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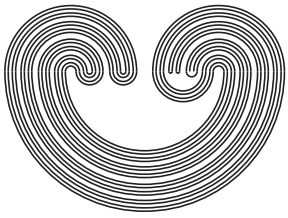

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BOOLEAN ALGEBRAS AND LOW SEPARATION AXIOMS

M. L. COLASANTE, C. UZCÁTEGUI, AND J. VIELMA

Abstract. Let $\mathbb{B}(\tau)$ be the smallest complete Boolean algebra containing the topology τ. We present results of the following type: τ satisfies certain separation axioms if and only if $\mathbb{B}(\tau)$ is equal to (naturally defined) subcollections of $\mathbb{B}(\tau)$. Examples of such collections are the kerneled sets, the λ-closed sets, and the $\overline{\tau}$-locally closed sets (where $\overline{\tau}$ is the smallest Alexandroff topology containing τ).

1. Introduction

Let τ be a topology over a set X and let τ^* be the τ-closed sets. By $\mathbb{B}(\tau)$ we denote the smallest complete Boolean algebra containing τ. In this paper we show that several low separation axioms (i.e., at most $\mathbf{T}_2$) are characterized by properties of $\mathbb{B}(\tau)$. In order to state our results we need to introduce some natural subclasses of $\mathbb{B}(\tau)$.

A topology is said to be an *Alexandroff topology* if it is closed under arbitrary intersections. Juris Steprāns and Stephen Watson [13] attributed this notion to both Alexandroff and Tucker, and thus called them AT topologies. We will use their notation here. This class of topologies plays an important role in the study of low separation axioms. Notice that only the Alexandroff $\mathbf{T}_1$ topology is the discrete topology.

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We denote by $\overline{\tau}$ the smallest Alexandroff topology containing τ. Let $CO(\tau)$ be the collection of τ-clopen subsets of X. By λ_τ^* we denote the collection of sets of the form $K \cap C$ where $K \in \overline{\tau}$ and C is τ-closed (i.e., the collection of λ-*closed* sets [1]), and by $CX(\tau)$ the collection of $\overline{\tau}$-locally closed sets (i.e., sets of the form $K \cap S$ where $K \in \overline{\tau}$ and $S \in \overline{\tau}^*$).

Let $\tau(x)$ be the collection of open sets containing $x \in X$. The θ-*closure* of a subset A of X is defined in [2] as $cl_\theta(A) = \{x \in X : cl(V) \cap A \neq \emptyset \text{ for all } V \in \tau(x)\}$, where $cl(V)$ is the τ-closure operator on X. Observe that $cl_\theta(A)$ is the intersection of the closure of all open sets containing A; hence, it is a closed set containing $Ker(A)$. In particular, for any $x, y \in X$, $x \in cl_\theta(y)$ iff $y \in cl_\theta(x)$ iff $V \cap W \neq \emptyset\ \forall V \in \tau(x)$ and $\forall W \in \tau(y)$. A set A is said to be θ-*closed* if $cl_\theta(A) = A$, equivalently if for all $x \notin A$, there are disjoint open sets V and W such that $x \in V$ and $A \subseteq W$. The set $cl_\theta(A)$ is not in general θ-closed. Complements of θ-closed sets are called θ-open. Thus, A is θ-open iff for each $x \in A$ there is $V \in \tau(x)$ such that $cl(V) \subseteq A$. The family of all θ-open sets forms a topology τ_θ on X which is clearly weaker than τ. Denoting by $cl_{\tau_\theta}(A)$ the closure of A in the τ_θ topology, it is straightforward to prove that $cl(A) \subseteq cl_\theta(A) \subseteq cl_{\tau_\theta}(A)$, for any $A \subseteq X$. Since any θ-open set can be written as a union of τ-closed sets, then $\tau_\theta \subseteq CO(\overline{\tau})$. Moreover, τ and τ_θ have the same clopen sets. The following inclusion holds for any topology τ.

$$\begin{array}{ccccccccc} CO(\tau) & \subseteq & \tau_\theta & \subseteq & \mathbb{B}(\tau_\theta) & \subseteq & CO(\overline{\tau}) & & \\ CO(\overline{\tau}) & \subseteq & \overline{\tau} & \subseteq & \lambda_\tau^* & \subseteq & CX(\tau) & \subseteq & \mathbb{B}(\tau). \end{array}$$

Let $\mathcal{A}(\tau)$ denote any of these subclasses of $\mathbb{B}(\tau)$. Several low separation axioms are characterized by the fact that $\mathcal{A}(\tau) = \mathcal{P}(X)$ (the power set of X). For instance, τ is $\mathbf{T}_{1/2}$ iff every set is λ-closed [1], and τ is $\mathbf{T}_{1/4}$ iff every set is $\overline{\tau}$-locally closed [4]. In this paper we put these facts in a general framework. Instead of requiring that $\mathcal{A}(\tau) = \mathcal{P}(X)$, we ask only that $\mathcal{A}(\tau)$ is the complete Boolean algebra $\mathbb{B}(\tau)$ and show that this requirement corresponds to a separation axiom. We use axioms $\mathbf{S}_i$ such that τ is $\mathbf{T}_i$ iff it is $\mathbf{S}_i$ and $\mathbf{T}_0$, for $i = 1/4$, $1/2$, 1, or 2. An example of our results is that τ is $\mathbf{S}_{1/4}$ iff $CX(\tau) = \mathbb{B}(\tau)$.

The original motivation for this paper was [10], where results of this type were shown for countable spaces (in this case $\mathbb{B}(\tau)$ is the σ-algebra of Borel sets). Independently, Boolean algebras were used in [4] to analyze some separation axioms.

