TOPOLOGY PROCEEDINGS
Volume 34, 2009
Pages 77–96
http://topology.auburn.edu/tp/

Dendrites with Unique Hyperspace $C_2(X)$, II

by

Alejandro Illanes

Electronically published on April 21, 2009

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 34 (2009)
Pages 77-96

E-Published on April 21, 2009

DENDRITES WITH UNIQUE HYPERSPACE $C_2(X)$, II

ALEJANDRO ILLANES

Abstract. Let Y be a metric continuum. Let $C_n(Y)$ be the hyperspace of nonempty closed subsets of Y with at most n components. In this paper we show that if X is a dendrite with a closed set of end points and $C_2(X)$ is homeomorphic to $C_2(Y)$, for some continuum Y, then X is homeomorphic to Y. This completes the work by David Herrera-Carrasco and Fernando Macías-Romero who previously proved the corresponding result for each $n \neq 2$.

1. Introduction

All concepts not defined here will be taken as in [30] or [23]. A *continuum* is a nonempty compact and connected metric space. For a continuum X and a positive integer n, consider the following hyperspaces:

$2^X = \{A \subset X : A \text{ is closed and nonempty}\}$,
$C(X) = \{A \in 2^X : A \text{ is connected}\}$,
$F_n(X) = \{A \in 2^X : A \text{ contains at most } n \text{ points}\}$, and
$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$.

All the hyperspaces are considered with the Hausdorff metric H_X.

2000 *Mathematics Subject Classification.* Primary 54B20. Secondary 54F15, 54F50.

Key words and phrases. continuum, dendrite, hyperspace, unique hyperspace.

Let $\mathcal{H}(X)$ denote one of the hyperspaces 2^X, $C(X)$, $F_n(X)$, or $C_n(X)$. We say that a continuum X has *unique hyperspace* $\mathcal{H}(X)$ provided that the following implication holds: If Y is a continuum and $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y.

A *dendrite* is a locally connected continuum without simple closed curves.

The topic of this paper is contained in the following problem.

Problem 1.1. *Find conditions on the continuum X in order that X has unique hyperspace $\mathcal{H}(X)$.*

Generalizing results by R. Duda [10, 9.1]; Carl Eberhart and Sam B. Nadler, Jr., (see [29, 0.60]; and Sergio Macías [25, Theorem 3], Gerardo Acosta proved that the following continua have unique hyperspace $C(X)$:

(a) finite graphs different from an arc and a simple closed curve [2, Theorem 1],
(b) hereditarily indecomposable continua [2, Theorem 2],
(c) indecomposable continua such that all their proper nondegenerate subcontinua are arcs [3, Theorem 2.3].

In [2], Acosta also proved that the metric compactifications of the ray $[0, \infty)$, different from an arc, have unique hyperspace $C(X)$.

Macías [26] has shown that hereditarily indecomposable continua have unique hyperspace 2^X. Enrique Castañeda and Alejandro Illanes [7] have proved that finite graphs have unique hyperspace $F_n(X)$ for each n. Illanes [19], [20] proved that finite graphs have unique hyperspaces $C_n(X)$, for each $n > 1$. Related results to the subject of this paper can be found in [1], [2], [3], [4], [5], [6], [12], [13], [15], [16], [17], [18], [19], [21], and [22]. The non-metric case has been considered in [24]. Related problems can be found in [21].

In [13] and [14], it has been shown that if X is a dendrite with closed set of end points, then X has unique hyperspace $C_n(X)$ for each $n \neq 2$. In this case ($n \neq 2$), this result can be proved by using the fact that it is possible to give topological properties that characterize the set $F_1(X)$ inside $C_n(X)$. This is not the case for $n = 2$, and other techniques are needed. An important tool we use is the formula proved in [28, Theorem 2.4] which was adapted for dendrites in Lemma 1 of [14].

In [14], David Herrera-Carrasco, Alejandro Illanes, María de J. López, and Fernando Macías-Romero have shown that if X is a dendrite with a closed set of end points, Y is a dendrite, and $C_2(X)$ is homeomorphic to $C_2(Y)$, then X is homeomorphic to Y. In this paper we use the main result and the techniques used in [14] to extend the mentioned result by proving that dendrites X with a closed set of end points have unique hyperspace $C_2(X)$.

2. Preliminary results

The set of positive integers is denoted by $\mathbb{N}$. For each $n \in \mathbb{N}$, $n \geq 3$, we define a *simple n-od*, as a continuum X for which there is a point v, called the *vertex* of X, such that X is the union of n arcs which meet by pairs at the set $\{v\}$ and v is an end point of each one of these arcs. Let Y be a continuum and let $q \in Y$. Let β be a cardinal number. We say that *q has order in Y less than or equal to β*, written $\mathrm{ord}_Y(q) \leq \beta$, provided that q has a basis of neighborhoods $\mathfrak{B}$ in Y such that the cardinality of $\mathrm{bd}_Y(U) \leq \beta$, for each $U \in \mathfrak{B}$. We say that *q has order in Y equal to β* ($\mathrm{ord}_Y(q) = \beta$) provided that $\mathrm{ord}_Y(q) \leq \beta$ and $\mathrm{ord}_Y(q) \nleq \alpha$ for any cardinal number $\alpha < \beta$. Clearly, in a finite graph G, $\mathrm{ord}_G(q) \in \mathbb{N}$ and $\mathrm{ord}_G(q) = n \geq 3$ if and only if q has a neighborhood Q in G that is a simple n-od, where q is the vertex of Q [30, Lemma 9.9].

