]$ we examine an ultrafilter $\sigma_{i_k}^{j_k}$, etc. ; $\tau_{i_\ell}^{j_\ell} = \sigma_e^{j_r}$ (see Section 5.12). We assume that the rows $j_{r+1}, \dots, j_\ell$ are special (of the first or second kind), because if at least one of them is ordinary, then our bunch is non-degenerate. If $\xi \in [r, \ell-1]$, the row j_ξ is special (of the first or second kind), and $\sigma_1^{j_\xi}, \sigma_1^{j_{\xi+1}}$ do not form a pair of special ultrafilters of the first kind, then we arrive at a contradiction: By virtue of our constructions $j_\xi > j_{\xi+1}$ and among the rows $j_{r+1}, \dots, j_\ell$ there is an ordinary one. Therefore the following four cases are possible.

- Case (a). *The row j_r and j_{r+1} are special of the first kind, and $\ell = r+1$.* Clearly, $\sigma_1^{j_r}$ and $\sigma_1^{j_\ell}$ constitute a pair of special ultrafilters of the first kind. This is Case I from Section 5.12.
- Case (b). *The row j_r is ordinary, the row j_{r+1} is special of the first kind, and $\ell = r+1$.* This is Case I or Case II from Section 5.12.
- Case (c). *The row j_r is ordinary, the rows j_{r+1}, j_{r+2} are special of the first kind, and $\ell = r+2$.* Clearly, $\sigma_1^{j_{r+1}}$ and $\sigma_1^{j_\ell}$ constitute a pair of special ultrafilters of the first kind. Since we have eliminated ternary cycles, this is Case I from Section 5.12.
- Case (d). *The row j_r is ordinary, the row j_{r+1} is special of the second kind, and $\ell = r+1$.* This is Case I or Case II from Section 5.12.

Thus, having considered $\mathfrak{S}_n$, we do not find any degenerate bunches. As in Section 5.12 we mark all $\sigma_i^k \in \mathfrak{S}_n$ by 0 and 1. We denote by S_n the set of all ultrafilters marked by 1 (clearly, $S_* \subset S_n$). Let

$$\mathcal{A}_k \notin \{\mathcal{L}_{n+1}, \mathcal{L}_{n+2}, \dots, \mathcal{L}_{n+m}, \dots\} .$$

There exist $\mathcal{A}_k$-equivalent ultrafilters $s_k^{(n)}$ and $t_k^{(n)}$ such that $s_k^{(n)} \in S_n$, $t_k^{(n)} \notin S_n$. It is noteworthy that $s_k^{(n)}$ and $t_k^{(n)}$ are irregular ultrafilters. It is clear that if $\mathcal{A}_k \notin \{\mathcal{L}_1, \dots, \mathcal{L}_m, \dots\}$, then $s_k^{(n)} \in S_*, t_k^{(n)} \notin \mathfrak{S}^* \cup S_0$, and $s_k^{(n)}, t_k^{(n)}$ are independent of n. Therefore in this case we denote $s_k^{(n)}$ by s_k, and $t_k^{(n)}$ by t_k. Let $\mathcal{A}_k = \mathcal{L}_p$ and $p \le n$. If there is only one ultrafilter σ_1^p in the $p-th$ row of $\mathfrak{S}^*$, then

$$\{s_k^{(n)}, t_k^{(n)}\} = \{\sigma_1^p, \tau_1^p\}\ .$$

If there are two ultrafilters σ_1^p and σ_2^p in the $p-th$ row of $\mathfrak{S}^*$, then

$$\{s_k^{(n)}, t_k^{(n)}\} \subset \{\sigma_1^p, \sigma_2^p, \tau_1^p, \tau_2^p\}\ .$$

Now it is clear that for each algebra $\mathcal{A}_k = \mathcal{L}_p$, where $p \leq n$, the respective ultrafilters $s_k^{(n)}$ and $t_k^{(n)}$ independent of n can be found. Therefore, in this case we denote $s_k^{(n)}$ by s_k, and $t_k^{(n)}$ by t_k. Thus, for each $k \in \mathbb{N}^+$ we have found $\mathcal{A}_k$-equivalent ultrafilters s_k, t_k. Let S be the set of all ultrafilters s_k. Let T be the set of all ultrafilters t_k. Since all ultrafilters from S marked by 1, and all ultrafilters from T are not marked by 1, then $S \cap T = \emptyset$. Since all ultrafilters from $S \cup T$ are irregular, and $|S \cup T| \leq \aleph_0$, then $\overline{S} \cap \overline{T} = \emptyset$. Owing to the Main Statement from Section 1.6, $\bigcup_{k=1}^{\infty} \mathcal{A}_k \neq \mathcal{P}(X)$. □

Acknowledgements. I would like to thank Professors N. Alon and J. Bernstein who were a great help to me in preparing the article. I would also like to thank the referee for his suggestions.

References

[E] P. Erdös. *Some remarks on set theory*, Proc. Amer. Math. Soc. **1**(1950), 127-141.

[F] D.H. Fremlin. *Real-valued-measurable cardinals*, Set theory of the reals (Ramat Gan, 1991). Israel Mathematical Conference Proceedings, **6**(1993), 151-304.

[G-S] M. Gitik and S. Shelah. *Forcing with ideals and simple forcing notions*, Israel J. Math. **68**(1989), 129-160.

[Gr1] L.Š. Grinblat. *Algebras of sets and combinatorics*, Translations of Math. Monographs, **214**, AMS, Providence, RI, 2002.

[Gr2] L.Š. Grinblat. *Theorems on sets not belonging to algebras*, ERA of AMS, **10**(2004), 51-57.

[Gr3] L.Š. Grinblat. *On sets not belonging to algebras*, J. of Sym. Logik, **72**(2007), 483-500.

[H] Ph. Hall. *On representatives of subsets*, J. London Math. Soc. **10**(1935), 26-30.

[K] A. Kamburelis. *A new proof of the Gitik-Shelah theorem*, Israel J. Math. **72**(1990), 373-380.

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