2. Terminology and preliminaries

If $\mathcal{A}$ is a collection of subsets of X, we will denote by $\mathcal{A}^*$ the collection $\{X \setminus A : A \in \mathcal{A}\}$. The kernel of a set $A \subseteq X$, denoted by $\ker(A)$, is the intersection of all open sets containing A. It is easy to check that $A \subseteq \ker(A)$ and that $\ker(\ker(A)) = \ker(A)$ for any $A \subseteq X$. Moreover, if $A \subseteq B$ then $\ker(A) \subseteq \ker(B)$. For any $x \in X$, we denote $ker(\{x\}) = ker(x)$. It is obvious that $x \in cl(y)$ iff $y \in \ker(x)$. A set A is said to be *τ-kerneled* (or just kerneled) if $A = \ker(A)$. Equivalently, A is kerneled iff $A = \bigcup_{x \in A} \ker(x)$. Kerneled sets are also called Λ-sets [11]. The family of all kerneled subsets of X is closed under arbitrary unions and intersections, so it is an AT topology. Moreover, it coincides with $\overline{\tau}$. In fact, since every open set is kerneled and $\overline{\tau}$ is the smallest AT topology containing τ, then every member of $\overline{\tau}$ is kerneled. On the other hand, since $\overline{\tau}$ is closed under arbitrary intersections and it contains τ, then every kerneled set belongs to $\overline{\tau}$.

A set $A \subseteq X$ is said to be *τ-saturated* (or just saturated) if it contains the closure of all its points. It is clear that every closed set is saturated, and the concepts of saturated and closed coincide on finite sets. Notice that a set is saturated iff its complement is kerneled. Thus, the family $\overline{\tau}^*$ of the $\overline{\tau}$-closed sets is precisely the family of the saturated sets.

If A is the intersection of a kerneled set and a saturated set, then A is said to be *λ-closed* [1]. Equivalently, A is λ-closed iff $A = ker(A) \cap cl(A)$. The set $ker(A) \cap cl(A)$ is denoted in [1] by $cl_\lambda(A)$. Thus, A is λ-closed iff $cl_\lambda(A) = A$. The operator cl_λ is monotone and idempotent. The complement of a λ-closed set is called λ-open. The collection of all λ-open sets is denoted by λ_τ and it is clear that a set is λ-open iff it is the union of an open set and a saturated set. The family λ_τ^* is closed under arbitrary unions, but it is not in general closed under finite intersection (see, for example, Example 4.3).

Each topology τ is associated with the following binary relation.

$$x \preceq_\tau y \iff x \in cl(y)$$

The relation $\preceq_\tau$ is transitive and reflexive (but in general, it is not antisymmetric) and is called the specialization preorder of τ. An AT topology τ is uniquely determined by its associated preorder $\preceq_\tau$ in the sense that a set V is τ-open iff V contains all y such that $x \preceq_\tau y$ for all $x \in V$ (for a general presentation of AT topologies see [8]). Since $cl_\tau(x) = cl_{\overline{\tau}}(x)$ for any $x \in X$, then $\preceq_\tau = \preceq_\rho$ iff $\overline{\tau} = \overline{\rho}$ for any pair of topologies τ and ρ on X.

A set $A \subseteq X$ is said to be τ-*convex* (or just convex) if the following holds: For all $x, y, z \in X$, if $x, z \in A$ and $x \preceq_\tau y \preceq_\tau z$, then $y \in A$, that is to say, A is convex with respect to the preorder $\preceq_\tau$ of τ. The collection of convex sets coincides with $CX(\tau)$. In fact, let $K \in \overline{\tau}$ and $S \in \overline{\tau}^*$, and let $x, z \in K \cap S$. If $x \preceq_\tau y \preceq_\tau z$, then $y \in \ker(x) \subseteq K$ and $y \in cl(z) \subseteq S$. Thus, $y \in K \cap S$, which shows that $K \cap S$ is convex. On the other hand, let A be a convex set. Take $K = \bigcup_{x \in A} \ker(x)$ and $S = \bigcup_{x \in A} cl(x)$. It is clear that $A \subseteq K \cap S$. If $y \in K \cap S$, then $y \in \ker(x) \cap cl(z)$ for some $x, z \in A$ and thus, $x \preceq_\tau y \preceq_\tau z$. Therefore, $y \in A$.

3. Main results

In this section we prove the main result of this paper which is a characterization of some low separation axioms in terms of the Boolean algebra $\mathbb{B}(\tau)$. First, we introduce an equivalence relation over (X, τ):

$$\begin{aligned} (x, y) \in E_\tau &\Leftrightarrow cl(x) = cl(y) \\ &\Leftrightarrow ker(x) = ker(y). \end{aligned}$$

We denote the E_τ-equivalence classes by $[x]_\tau$ for $x \in X$. We will show that they are the atoms of $\mathbb{B}(\tau)$. Note that

$$[x]_\tau = \{y \in X : x \preceq_\tau y \preceq_\tau x\}.$$

Now we introduce the separation axioms we are going to use in this article.

Definition 3.1. Let τ be a topology on X. Then τ is said to be

- ($\mathbf{T}_0$) If for all $x, y \in X$ with $x \neq y$, there is an open set containing x or y but not both.
- ($\mathbf{T}_{1/4}$) If every point of X is closed or kerneled.

($\mathbf{T}_{1/2}$) If every point of X is open or closed.
($\mathbf{T}_1$) If every point of X is closed.
($\mathbf{T}_2$) If for all $x, y \in X$ with $x \neq y$ there are disjoint open sets $U \in \tau(x)$ and $V \in \tau(y)$.
($\mathbf{S}_{1/4}$) If $[x]_\tau$ is closed or kerneled for every $x \in X$.
($\mathbf{S}_{1/2}$) If $[x]_\tau$ is closed or open for every $x \in X$.
($\mathbf{S}_1$) If for all $x, y \in X$, $x \in cl(y)$ iff $y \in cl(x)$, i.e. the closure of points forms a partition of X.
($\mathbf{S}_2$) If for all $x, y \in X$ with $cl(x) \neq cl(y)$, there are disjoint open sets $U \in \tau(x)$ and $V \in \tau(y)$ such that $cl(x) \subseteq U$ and $cl(y) \subseteq V$.