Given a continuum Y, let $E(Y) = \{p \in Y : \mathrm{ord}_Y(p) = 1\}$ and $R(Y) = \{p \in Y : \mathrm{ord}_Y(p) \geq 3\}$. The elements of $E(Y)$ ($R(Y)$, respectively) are called *end points* (*ramification points*, respectively) of Y. Let $R_f(Y) = \{p \in Y :$ there exists a neighborhood Q of p in Y such that Q is a simple n-od, for some $n \geq 3$, and p is the vertex of $Q\}$. Let $E_i(Y) = \{p \in E(Y) : p$ is an isolated point of $E(Y)\}$ and $E_a(Y) = E(Y) - E_i(Y)$. A *free arc in Y* is an arc A in Y, joining two points p and q, such that $A - \{p, q\}$ is an open subset of Y. A *maximal free arc in Y* is a free arc which is maximal with respect to the inclusion. A *free circle S* is a simple closed curve S in Y for which there exists $p \in S$ such that $S - \{p\}$ is open in Y. Let $\mathfrak{P}(Y) = \{A \in C_2(Y) : A$ has a neighborhood in $C_2(Y)$ that is a 4-cell$\}$, $\mathfrak{A}(Y) = \{J \subset Y : J$ is a maximal free arc in $Y\}$, $\mathfrak{A}_{R_f}(Y) = \{J \in \mathfrak{A}(Y) :$ the end points of J are in $R_f(Y)\}$, and $\mathfrak{A}_S(Y) = \mathfrak{A}(Y) \cup \{S \subset Y : S$ is a free circle in $Y\}$. Given a subset A of Y, we denote the interior of A in Y by A^{o}. Given subsets

$U_1, \ldots, U_n$ of Y, let $\langle U_1, \ldots, U_n \rangle = \{A \in C_2(Y) : A \subset U_1 \cup \ldots \cup U_n$ and $A \cap U_i \neq \emptyset$ for each $i \in \{1, \ldots, n\}\}$. It is known that the family of all sets of the form $\langle U_1, \ldots, U_n \rangle$, where $n \in \mathbb{N}$ and each U_i is open in Y, is a basis for the topology in $C_2(Y)$ [30, 4.24].

Let $\mathfrak{D} = \{X : X$ is a dendrite and $E(X)$ is closed$\}$.

Lemma 2.1. *Let Y be a locally connected continuum such that Y is not a simple closed curve. Suppose that $J, K \in \mathfrak{A}_S(Y)$. Then $\langle J^o, K^o \rangle$ is a component of $\mathfrak{P}(Y)$.*

Proof: Observe that if L is a free circle and $p \in L$ is such that $L - \{p\}$ is open in Y, then $L \neq Y$ and p is the vertex of a 3-od contained in Y.

First, we prove that $\langle J^o, K^o \rangle \subset \mathfrak{P}(Y)$.

If $J = K$, then $\langle J^o, K^o \rangle \subset C_2(J)$. In the case that J is an arc, $C_2(J)$ is a 4-cell ([19, Lemma 2.2]) and $\langle J^o, K^o \rangle$ is open in $C_2(Y)$, so $\langle J^o, K^o \rangle \subset \mathfrak{P}(Y)$. In the case that J is a free circle and $p \in J$ is such that $J - \{p\}$ is open, we have $J \neq Y$ and $\langle J^o, K^o \rangle = \{A \in C_2(J) : p \notin A\}$. Given an element $A \in C_2(J)$ such that $p \notin A$, there exists an arc $L \subset J - \{p\}$ such that $A \subset L^o$. Then $C_2(L)$ is a 4-cell such that $C_2(L)$ is a neighborhood of A in $C_2(Y)$. Thus, $A \in \mathfrak{P}(Y)$. Hence, $\langle J^o, K^o \rangle \subset \mathfrak{P}(Y)$.

Now suppose that $J \neq K$. We claim that $J^o \cap K^o = \emptyset$. In the case that J and K are arcs, this claim follows from Lemma 3(b) of [14]. Thus, suppose that J is a free circle. Let $p \in J$ be such that $J - \{p\}$ is open in Y. Suppose that $J^o \cap K^o \neq \emptyset$. Fix a point $x \in J^o \cap K^o$. Given $y \in K^o$, no matter if K is a free arc or a free circle, there exists a free arc L of Y, contained in K, such that $x, y \in L^o$. By [14, Lemma 3(a)], $L^o \cap R(Y) = \emptyset$. Since $p \in R(Y)$ and $Y - \{p\} = (Y - J) \cup (J - \{p\})$ is a separation of $Y - \{p\}$, $L^o \subset J - \{p\}$. We have shown that $K^o \subset J - \{p\}$. Thus, $K \subset J$. This implies that K is not a maximal free arc and K is a free circle. Hence, $K = J$, a contradiction. We have proved that $J^o \cap K^o = \emptyset$.