We prove in Theorem 3.11 that the separation axioms $\mathbf{S}_i$, for $i = 2, 1, 1/2$, and $1/4$, can be characterized in terms of $\mathbb{B}(\tau)$:

τ is $\mathbf{S}_2$	$\Leftrightarrow$	$\mathbb{B}(\tau) = \mathbb{B}(\tau_\theta)$
τ is $\mathbf{S}_1$	$\Leftrightarrow$	$\mathbb{B}(\tau) = \overline{\tau}$
τ is $\mathbf{S}_{1/2}$	$\Leftrightarrow$	$\mathbb{B}(\tau) = \lambda^*_\tau$
τ is $\mathbf{S}_{1/4}$	$\Leftrightarrow$	$\mathbb{B}(\tau) = CX(\tau)$

The most common notations for $\mathbf{S}_1$ and $\mathbf{S}_2$ are $\mathbf{R}_0$ and $\mathbf{R}_1$, respectively. However, the notation we use is more uniform (it is taken from [5]). The separation axioms $\mathbf{R}_0$ and $\mathbf{R}_1$ were introduced by N. A. Shanin [12] and A. S. Davis [7], respectively. To the best of our knowledge, the axioms $\mathbf{S}_{1/2}$ and $\mathbf{S}_{1/4}$ are introduced in the literature for the first time in this paper.

It is easy to verify that $\mathbf{S}_2 \Rightarrow \mathbf{S}_1$ and that $\mathbf{S}_1$ is equivalent to asking that $[x]_\tau$ is closed for every $x \in X$. Thus, $\mathbf{S}_2 \Rightarrow \mathbf{S}_1 \Rightarrow \mathbf{S}_{1/2} \Rightarrow \mathbf{S}_{1/4}$. The following example shows that the separation axioms $\mathbf{S}_{1/2}$ and $\mathbf{S}_{1/4}$ are not in general equivalent. (They turn out to be equivalent in AT spaces, as shown in §4.)

Example 3.2. Let X be an infinite set and x_0, x_1 be two different points of X. Let Cof be the cofinite topology on X, and consider the topology $E = \{G \subseteq X : G \subseteq X \backslash \{x_0, x_1\}\} \cup \{X\}$. The topology $\tau = Cof \cap E$ is $S_{1/4}$ but is not $S_{1/2}$. In fact, $[x_0]_\tau = [x_1]_\tau = \{x_0, x_1\}$ is a closed set, and for all $y \notin \{x_0, x_1\}$, the set $[y]_\tau = \{y\}$ is kerneled but not open. Note that X is the only open set containing x_i $(i = 0, 1)$; hence, τ is not T_0.

It is worth notice that the separation axioms $\mathbf{S}_i$ defined above are weaker than the regularity property. Recall that a topology τ is said to be regular if for each $x \in X$ and $V \in \tau(x)$, there exists $U \in \tau(x)$ such that $cl(U) \subseteq V$. Thus, τ is regular iff $\tau = \tau_\theta$. From our result, τ is $\mathbf{S}_2$ iff $\mathbb{B}(\tau) = \mathbb{B}(\tau_\theta)$; it follows immediately that if τ is regular, then τ satisfies $\mathbf{S}_2$.

To prove our main result, we need some facts concerning the classes $[x]_\tau$.

Lemma 3.3. *The following hold for any topology τ on X.*

(i) $[x]_\tau = cl(x) \cap \ker(x)$, *for all $x \in X$ (i.e., $[x]_\tau$ is λ-closed);*
(ii) *if A is kerneled or saturated and $x \in A$, then $[x]_\tau \subseteq A$;*
(iii) $cl([x]_\tau) = cl(x)$, $\ker([x]_\tau) = \ker(x)$, *and* $cl_\theta([x]_\tau) = cl_\theta(x)$, *for all $x \in X$;*
(iv) $[x]_\tau \subseteq cl_\theta(x) \subseteq [x]_{\tau_\theta}$, *for all $x \in X$.*

Proof: (i) Let $x \in X$. By definition, $y \in [x]_\tau$ iff $cl(x) = cl(y)$ iff $\ker(x) = \ker(y)$ iff $y \in cl(x) \cap \ker(x)$.

(ii) If A is kerneled (saturated, respectively) and $x \in A$, then $\ker(x) \subseteq A$ ($cl(x) \subseteq A$, respectively). Thus, $[x]_\tau \subseteq A$.

(iii) Let $x \in X$. Since $x \in [x]_\tau$ and since cl, ker, and cl_θ are monotone operators, it is enough to prove that $cl([x]_\tau) \subseteq cl(x)$, $\ker([x]_\tau) \subseteq \ker(x)$, and $cl_\theta([x]_\tau) \subseteq cl_\theta(x)$. If $y \notin cl(x)$, there exists $V \in \tau(y)$ such that $x \notin V$. Then $x \in X\backslash V$, a closed set and so a saturated set. By (ii), $[x]_\tau \subseteq X\backslash V$ and thus, $[x]_\tau \cap V = \emptyset$. Therefore, $y \notin cl([x]_\tau)$. Similarly, it can be proved that $ker([x]_\tau) = \ker(x)$, for any $x \in X$. Now if $y \notin cl_\theta(x)$, there exist $U \in \tau(x)$ and $V \in \tau(y)$ such that $U \cap cl(V) = \emptyset$. By (ii), $[x]_\tau \subseteq U$ and thus, $[x]_\tau \cap cl(V) = \emptyset$. Then $y \notin cl_\theta([x]_\tau)$.

(iv) Let $x \in X$. The first inclusion follows from the facts that $[x]_\tau \subseteq cl_\theta([x]_\tau)$ and that $[x]_\tau$. Now given $y \in X$, if $y \notin cl_{\tau_\theta}(x)$, there exists a θ-closed set B containing x such that $y \notin B$. Since $X\backslash B$ is θ-open containing y, there is $V \in \tau(y)$ such that $cl(V) \subseteq X\backslash B$. Then $x \notin cl(V)$ and thus, $y \notin cl_\theta(x)$. Therefore, $cl_\theta(x) \subseteq cl_{\tau_\theta}(x)$ for all $x \in X$. On the other hand, if $y \in cl_\theta(x)$, then $x \in cl_\theta(y) \subseteq cl_{\tau_\theta}(y)$. Thus, $y \in \ker_{\tau_\theta}(x)$, which proves that $cl_\theta(x) \subseteq \ker_{\tau_\theta}(x)$ for all $x \in X$. Applying (i) to the topology τ_θ, one has that $[x]_{\tau_\theta} = cl_{\tau_\theta}(x) \cap \ker_{\tau_\theta}(x)$, for all $x \in X$. Therefore, $[x]_\tau \subseteq cl_\theta(x) \subseteq [x]_{\tau_\theta}$, for all $x \in X$. □

Remark 3.4. It was proved in [1] that τ is $\mathbf{T}_0$ iff every point is λ-closed, that is to say, $[x]_\tau = x$ for every $x \in X$. From this, it is clear that τ is $\mathbf{T}_i$ iff it is $\mathbf{S}_i$ and $\mathbf{T}_0$, for $i = 1/4$, $1/2$, 1, or 2.

As $\mathbb{B}(\tau)$ is complete, then it is atomic and its atoms are given by the following result.

Proposition 3.5. *The atoms of* $\mathbb{B}(\tau)$ *are the sets* $[x]_\tau$, *for* $x \in X$.

Proof: Let $\mathbb{C}$ be the complete Boolean algebra generated by the equivalence classes $[x]_\tau$ with $x \in X$. Since the equivalence classes form a partition of X, then they are the atoms of $\mathbb{C}$. Thus, it suffices to show that $\mathbb{B}(\tau) = \mathbb{C}$. By Lemma 3.3(i), $[x]_\tau$ is λ-closed for all $x \in X$, so $[x]_\tau \in \mathbb{B}(\tau)$ for all $x \in X$. Thus, $\mathbb{C} \subseteq \mathbb{B}(\tau)$. On the other hand, if A is open and $x \in A$, then $[x]_\tau \subseteq A$ (Lemma 3.3(ii)). Thus, $\tau \subseteq \mathbb{C}$ and therefore, $\mathbb{B}(\tau) \subseteq \mathbb{C}$. And we are done. □

Corollary 3.6. τ *is* $\mathbf{T}_0$ *iff* $\mathbb{B}(\tau) = \mathcal{P}(X)$.

Proof: Since τ is $\mathbf{T}_0$ iff $[x]_\tau = \{x\}$ for all $x \in X$ and since the singletons are the atoms of $\mathcal{P}(X)$, then the result follows from Proposition 3.5. □

Theorem 3.7. $\mathbb{B}(\tau) = \overline{\tau} \vee \overline{\tau}^*$.

Proof: Since $\overline{\tau} \vee \overline{\tau}^* = \{A : A$ is an arbitrary union of finite intersections of kerneled and saturated sets$\}$, then it is clear that $\overline{\tau} \vee \overline{\tau}^* \subseteq \mathbb{B}(\tau)$. Now since $[x]_\tau = cl(x) \cap \ker(x)$, it follows that $[x]_\tau \in \overline{\tau} \vee \overline{\tau}^*$ for all $x \in X$. Then, if $A \in \mathbb{B}(\tau)$, $A = \cup_{x \in A}[x]_\tau$ (Proposition 3.5) and thus, $A \in \overline{\tau} \vee \overline{\tau}^*$ since $\overline{\tau} \vee \overline{\tau}^*$ is a topology on X. Therefore, $\mathbb{B}(\tau) \subseteq \overline{\tau} \vee \overline{\tau}^*$. □

The following lemma shows other properties of the E_τ-equivalence classes $[x]_\tau$, which are crucial to the remainder of this paper.

Lemma 3.8. *Let* $x \in X$.

(i) $[x]_\tau$ *is closed iff* $[x]_\tau$ *is saturated.*
(ii) $[x]_\tau$ *is closed or open iff* $[x]_\tau$ *is* λ*-open.*
(iii) $[x]_\tau$ *is closed or kerneled iff* $[x]_\tau \in CX(\tau)^*$.
(iv) $[x]_\tau$ *is* θ*-closed iff* $[x]_\tau \in \mathbb{B}(\tau_\theta)$.

Proof: (i) If $[x]_\tau$ is saturated, then $cl([x]_\tau) = cl(x) \subseteq [x]_\tau$, so that $[x]_\tau$ is closed. The other direction is obvious, as every closed set is saturated.

(ii) Since A is λ-open iff it is the union of an open set and a saturated set, then every open set and every closed set is λ-open. Now if $[x]_\tau$ is λ-open, then $[x]_\tau = V \cup S$, where V is open and S is saturated. Thus, $x \in V$ or $x \in S$ and, by Lemma 3.3(ii), $[x]_\tau = V$ or $[x]_\tau = S$, respectively. Therefore, $[x]_\tau$ is open or saturated. Equivalently, by (i), $[x]_\tau$ is open or closed.

The proof of (iii) is similar to that of (ii), noting that a set belongs to $CX(\tau)^*$ iff it is the union of a kerneled set and a saturated set.

(iv) Since $B(\tau_\theta)$ is a complete Boolean algebra containing the θ-open sets, then it is obvious that $[x]_\tau \in B(\tau_\theta)$ if $[x]_\tau$ is θ-closed. Now, let $[x]_\tau \in B(\tau_\theta)$. By Proposition 3.5, the atoms of $B(\tau_\theta)$ are the sets $[x]_{\tau_\theta}$. Then $[x]_{\tau_\theta} \subseteq [x]_\tau$. By Lemma 3.3(iv), $[x]_\tau \subseteq cl_\theta(x) \subseteq [x]_{\tau_\theta}$; thus, in particular, $[x]_\tau = cl_\theta(x)$. Finally, by Lemma 3.3(iii), $cl_\theta(x) = cl_\theta([x]_\tau)$. Therefore, $[x]_\tau$ is θ-closed. □

Proposition 3.9. *τ satisfies $\mathbf{S}_2$ iff $[x]_\tau$ is θ-closed for all $x \in X$.*

Proof: Suppose τ satisfies $\mathbf{S}_2$. We will show that $cl_\theta([x]_\tau) = [x]_\tau$ for all $x \in X$. Let $x \in X$ and $y \in cl_\theta([x]_\tau)$. Since $cl_\theta([x]_\tau) = cl_\theta(x)$ (Lemma 3.3(iii)), it follows that $V \cap W \neq \emptyset$ for all $V \in \tau(x)$ and all $W \in \tau(y)$. It must be then that $cl(x) = cl(y)$, for otherwise, $cl(x)$ and $cl(y)$ can be separated by disjoint open sets, a contradiction. Thus, $y \in [x]_\tau$, and it follows that $cl_\theta([x]_\tau) \subseteq [x]_\tau$. The reverse inclusion is obvious by the definition of the θ-closure of a set.