Let $A \in \langle J^o, K^o \rangle$. Then $A \subset J^o \cup K^o$, $A \cap J^o \neq \emptyset$, and $A \cap K^o \neq \emptyset$. Thus, the components of A are $A_1 = A \cap J^o$ and $A_2 = A \cap K^o$. Let J_1 and K_1 be subarcs of J and K, respectively, such that $A_1 \subset \mathrm{int}_J(J_1) \subset J_1 \subset J^o$ and $A_2 \subset \mathrm{int}_K(K_1) \subset K_1 \subset K^o$. Then $\mathrm{int}_J(J_1)$ and $\mathrm{int}_K(K_1)$ are open in Y. Let $\psi : C(J_1) \times C(K_1) \to \langle J_1, K_1 \rangle$ be given by $\psi(M, N) = M \cup N$. It is easy to show that ψ is an embedding and $A \in \langle \mathrm{int}_J(J_1), \mathrm{int}_K(K_1)\rangle \subset \psi(C(J_1) \times C(K_1))$. Since $C(J_1)$ and $C(K_2)$ are 2-cells and $\langle \mathrm{int}_J(J_1), \mathrm{int}_K(K_1)\rangle$ is open in $C_2(Y)$, we conclude that $A \in \mathfrak{P}(Y)$.

This completes the proof that $\langle J^o, K^o\rangle \subset \mathfrak{P}(Y)$.

In Lemma 4 of [14], it was proved that if L is a maximal free arc of Y, $p \in L - L^o$, and $A \in \mathfrak{P}(Y)$, then $p \notin A$. We extend this result for free circles. Let L be a free circle in Y, $p \in L - L^o$, and $A \in \mathfrak{P}(Y)$. Notice that p is the point of L such that $L - \{p\}$ is open and p is the vertex of a 3-od contained in Y. By [14, Lemma 2], $p \notin A$.

It is easy to show that $\langle J^o, K^o\rangle$ is an arcwise open subset of $\mathfrak{P}(Y)$. Suppose that $\langle J^o, K^o\rangle$ is not a component of $\mathfrak{P}(Y)$. Let $\mathfrak{C}$ be the component of $\mathfrak{P}(Y)$ that contains $\langle J^o, K^o\rangle$ and let $E \in \mathfrak{C} - \langle J^o, K^o\rangle$. Notice that $\mathfrak{P}(Y)$ is open in $C_2(Y)$. Since $C_2(Y)$ is locally connected (see [27, Theorem 3.2]), $\mathfrak{C}$ is open, so $\mathfrak{C}$ is arcwise connected ([30, Theorem 8.26]). Fix an element $A \in \langle J^o, K^o\rangle$. Let $\alpha : [0, 1] \to \mathfrak{C}$ be a one-to-one map such that $\alpha(0) = A$ and $\alpha(1) = E$. Since $\langle J^o, K^o\rangle$ is an open subset of $C_2(Y)$ and it does not contain $\alpha(1)$, there exists $t = \min\{s \in [0, 1] : \alpha(s) \notin \langle J^o, K^o\rangle\}$. Let $B = \alpha(t)$. Then $B \notin \langle J^o, K^o\rangle$, so $0 < t$. Thus, $B \in \mathrm{cl}_{C_2(Y)}(\langle J^o, K^o\rangle)$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $\langle J^o, K^o\rangle$ such that $\lim B_n = B$. In the case that $J = K$, we have that $B_n \subset J^o$, for each $n \in \mathbb{N}$. Thus, $B \subset J$. If $B \subset J^o$, then $B \in \langle J^o, K^o\rangle$, a contradiction. Thus, $B \cap (J - J^o) \neq \emptyset$. By the previous paragraph, $B \notin \mathfrak{P}(Y)$; this is impossible since $B \in \mathfrak{C} \subset \mathfrak{P}(Y)$. We have shown that $J \neq K$. From what we proved before, J^o and K^o are disjoint open subsets of Y. Thus, for each $n \in \mathbb{N}$, the components of B_n are the sets $C_n = B_n \cap J^o$ and $D_n = B_n \cap K^o$. We may assume that $\lim C_n = C$ and $\lim D_n = D$, for some $C, D \in C(Y)$. Then $B = C \cup D$, $C \subset J$, and $D \subset K$. If $C \subset J^o$ and $D \subset K^o$, then $B \in \langle J^o, K^o\rangle$, a contradiction. Thus, $C \cap (J - J^o) \neq \emptyset$ or $D \cap (K - K^o) \neq \emptyset$. Thus, $B \cap ((J - J^o) \cup (K - K^o)) \neq \emptyset$. This contradicts what we showed in the previous paragraph and completes the proof that $\langle J^o, K^o\rangle$ is a component of $\mathfrak{P}(Y)$ □

Lemma 2.2. *Let Y be a locally connected continuum and let $F \in C_2(Y)$ be such that $\dim_F[C_2(Y)]$ is finite. Then there exists a finite graph D contained in Y such that $F \in int_{C_2(Y)}(C_2(D))$.*

Proof: As an intermediate step, we prove the following claim.

Claim 1. If $A \in C(Y)$ and $\dim_A[C(Y)]$ is finite, then there exists a finite graph D contained in Y such that $A \subset D^{\circ}$.