Conversely, suppose that $[x]_\tau$ is θ-closed for all $x \in X$. Since $[x]_\tau \subseteq \ker(x)$, $cl(x) \subseteq cl_\theta(x) = cl_\theta([x]_\tau)$, then $[x]_\tau = \ker(x) = cl(x) = cl_\theta(x)$ for all $x \in X$. Let $x, y \in X$ such that $cl(x) \neq cl(y)$. Then $y \notin [x]_\tau$, and thus, there exist $V \in \tau(x)$ and $W \in \tau(y)$ such that $V \cap W = \emptyset$. It is clear that $cl(x) = \ker(x) \subseteq V$ and $cl(y) = \ker(y) \subseteq W$. Therefore, τ satisfies $\mathbf{S}_2$. □

From the above results, the separation axioms $\mathbf{S}_i$, $i = 2, 1, 1/2$, and $1/4$, can be characterized as follows.

Proposition 3.10. *Let τ be a topology on X. Then τ satisfies*

(i) $\mathbf{S}_2$ *iff* $[x]_\tau \in \mathbb{B}(\tau_\theta)$, *for every* $x \in X$;
(ii) $\mathbf{S}_1$ *iff* $[x]_\tau \in \overline{\tau}^*$, *for every* $x \in X$;
(iii) $\mathbf{S}_{1/2}$ *iff* $[x]_\tau \in \lambda_\tau$, *for every* $x \in X$;
(iv) $\mathbf{S}_{1/4}$ *iff* $[x]_\tau \in CX(\tau)^*$, *for every* $x \in X$.

Proof: (i) follows immediately by Lemma 3.8(iv) and Proposition 3.9.

(ii) Recall that $\overline{\tau}^*$ is the family of the saturated sets. Since τ is $\mathbf{S}_1$ iff $[x]_\tau$ is closed for every $x \in X$, the result follows from Lemma 3.8(i).

Finally, (iii) and (iv) follow directly from the definition of the separation axioms $\mathbf{S}_{1/2}$ and $\mathbf{S}_{1/4}$ and Lemma 3.8(ii) and (iii), respectively. □

Now we are ready to prove the main result of this section.

Theorem 3.11. *Let τ be a topology on X. Then τ satisfies*

(i) $\mathbf{S}_2$ *iff* $\mathbb{B}(\tau) = \mathbb{B}(\tau_\theta)$;
(ii) $\mathbf{S}_1$ *iff* $\mathbb{B}(\tau) = \overline{\tau}$;
(iii) $\mathbf{S}_{1/2}$ *iff* $\mathbb{B}(\tau) = \lambda^*_\tau$;
(iv) $\mathbf{S}_{1/4}$ *iff* $\mathbb{B}(\tau) = CX(\tau)$.

Proof: Let $\mathcal{A} \subseteq \mathbb{B}(\tau)$ such that $\emptyset \in \mathcal{A}$ and it is closed under arbitrary unions. Since the atoms of $\mathbb{B}(\tau)$ are the sets $[x]_\tau$, for $x \in X$, it follows immediately that $[x]_\tau \in \mathcal{A}$ for all $x \in X$ iff $\mathbb{B}(\tau) = \mathcal{A}$. The result follows directly from this observation, Proposition 3.10, and the fact that $\mathbb{B}(\tau)$ is a complete Boolean algebra. □

Corollary 3.12. *$\tau \vee \overline{\tau}^*$ is $\mathbf{S}_{1/4}$ for any topology τ.*

Proof: We show first that the smallest AT topology containing $\tau \vee \overline{\tau}^*$ is $\overline{\tau} \vee \overline{\tau}^*$. In fact, since the supremum of two AT topologies is AT (see [15]), and since $\overline{\tau} \vee \overline{\tau}^* \supseteq \tau \vee \overline{\tau}^*$, then $\overline{\tau} \vee \overline{\tau}^* \supseteq \overline{\tau \vee \overline{\tau}^*}$. The reverse inclusion follows from the fact that $\tau \vee \overline{\tau}^*$ contains τ and $\overline{\tau}^*$. Thus, $\overline{\tau \vee \overline{\tau}^*} = \overline{\tau} \vee \overline{\tau}^*$. Now by Theorem 3.7, $\overline{\tau} \vee \overline{\tau}^*$ is a Boolean algebra, so $(\overline{\tau \vee \overline{\tau}^*})^* = \overline{\tau \vee \overline{\tau}^*}$. Hence, $\overline{\tau \vee \overline{\tau}^*} = \mathbb{B}(\tau \vee \overline{\tau}^*)$. Note that the family of convex sets for the topology $\tau \vee \overline{\tau}^*$ is precisely $\overline{\tau} \vee \overline{\tau}^*$. Therefore, $\mathbb{B}(\tau \vee \overline{\tau}^*) = \overline{\tau} \vee \overline{\tau}^* = CX(\tau \vee \overline{\tau}^*)$. The result follows from Theorem 3.11(iv). □

Corollary 3.13. *τ is $\mathbf{S}_1$ iff $\overline{\tau} = \overline{\tau}^*$ (i.e., the kerneled sets and saturated sets of τ coincide).*

Proof: By Theorem 3.11, if τ is $\mathbf{S}_1$, then $\mathbb{B}(\tau) = \overline{\tau}$. Since $\mathbb{B}(\tau) = \overline{\tau} \vee \overline{\tau}^*$ (Theorem 3.7), then $\overline{\tau}^* \subseteq \overline{\tau}$ and thus, $\overline{\tau} = \overline{\tau}^*$. Reciprocally, if $\overline{\tau} = \overline{\tau}^*$, then $\mathbb{B}(\tau) = \overline{\tau} \vee \overline{\tau}^* = \overline{\tau}$ and so, τ is $\mathbf{S}_1$. □

The next result is an immediate consequence of Corollary 3.6, Theorem 3.11, and the fact that a topology τ is $\mathbf{T}_i$ iff it is $\mathbf{S}_i$ and $\mathbf{T}_0$, for $i = 2, 1, 1/2$, and $1/4$.

Corollary 3.14. *Let τ be a topology on X. Then τ satisfies*

(i) $\mathbf{T}_2$ *iff* $P(X) = \mathbb{B}(\tau_\theta)$;
(ii) $\mathbf{T}_1$ *iff* $P(X) = \overline{\tau}$;
(iii) $\mathbf{T}_{1/2}$ *iff* $P(X) = \lambda_\tau^*$;
(iv) $\mathbf{T}_{1/4}$ *iff* $P(X) = CX(\tau)$.

By definitions of λ-closed sets and convex sets, it is clear that these concepts coincide on finite sets. We prove in the following lemma that these two concepts also coincide for finite unions of atoms.

Lemma 3.15. *For every finite set $F \subset X$, $\bigcup_{x\in F}[x]_\tau$ is λ-closed iff it is convex.*

Proof: Let $F = \{x_1, x_2, ..., x_n\}$, and suppose that $\bigcup_{i=1}^n [x_i]_\tau$ is convex. Then $\bigcup_{i=1}^n [x_i]_\tau = K \cap S$ for some $K \in \overline{\tau}$ and $S \in \overline{\tau}^*$. Since $cl(F) = \bigcup_{i=1}^n cl(x_i) \supseteq \bigcup_{i=1}^n [x_i]_\tau$, then $K \cap cl(F) \supseteq K \cap S$. On the other hand, since S is saturated and contains F, it contains $cl(F)$. Thus, $K \cap S \supseteq K \cap cl(F)$. Therefore, $\bigcup_{i=1}^n [x_i]_\tau = K \cap cl(F)$ is a λ-closed set. The converse is obvious as every λ-closed set is convex. □

Part (ii) of the following theorem was proved differently in [1].

Theorem 3.16. (i) *τ is $\mathbf{S}_{1/4}$ iff $\bigcup_{x\in F}[x]_\tau$ is convex for every finite set $F \subset X$.*

(ii) *τ is $\mathbf{T}_{1/4}$ iff every finite set is λ-closed.*

Proof: For each $F \subseteq X$, we will denote $\bigcup_{x\in F}[x]_\tau$ by $[F]$.

(i) Suppose τ is $\mathbf{S}_{1/4}$ and $F \subseteq X$ is a finite set. By Theorem 3.11(iv), $CX(\tau) = \mathbb{B}(\tau)$. Since $[x]_\tau \in \mathbb{B}(\tau)$ for each $x \in X$, and since $\mathbb{B}(\tau)$ is a Boolean algebra, then $[F] \in \mathbb{B}(\tau)$ and thus, $[F]$ is convex. Conversely, suppose that $[F]$ is convex for every finite set F. To show that τ is $\mathbf{S}_{1/4}$, it is enough to show that $\mathbb{B}(\tau) \subseteq CX(\tau)$. If $A \notin CX(\tau)$, there exist $x, z \in A$ and $y \in X \backslash A$ such that $x \preceq_\tau y \preceq_\tau z$. Since the set $[x]_\tau \cup [z]_\tau$ is convex and contains x and z, then it contains y. It follows that $A \notin \mathbb{B}(\tau)$.

(ii) follows from (i) and Lemma 3.15. □

The following proposition is a folklore fact. We give a proof using results from this section.

Proposition 3.17. *τ is $\mathbf{T}_2$ iff τ_θ is $\mathbf{T}_1$ iff τ_θ is $\mathbf{T}_{1/2}$ iff τ_θ is $\mathbf{T}_0$* [6].

Proof: By Corollary 3.11, τ is $\mathbf{T}_2$ iff $P(X) = \mathbb{B}(\tau_\theta)$, and by Corollary 3.6, this equality holds iff τ_θ is $\mathbf{T}_0$. On the other hand, if τ is $\mathbf{T}_2$, then τ is $\mathbf{T}_0$ and $\mathbf{S}_2$. Thus, $\{x\} = [x]_\tau$ is θ-closed for all $x \in X$, and therefore, τ_θ is $\mathbf{T}_1$. The rest of the equivalences are obvious. □

4. Additional results

As an application of the previous section, we include some results about λ-spaces, AT spaces, and the order theoretical property.

4.1. λ-spaces

In [1], λ-spaces were defined as those spaces (X, τ) for which the family λ_τ of the λ-open sets is a topology. It is easy to prove that (X, τ) is a λ-space iff $V \cap S \in \lambda_\tau$ for all $V \in \tau$ and $S \in \overline{\tau}^*$. Since τ is $\mathbf{S}_{1/2}$ iff $\mathbb{B}(\tau) = \lambda_\tau$, every $\mathbf{S}_{1/2}$ space is a λ-space. Moreover, every λ-space is $\mathbf{S}_{1/4}$ as shown in next result.

Proposition 4.1. *Every λ-space is $\mathbf{S}_{1/4}$. Hence, every $\mathbf{T}_0$, λ-space is $\mathbf{T}_{1/4}$.*

Proof: If (X, τ) is a λ-space, then the finite union of λ-closed sets is a λ-closed set. Thus, for every finite set $F \subset X$, the set $\bigcup_{x \in F} [x]_\tau$ is λ-closed, therefore convex. By Theorem 3.16(i), τ is $\mathbf{S}_{1/4}$. □

Let $LC(\tau)$ be the family of τ-locally closed sets in X, i.e., $LC(\tau) = \{V \cap F$, with $V \in \tau$ and $F \in \tau^*\}$. The following theorem gives a characterization of λ-spaces.

Theorem 4.2. *The following are equivalent.*

(i) *(X, τ) is a λ-space.*
(ii) *$LC(\tau) \subseteq \lambda_\tau$.*
(iii) *$\lambda_\tau = \tau \vee \overline{\tau}^*$.*
(iv) *cl_λ is a Kuratowski closure operator.*

Proof: Suppose (X, τ) is a λ-space. Then λ_τ is a topology on X and thus, the intersection of two λ-open sets is a λ-open set. Since

$\tau \cup \tau^* \subseteq \lambda_\tau$, then $A \in LC(\tau)$ implies A is λ-open. Therefore, (i) $\Rightarrow$ (ii).