Proof of Claim 1: In the case that $A = Y$, $\dim_Y[C(Y)]$ is finite; by [10, 1.1 and §7], Y is a finite graph. Thus, we may assume that $A \neq Y$. Let $m = \dim_A[C(Y)]$. First, we show that A is locally connected. Suppose to the contrary that there exists a point $p \in A$ such that A is not connected im kleinen at p. Then there exists an open set U of A such that, if C is the component of U containing p, then $p \notin \text{int}_A(C)$. Let V be an open subset of A such that $p \in V \subset \text{cl}_A(V) \subset U$ and let D be the component of $\text{cl}_A(V)$ such that $p \in D$. Then $p \notin \text{int}_A(D)$. Thus, it is possible to find a sequence of points $\{p_n\}_{n=1}^{\infty}$ in V and a sequence of pairwise different components $\{D_n\}_{n=1}^{\infty}$ of $\text{cl}_A(V)$ such that $p_n \in D_n$, for each $n \geq 1$ and $\lim p_n = p$. We may assume that $\lim D_n = D_0$ for some $D_0 \in C(A)$. Notice that $D_0 \subset D$. By [30, Theorem 5.4], for each $n \geq 1$, $D_n \cap \text{bd}_A(V) \neq \emptyset$. This implies that $D_0 \cap \text{bd}_A(V) \neq \emptyset$. Hence, D_0 is nondegenerate, and then D_0 is infinite.

Fix pairwise different points $q_1, \dots, q_{m+1} \in D_0 \cap V$ and fix pairwise disjoint closed connected subsets $Q_1, \dots, Q_{m+1}$ of Y such that $q_i \in Q_i^{\circ}$ and $Q_i \cap A \subset V$, for each $i \in \{1, \dots, m+1\}$. Let $B = A \cup Q_1 \cup \dots \cup Q_{m+1}$. Then B is a subcontinuum of Y such that $B - A = (Q_1 - A) \cup \dots \cup (Q_{m+1} - A)$. Notice that each Q_i intersects D_n for infinitely many numbers n; therefore, Q_i cannot be contained in V. Thus, each set $Q_i - A$ is nonempty. This implies that $B - A$ has at least $m+1$ components. By the proof of Theorem 1.100 [29], there exists an $(m+1)$-cell $\mathfrak{Q}$ in $C(Y)$ such that $A \in \mathfrak{Q}$. This contradicts the fact that $m = \dim_A[C(Y)]$ and proves that A is locally connected.

Since $\dim_A[C(A)] \leq \dim_A[C(Y)] < \infty$, by [10, 1.1 and §7], A is a finite graph. Now we see that $\text{bd}_Y(A)$ contains at most m points. Suppose to the contrary that there exist pairwise different points $p_1, \dots, p_{m+1} \in \text{bd}_Y(A)$. Fix pairwise disjoint closed connected subsets $P_1, \dots, P_{m+1}$ of Y such that $p_i \in P_i^{\circ}$ for each $i \in \{1, \dots, m+1\}$. Let $B_1 = A \cup P_1 \cup \dots \cup P_{m+1}$. Then B_1 is a subcontinuum of Y such that $B - A$ has at least $m+1$ components. Again, this implies that there exists an $(m+1)$-cell $\mathfrak{Q}$ in $C(Y)$ such that $A \in \mathfrak{Q}$, a contradiction. Therefore, $\text{bd}_Y(A)$ contains at most m points. Let $\text{bd}_Y(A) = \{x_1, \dots, x_r\}$, where $r \leq m$. Notice that if α is an arc in

Y such that $A\cap\alpha \neq \emptyset$ and $\alpha - A \neq \emptyset$, then there exists $i \in \{1,\ldots,r\}$ such that $x_i \in A$.

For each $i \in \{1,\ldots,r\}$, let $k_i = \sup\{s \geq 1$: there exist arcs $\alpha_1,\ldots,\alpha_s$ in Y such that x_i is an end point of α_j and $A\cap\alpha_j = \{x_i\}$ for each $j \in \{1,\ldots,s\}$, and $\alpha_j\cap\alpha_l = \{x_i\}$ if $j \neq l\}$. In the case that $k_1+\ldots+k_r \geq m+1$, it is possible to construct a subcontinuum B_2 of Y such that $A \subset B_2$ and $B_2 - A$ has at least $m+1$ components, since this contradicts the choice of m, we have that $k_1+\ldots+k_r \leq m$.

Given $i \in \{1,\ldots,r\}$, let $\alpha_1,\ldots,\alpha_{k_i}$ be k_i arcs in Y satisfying the conditions described in the definition of k_i. We see that x_i is not an accumulation point of $R(Y)$. Suppose to the contrary that there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in $R(Y) - \{x_i\}$ such that $\lim z_n = x_i$ and $z_n \neq z_l$, if $n \neq l$. Since Y is locally connected, it is possible to choose a sequence of open connected sets $\{U_n\}_{n=1}^{\infty}$ such that $x_i, z_n \in U_n$ for each $n \geq 1$ and $\lim \mathrm{cl}_Y(U_n) = \{x_i\}$. Since A is a finite graph, we may assume that $z_n \notin A$ and $U_n \cap \{x_1,\ldots,x_r\} = \{x_i\}$ for each $n \geq 1$. For each $n \geq 1$, let β_n be an arc in U_n joining z_n and x_i. Notice that $\beta_n \cap A = \{x_i\}$. Thus, there exists $j \in \{1,\ldots,k_i\}$ such that $\beta_n \cap \alpha_j - \{x_i\} \neq \emptyset$. Thus, we may assume that $\beta_n \cap \alpha_1 - \{x_i\} \neq \emptyset$ for each $n \geq 1$. Since $z_n \in R(Y)$, it is possible to construct an arc γ_n in U_n such that $x_i \notin \gamma_n$ and $\gamma_n \cap \alpha_1$ is a one-point set. Since $\lim \mathrm{cl}_Y(U_n) = \{x_i\}$, we may assume that the arcs $\gamma_1, \gamma_2, \ldots$ are pairwise disjoint.