Recall that $A \in \lambda_\tau$ implies that $A = V \cup S$ for some $V \in \tau$ and $S \in \overline{\tau}^*$; thus, it is obvious that $\lambda_\tau \subseteq \tau \vee \overline{\tau}^*$. On the other hand, since any set of the form $V \cap S$, with $V \in \tau$ and $S \in \overline{\tau}^*$, can be written as an arbitrary union of locally closed sets, (namely $V \cap S = \bigcup_{x \in S} V \cap cl(x)$), it follows that $LC(\tau)$ is a basis for the topology $\tau \vee \overline{\tau}^*$. Thus, $LC(\tau) \subseteq \lambda_\tau$ implies $\tau \vee \overline{\tau}^* \subseteq \lambda_\tau$, since the family λ_τ is closed under arbitrary unions. Therefore, (ii) $\Rightarrow$ (iii).

Now suppose that $\lambda_\tau = \tau \vee \overline{\tau}^*$. Then λ_τ is a topology. To prove that cl_λ is a Kuratowski operator, one has to prove that it is additive on X. Let $A, B \subseteq X$. Since $cl_\lambda(A) \cup cl_\lambda(B)$ is a λ-closed set containing $A \cup B$, then $cl_\lambda(A \cup B) \subseteq cl_\lambda(A) \cup cl_\lambda(B)$. The reverse inclusion follows from the monotonicity of the cl_λ operator. Therefore, (iii)$\Rightarrow$ (iv). The equivalence of (i) and (iv) is trivial. $\square$

The property of being a λ-space is strictly placed between $\mathbf{S}_{1/2}$ and $\mathbf{S}_{1/4}$, as the following examples show.

Example 4.3 ($\mathbf{S}_{1/4} \not\Rightarrow \lambda$-space)**.** Consider the $\mathbf{S}_{1/4}$ space (X, τ) given in Example 3.2. Let A be an infinite proper subset of $X \setminus \{x_0, x_1\}$ and let $B = A \cup \{x_0, x_1\}$. Note that $A = \cap\{X \setminus \{y, x_0, x_1\} : y \notin B\}$ is kerneled and so λ-closed. Also, the set $\{x_0, x_1\}$ is λ-closed since it is closed. But B is not λ-closed, since $cl(B) = X = \ker(B)$, and hence, $cl_\lambda(B) = X \neq B$. Therefore, (X, τ) is not a λ-space.

Example 4.4 (λ-space $\not\Rightarrow \mathbf{S}_{1/2}$)**.** Let ρ be the topology of the digital line on the set of integers numbers Z (i.e., the topology generated by the sets $\{2n-1, 2n, 2n+1\}_{n \in Z}$). This is an AT and $\mathbf{T}_{1/2}$ topology. Let τ be the topology on Z generated by $Z \setminus cl_\rho(F)$ for $F \subseteq Z$ finite. Note that $\overline{\tau} = \rho$. In fact, since $\tau \subseteq \rho$ and ρ is an AT topology, then $\overline{\tau} \subseteq \rho$. On the other hand, since any basic ρ-open set $\{2n-1, 2n, 2n+1\}$ can be written as $\bigcap_{k \neq n} Z \setminus \{2k\}$, a τ-kerneled set, then $\rho \subseteq \overline{\tau}$. The topology τ is $\mathbf{T}_0$ but not $\mathbf{T}_{1/2}$, as the open points of (Z, ρ) are neither τ-open nor τ-closed. Thus, τ is not $\mathbf{S}_{1/2}$. It is straightforward to check that $LC(\tau) \subseteq \tau^*$. Since closed sets are λ-open, then (X, τ) is a λ-space by Theorem 4.2.

4.2 AT spaces

If τ is an AT topology (i.e., $\tau = \overline{\tau}$), then the smallest complete Boolean algebra $\mathbb{B}(\tau)$ containing τ is just the topology $\tau \vee \tau^*$. We prove below that some of the low separation axioms defined in section 3 are indistinguishable for AT topologies. For instance, $\mathbf{S}_{1/2}$ and $\mathbf{S}_{1/4}$ are equivalent, and $\mathbf{S}_1$ and $\mathbf{S}_2$ are equivalent to the regularity property. Note that, for AT topologies, $CO(\tau) = CO(\tau_\theta) = \tau_\theta = \mathbb{B}(\tau_\theta)$. This shows, in particular, that τ_θ is a complete Boolean algebra.

Proposition 4.5. *Let τ be an AT topology. Then,*

(i) *τ is $\mathbf{S}_2$ iff τ is $\mathbf{S}_1$ iff τ is regular;*
(ii) *τ is $\mathbf{S}_{1/2}$ iff τ is $\mathbf{S}_{1/4}$ iff (X, τ) is a λ-space.*

Proof: (i) If τ is $\mathbf{S}_1$, then $\mathbb{B}(\tau) = \tau$ (Theorem 3.11) and so $\mathbb{B}(\tau) = CO(\tau)$. Therefore, τ is regular. Since $regular \Rightarrow \mathbf{S}_2 \Rightarrow \mathbf{S}_1$, the conclusion follows.

(ii) Since, for AT topologies, saturated sets are closed, then $\lambda_\tau^* = CX(\tau)$. Thus, the equivalence $\mathbf{S}_{1/2}$ iff τ is $\mathbf{S}_{1/4}$ is a consequence of Theorem 3.11. The other equivalence follows from the fact that λ-space property is placed between $\mathbf{S}_{1/2}$ and $\mathbf{S}_{1/4}$. □

The separation axioms $\mathbf{S}_{1/2}$ and $\mathbf{S}_1$ are not equivalent in AT spaces. In fact, the topology ρ of the digital line (see Example 4.4) is an AT and $\mathbf{T}_{1/2}$ topology, thus $\mathbf{S}_{1/2}$ and $\mathbf{T}_0$. But, being not discrete, ρ is not $\mathbf{T}_1$. Therefore, it is not $\mathbf{S}_1$.