We are going to obtain a contradiction by showing that $\dim_A[C(Y)] = \infty$. Let $\mathfrak{U}$ be an open subset of $C(Y)$ such that $A \in \mathfrak{U}$, and if W is a subcontinuum of Y containing $A \cup \alpha_1$, then W does not belong to $\mathrm{cl}_{C(Y)}(\mathfrak{U})$. Since $\lim \gamma_n = \{x_i\}$, it is possible to choose a proper subarc α_0 of α_1 such that $x_i \in \alpha_0$, the set $M = \{n \geq 1 : \alpha_0 \cap \gamma_n \neq \emptyset\}$ is infinite and the subcontinua $B_0 = A \cup \alpha_0 \cup (\bigcup\{\gamma_n : n \in M\})$ and $A \cup \alpha_0$ of Y belongs to $\mathfrak{U}$. Notice that $B_0 \cup \alpha_1$ is a subcontinuum of Y, $B_0 \cup \alpha_1 \notin \mathfrak{U}$, and $(B_0\cup\alpha_1) - (A\cup\alpha_0) = (\alpha_1 - (A\cup\alpha_0)) \cup (\bigcup\{\gamma_n - (A\cup\alpha_0) : n \in M\})$ has infinitely many components. By an argument similar to that in the proof of Theorem 1.100 of [29], there exists a homeomorphic copy of the Hilbert cube $\mathfrak{Q}$ in $C(Y)$ such that $B_0 \cup \alpha_1, A \cup \alpha_0 \in \mathfrak{Q}$. Thus, $\mathfrak{Q} \cap \mathfrak{U} \neq \emptyset$ and $\mathfrak{Q} - \mathrm{cl}_{C(Y)}(\mathfrak{U}) \neq \emptyset$. This implies that $\dim(\mathrm{bd}_{C(Y)}(\mathfrak{U}))$ is infinite. Therefore, $\dim_A[C(Y)] = \infty$, a contradiction. This completes the proof that x_i is not an accumulation point of $R(Y)$.

Thus, we may assume that $\alpha_j \cap R(Y) \subset \{x_i\}$ for each $j \in \{1, \ldots, k_i\}$. Given $j \in \{1, \ldots, k_i\}$, let z_j be the end point of α_j such that $z_j \neq x_i$. Let W_i be a closed arcwise connected neighborhood of x_i in Y such that $W_i \cap \{x_1, \ldots, x_r\} = \{x_i\}$ and $W_i \cap (\{z_1, \ldots, z_{k_i}\} \cup (R(Y) - \{x_i\})) = \emptyset$. We claim that $W_i \subset A \cup \alpha_1 \cup \ldots \cup \alpha_{k_i}$. Suppose to the contrary that there exists a point $z \in W_i - (A \cup \alpha_1 \cup \ldots \cup \alpha_{k_i})$. Let $\gamma : [0,1] \to W_i$ be a one-to-one map such that $\gamma(0) = z$ and $\gamma(1) = x_i$, and let $t_0 = \min \gamma^{-1}(A \cup \alpha_1 \cup \ldots \cup \alpha_{k_i})$. If $\gamma(t_0) \in A$, then $\gamma(t_0) = x_i$; this contradicts the choice of $\alpha_1, \ldots, \alpha_{k_i}$, and if $\gamma(t_0) \notin A$, then $\gamma(t_0) \in R(Y) \cap W_i - \{x_i\}$, which contradicts the choice of W_i. Therefore, $W_i \subset A \cup \alpha_1 \cup \ldots \cup \alpha_{k_i}$.

Let $D = A \cup (\bigcup\{W_i : i \in \{1, \ldots, r\}\})$. Clearly, D is a finite graph and $A \subset D^{\mathrm{o}}$. This completes the proof of Claim 1.

Now we return to the proof of the lemma. Let $F \in C_2(Y)$ be such that $\dim_F[C_2(Y)]$ is finite. In the case that F is connected, since $\dim_F[C(Y)] \leq \dim_F[C_2(Y)]$, by the claim there exists a finite graph D contained in Y such that $F \subset D^{\mathrm{o}}$. Hence, $F \in \mathrm{int}_{C_2(Y)}(C_2(D))$.

Now suppose that F is disconnected. Let F_1 and F_2 be the components of F. Let Z_1 and Z_2 be disjoint subcontinua of Y such that $F_1 \subset Z_1^{\mathrm{o}}$ and $F_2 \subset Z_2^{\mathrm{o}}$. Since the map $\varphi : C(Z_1) \times C(Z_2) \to \langle Z_1, Z_2 \rangle$ given by $\varphi(B_1, B_2) = B_1 \cup B_2$ is a homeomorphism, we obtain that for each $i \in \{1, 2\}$, $\dim_{F_i}[C(Z_i)] \leq \dim_{(F_1,F_2)}[C(Z_1) \times C(Z_2)] = \dim_F[\langle Z_1, Z_2 \rangle] = \dim_F[C_2(Y)] < \infty$. Since $C(Z_i)$ is a neighborhood of F_i in $C(Y)$, $\dim_{F_i}[C(Y)] = \dim_{F_i}[C(Z_i)]$. By Claim 1, there exists a finite graph D_i contained in Y such that $F_i \subset D_i^{\mathrm{o}}$. We may assume that $D_i \subset Z_i$. Let α be an arc in Y such that $D_1 \cap \alpha$ and $D_2 \cap \alpha$ are one-point sets. Let $D_0 = D_1 \cup \alpha \cup D_2$. Then D_0 is a finite graph and $F \in \langle D_1^{\mathrm{o}}, D_2^{\mathrm{o}} \rangle \subset \mathrm{int}_{C_2(Y)} C_2(D_0)$.

This completes the proof of the lemma. □

3. Main result

Theorem 3.1. *Let $X \in \mathfrak{D}$ and let Y be a continuum. Suppose that $C_2(X)$ and $C_2(Y)$ are homeomorphic, then X and Y are homeomorphic.*

Proof: By [27, Theorem 3.2], Y is a locally connected continuum. According to [14, Theorem 13], we need only to check that Y is a dendrite. That is, we need to prove that Y does not contain simple closed curves. By the main result of [19], we may assume that Y is not a finite graph.

Claim 1. Let $p \in Y$ be such that $\{p\} \in \mathfrak{P}(Y)$. Then there exists a set $L \subset Y$ such that $L \in \mathfrak{A}_S(Y)$ and the component $\mathfrak{C}$ of $\mathfrak{P}(Y)$ containing $\{p\}$ is $\langle L^{\circ} \rangle$.

Proof of Claim 1: Since $C_2(Y)$ is locally connected ([27, Theorem 3.2] and $\mathfrak{P}(Y)$ is open $C_2(Y)$, $\mathfrak{C}$ is open in $C_2(Y)$. Since locally connected continua are Kelley continua, by [8, Theorem 3.5], the set $U = \bigcup \mathfrak{C}$ is an open subset of Y. Since $\mathfrak{C} \cap C(X) \neq \emptyset$, the set U is connected (see [29, Lemma 1.43]). Therefore, U is an open connected subset of Y. By [15, Lemma 2], U cannot contain a simple 3-od. It is easy to prove that an open connected subset, containing no simple 3-ods, of a locally connected continuum must be a 1-dimensional manifold. Thus, U is a simple closed curve or U is homeomorphic to one of the following sets: $[0,1]$, $[0,1)$, or $(0,1)$. If U is homeomorphic to $[0,1]$, by the connectedness of Y, $Y = U$. This is a contradiction since we are assuming that Y is not a finite graph. Similarly, U is not a simple closed curve. Thus, U is homeomorphic either to $[0,1)$ or to $(0,1)$. Let $\varphi : G \to U$ be a homeomorphism, where G is either $[0,1)$ or $(0,1)$.

Now, we check that $\mathrm{bd}_Y(U)$ has at most two points. Suppose to the contrary that there exists three different points a, b, and c in $\mathrm{bd}_Y(U)$. We may assume that $a, b \in \mathrm{cl}_Y(\varphi([\frac{1}{2}, 1)))$. Let V and W be two disjoint open connected (and then arcwise connected; see [30, Theorem 8.26]) subsets of Y such that $a \in V$, $b \in W$, and $\varphi(\frac{1}{2}) \notin V \cup W$. Fix points $v \in V \cap \varphi((\frac{1}{2}, 1))$ and $w \in W \cap \varphi((\frac{1}{2}, 1))$. Let $s, t \in (\frac{1}{2}, 1)$ be such that $v = \varphi(s)$ and $w = \varphi(t)$. We may assume that $s < t$. Let α be an arc in V joining a to v. Since U does not contain simple 3-ods and $\varphi(\frac{1}{2}) \notin \alpha$, $\alpha \cap \varphi([\frac{1}{2}, s]) = \{v\}$. Similarly, if β is an arc in W joining b to w, then $\beta \cap \varphi([\frac{1}{2}, t]) = \{w\}$. Therefore, the arc α intersects the arc $\varphi([\frac{1}{2}, t])$ and α does not contain any of the end points of $\varphi([\frac{1}{2}, t])$. This implies that $\varphi([\frac{1}{2}, t])$ contains the vertex of a simple 3-od. This is a contradiction since U does not contain vertices of simple 3-ods. We have shown that $\mathrm{bd}_Y(U)$ contains at most two elements.

In a similar way, it can be proved that in the case that U is homeomorphic to $[0,1)$, then $\mathrm{bd}_Y(U)$ contains exactly one point.

In this case, $\text{cl}_Y(U)$ is the one-point compactification of $[0,1)$. Thus, $\text{cl}_Y(U)$ is an arc L which is a free arc in Y and $L^{\circ}=U$. In this case, we have that $\mathfrak{C}\subset\langle L^{\circ}\rangle$. Given an element $A\in\langle L^{\circ}\rangle$, $A\subset U$. Since U is homeomorphic to $[0,1)$, there exists a subarc L_1 of L such that $A\subset L_1^{\circ}\subset L_1\subset L^{\circ}$. By [19, Lemma 2.2], $C_2(L_1)$ is a 4-cell and it is a neighborhood of A in $C_2(Y)$. We have shown that $\langle L^{\circ}\rangle\subset\mathfrak{P}(Y)$. It is easy to check that $\langle L^{\circ}\rangle$ is connected. Therefore, $\mathfrak{C}=\langle L^{\circ}\rangle$.