We end this section with a result about a notion taken from [16]. A property P (like a separation axiom) is said to be *order theoretical* if the following holds: Given topologies τ and ρ such that $\preceq_\tau = \preceq_\rho$, if (X, τ) satisfies P, then (X, ρ) also satisfies P. Since $cl_\tau(x) = cl_{\overline{\tau}}(x)$ for any $x \in X$, then $\preceq_\tau = \preceq_\rho$ iff $\overline{\tau} = \overline{\rho}$ for any pair of topologies τ and ρ on X. Thus, a property P is order theoretical if the following holds: (X, τ) satisfies P iff $(X, \overline{\tau})$ satisfies P.

Theorem 4.6. *The separation axioms $\mathbf{T}_0$, $\mathbf{T}_1$, $\mathbf{T}_{1/4}$, $\mathbf{S}_1$, and $\mathbf{S}_{1/4}$ are order theoretical.*

Proof: The result follows immediately from Corollary 3.6, Theorem 3.11, Corollary 3.14, and the facts that $\mathbb{B}(\tau) = \mathbb{B}(\overline{\tau})$ and $CX(\tau) = CX(\overline{\tau})$. □

Remark 4.7. The separation axioms $\mathbf{S}_2$, $\mathbf{S}_{1/2}$, and λ-space are not order theoretical.

Since $\mathbb{B}(\tau_\theta) \subseteq CO(\overline{\tau}) = (\overline{\tau})_\theta = \mathbb{B}((\overline{\tau})_\theta) \subseteq \mathbb{B}(\tau)$, then if τ is $\mathbf{S}_2$, it follows that $\overline{\tau}$ is $\mathbf{S}_2$. Now let τ be the cofinite topology on an infinite set. The topology τ is not $\mathbf{S}_2$, but $\overline{\tau}$ is the discrete topology which is obviously $\mathbf{S}_2$ (note that any singleton $\{x\}$ can be written as $\bigcap_{y\neq x} X\backslash\{y\}$, a kerneled set). Therefore, $\mathbf{S}_2$ is not order theoretical.

Note that, for AT topologies, the family of λ-closed sets coincides with the family of convex sets; thus, $\lambda^*_{\overline{\tau}} = CX(\overline{\tau}) = CX(\tau)$. If τ is $\mathbf{S}_{1/2}$, then $\lambda^*_\tau = \mathbb{B}(\tau)$, and so $\lambda^*_{\overline{\tau}} = \mathbb{B}(\tau) = \mathbb{B}(\overline{\tau})$ which shows that $\overline{\tau}$ is $\mathbf{S}_{1/2}$. But the fact that $\overline{\tau}$ is $\mathbf{S}_{1/2}$ does not imply that τ is $\mathbf{S}_{1/2}$. Consider, for instance, the digital topology ρ on Z, and let τ be the non $\mathbf{S}_{1/2}$ topology given in Example 4.4. Since $\overline{\tau} = \rho$ and ρ is $\mathbf{T}_{1/2}$, then $\overline{\tau}$ is, in particular, $\mathbf{S}_{1/2}$. Therefore, $\mathbf{S}_{1/2}$ is not order theoretical.

If $(X,\ \tau)$ is a λ-space, then τ is $\mathbf{S}_{1/4}$ by Proposition 4.5. The above theorem implies that $\overline{\tau}$ is $\mathbf{S}_{1/4}$. Since $\overline{\tau}$ is an AT topology, then $(X, \overline{\tau})$ is λ-space by Theorem 4.2(ii). But Example 4.3 shows that λ-space and $\mathbf{S}_{1/4}$ are not equivalent. Then the fact that $(X, \overline{\tau})$ is a λ-space does not imply that $(X,\ \tau)$ is a λ-space. Hence, the property of being a λ-space is not order theoretical.

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References

[1] Francisco G. Arenas, Julian Dontchev, and Maximilian Ganster, *On λ-sets and the dual of generalized continuity*, Questions Answers Gen. Topology **15** (1997), no. 1, 3–13.

[2] M. Caldas, S. Jafari, and M. M. Kovár, *Some properties of θ-open sets*, Divulg. Mat. **12** (2004), no. 2, 161–169.

[3] J. Cao, M. Ganster, I. Reilly, and M. Steiner, *δ-closure, θ-closure and generalized closed sets*, Appl. Gen. Topol. **6** (2005), no. 1, 79–86.

[4] M. L. Colasante, C. Uzcátegui, and J. Vielma, *Low separation axioms via the diagonal*, Appl. Gen. Topol. **9** (2007), no. 1, 39–50.

[5] Á. Császár, *Separation axioms for generalized topologies*, Acta Math. Hungar. **104** (2004), no. 1-2, 63–69.

[6] Anna Di Concilio, *On θ-continuous convergence in function spaces*, Rend. Mat. (7) **4** (1984), no. 1, 85–94 (1985).

[7] A. S. Davis, *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly **68** (1961), no. 9, 886–893.

[8] Peter T. Johnstone, *Stone Spaces*. Reprint of the 1982 edition. Cambridge Studies in Advanced Mathematics, 3. Cambridge: Cambridge University Press, 1986.

[9] Dragan Janković, *On some separation axioms and θ-closure*, Mat. Vesnik **4(17)(32)** (1980), no. 4, 439–449

[10] Joseph T. H. Lo, *Borel sets and separation axioms*, Math. Proc. R. Ir. Acad. **101A** (2001), no. 2, 111–123.

[11] Haruo Maki, *Generalized Λ-sets and the associated closure operator*, in The Special Issue in Commemoration of Professor Kazusada IKEDA's Retirement. Japan, 1986, 139–146.

[12] N. A. Shanin, *On separation in topological spaces*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **38** (1943), 110–113.

[13] Juris Steprāns and Stephen Watson, *Mutually complementary families of T_1 topologies, equivalence relations and partial orders*, Proc. Amer. Math. Soc. **123** (1995), no. 7, 2237–2249.

[14] Carlos Uzcátegui, *Topologies generated by ideals*, Comment. Math. Univ. Carolin. **47** (2006), no. 2, 317–335.

[15] C. Uzcátegui and J. Vielma, *Alexandroff topologies viewed as closed sets in the Cantor cube*, Divulg. Mat. **13** (2005), no. 1, 45–53.

[16] Stephen Watson, *The construction of topological spaces: Order*. Unpublished, 2001. http://at.yorku.ca/z/a/a/a/12.dir/papers/98/1.htm.

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