In the case that U is homeomorphic to $(0,1)$, $\text{cl}_Y(U)$ is either a one-point or a two-point compactification of U. Thus, the set $L = \text{cl}_Y(U)$ is either a simple closed curve or an arc. Hence, L is either a free circle or a free arc, and $\mathfrak{C}\subset\langle L^{\circ}\rangle$. Proceeding as in the paragraph above, $\mathfrak{C}=\langle L^{\circ}\rangle$.

In the case that L is a free arc, we prove that L is a maximal free arc. Let K be a free arc such that $L\subset K$. Clearly, $\langle K^{\circ}\rangle$ is contained in $\mathfrak{P}(Y)$, so $\langle L^{\circ}\rangle=\langle K^{\circ}\rangle$. This implies that $L=K$.

Then Claim 1 is proved.

Claim 2. For each $y\in R_f(Y)$, there exist $p,q\in R(X)$ such that $h(\{p,q\})=\{y\}$ (p could be equal to q).

Proof of Claim 2: Let Q be a neighborhood of y in Y such that Q is a simple n-od for some $n\geq 3$, and y is the vertex of Q. Let $K_1,\ldots,K_n$ be arcs in Y such that $K_1\cup\ldots\cup K_n\subset Q^{\circ}$, and $K_i\cap K_j=\{y\}$ if $i\neq j$ and y is an end point of each K_i. For each $i\in\{1,\ldots,n\}$, let z_i be the end point of K_i which is different from y. Choose a point $y_i\in K_i-\{z_i,y\}$ and let $P_i=K_i-\{z_i,y\}$. Clearly, P_i is an open connected subset of Y, P_i is homeomorphic to $(0,1)$, and $\langle P_i\rangle\subset\mathfrak{P}(Y)$. By Claim 1, there exists a set $L_i\subset Y$ such that $L_i\in\mathfrak{A}_S(Y)$ and the component $\mathfrak{C}_i$ of $\mathfrak{P}(Y)$ containing $\{y_i\}$ is $\langle L_i^{\circ}\rangle$.

We claim that there exist $i,j\in\{1,\ldots,n\}$ such that $i\neq j$ and $L_i\neq L_j$. Suppose to the contrary that $L_1=\cdots=L_n$. Let $\varphi: J\to L_1^{\circ}$ be a homeomorphism, where J is either $[0,1)$ or $(0,1)$. Given $i\in\{1,\ldots,n\}$, $\langle P_i\rangle\subset\langle L_i^{\circ}\rangle$. This implies that $P_i\subset L_i^{\circ}$. Thus, $\varphi^{-1}(P_1)$, $\varphi^{-1}(P_2)$, and $\varphi^{-1}(P_3)$ are connected subsets of J. We may assume that $\varphi^{-1}(P_2)$ is between $\varphi^{-1}(P_1)$ and $\varphi^{-1}(P_3)$. Thus, $\text{cl}_{(0,1)}(\varphi^{-1}(P_2))$ is a compact subset of J. Hence, $\text{cl}_Y(P_2)$ is a compact subset of L_1°. This implies that $y\in\text{cl}_Y(P_2)\subset L_1^{\circ}$. For each $i\in\{1,2,3\}$, let α_i be the subarc of K_i joining y and

y_i. Then it is possible to choose a simple 3-od T contained in $\alpha_1 \cup \alpha_2 \cup \alpha_3 - \{y_1, y_2, y_3\} \subset L_1^{\circ}$. This is a contradiction since L_1° is homeomorphic to J. Therefore, we may assume that $L_1 \neq L_2$.

If there is a point $w \in L_1^{\circ} \cap L_2^{\circ}$, then $\{w\} \in \langle L_1^{\circ} \rangle \cap \langle L_2^{\circ} \rangle$. Thus, $\langle L_1^{\circ} \rangle = \langle L_2^{\circ} \rangle$ (both are components of $\mathfrak{P}(Y)$). This implies that $L_1^{\circ} = L_2^{\circ}$ and $L_1 = L_2$, a contradiction. We have shown that $L_1^{\circ} \cap L_2^{\circ} = \emptyset$. Therefore, we have the following possibilities for $L_1 \cap L_2$.

(a) $L_1 \cap L_2$ is a one-point set. In this case, $\mathrm{cl}_{C_2(Y)}(\langle L_1^{\circ} \rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle L_2^{\circ} \rangle) = \{y\}$.

(b) $L_1 \cap L_2$ is a set with exactly two elements y and y_0. In this case, $\mathrm{cl}_{C_2(Y)}(\langle L_1^{\circ} \rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle L_2^{\circ} \rangle) = \{\{y\}, \{y_0\}, \{y, y_0\}\}$.

Since h is a homeomorphism, $h(\mathfrak{P}(X)) = \mathfrak{P}(Y)$. By [14, Lemma 7], there exist $I_1, I_2, J_1, J_2 \in \mathfrak{A}(X)$ such that $h(\langle I_1^{\circ}, I_2^{\circ} \rangle) = \langle L_1^{\circ} \rangle$ and $h(\langle J_1^{\circ}, J_2^{\circ} \rangle) = \langle L_2^{\circ} \rangle$. Therefore,

$$\mathrm{cl}_{C_2(X)}(\langle I_1^{\circ}, I_2^{\circ} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}, J_2^{\circ} \rangle) =$$

$$h^{-1}(\mathrm{cl}_{C_2(Y)}(\langle L_1^{\circ} \rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle L_2^{\circ} \rangle)$$

is a nonempty finite set. Since $\{I_1, I_2\} \neq \{J_1, J_2\}$, we may assume that $I_1 \notin \{J_1, J_2\}$. Since $\mathrm{cl}_{C_2(X)}(\langle I_1^{\circ}, I_2^{\circ} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}, J_2^{\circ} \rangle)$ is nonempty, $I_1 \cap (J_1 \cup J_2) \neq \emptyset$. Hence, $I_1 \cap (J_1 \cup J_2)$ has at most two elements. We consider two cases.

Case 1. $\{I_1, I_2\} \cap \{J_1, J_2\} = \emptyset$.

In this case, $(I_1 \cup I_2) \cap (J_1 \cup J_2)$ is a finite subset of $R(X)$. Let $A = h^{-1}(\{y\})$, then $A \in \mathrm{cl}_{C_2(X)}(\langle I_1^{\circ}, I_2^{\circ} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}, J_2^{\circ} \rangle)$, so $A \subset (I_1 \cup I_2) \cap (J_1 \cup J_2) \subset R(X)$. Thus, there exist $p, q \in R(X)$ such that $A = \{p, q\}$ and $h(\{p, q\}) = \{y\}$.

Case 2. $\{I_1, I_2\} \cap \{J_1, J_2\} \neq \emptyset$.

We will check that this case leads to a contradiction. We may assume that $I_2 = J_2$.

In the case that $I_1 \cap J_1 \neq \emptyset$, we have that $I_1 \cap J_1 = \{x\}$ for some $x \in R(X)$. Thus, for each subcontinuum C of X with $C \subset I_2^{\circ}$, we have that $\{x\} \cup C \in \mathrm{cl}_{C_2(X)}(\langle I_1^{\circ}, I_2^{\circ} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}, J_2^{\circ} \rangle)$. This is a contradiction since this set is finite.

In the case that $I_1 \cap J_1 = \emptyset$, we have that $I_1 \neq I_2$ and $J_1 \neq J_2$. Let $A \in \mathrm{cl}_{C_2(X)}(\langle I_1^{\circ}, I_2^{\circ} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}, J_2^{\circ} \rangle)$. This implies that $A = A_1 \cup A_2 = B_1 \cup B_2$ for some $A_1 \in C(I_1)$, $A_2 \in C(I_2)$, $B_1 \in C(J_1)$,

$B_2 \in C(J_2)$. Since $A_1 \cap B_1 = \emptyset$, $A_1 \subset B_2$. Thus, $A_1 \subset I_1 \cap I_2$ and $A_1 = \{x_1\}$, for some $x_1 \in R(X)$. Similarly, $B_1 = \{x_2\} = J_1 \cap J_2$ and $x_2 \in R(X) \cap A_2$. Since $x_1 \neq x_2$, x_1 and x_2 are the end points of I_2. If A is not connected, then the components of A are, on the one hand, $\{x_1\}$ and A_2; and, on the other hand, they are $\{x_2\}$ and B_2. This implies that $\{x_1\} = B_2$ and $A = \{x_1, x_2\}$. If A is connected, then $x_1, x_2 \in A_2 \subset I_2$. Thus, $A = A_2 = I_2$. We have shown that $\mathrm{cl}_{C_2(X)}(\langle I_1^{\mathrm{o}}, I_2^{\mathrm{o}}\rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\mathrm{o}}, J_2^{\mathrm{o}}\rangle) \subset \{\{x_1, x_2\}, I_2\}$. The opposite inclusion is clear; thus, $\mathrm{cl}_{C_2(X)}(\langle I_1^{\mathrm{o}}, I_2^{\mathrm{o}}\rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_1^{\mathrm{o}}, J_2^{\mathrm{o}}\rangle)$ is a set containing exactly two elements. This is a contradiction since the number of elements of the set $\mathrm{cl}_{C_2(Y)}(\langle L_1^{\mathrm{o}}\rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle L_2^{\mathrm{o}}\rangle$ can be only one or three (by (a) and (b)).

This finishes the proof of Claim 2.

Claim 3. Let L be an arc in Y and let a and b be the end points of L. If $p, q \in L - (L^{\mathrm{o}} \cup \{a, b\})$ and $p \neq q$, then $\{p, q\} \cap \mathrm{cl}_Y(R_f(Y)) \neq \emptyset$.

Proof of Claim 3: By [15, 3.16], since $C_2(X)$ is homeomorphic to $C_2(Y)$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $C_2(Y)$ such that $\lim A_n = L$ and $\dim_{A_n}[C_2(Y)]$ is finite for each $n \geq 1$. Let d be a metric for Y. We fix an order in the arc L such that $a < b$, and we assume that $p < q$.

Let $\varepsilon > 0$. Let $m \geq 1$ and $L_1, \ldots, L_m$ be subarcs of L such that $a \in L_1$, $b \in L_m$, $\mathrm{diameter}(L_i) < \min\{\frac{d(p,q)}{5}, \varepsilon\}$ for each $i \in \{1, \ldots, m\}$, and $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Since Y is locally connected, there exist open connected subsets $U_1, \ldots, U_m$ of Y such that $L_i \subset U_i$ and $\mathrm{diameter}(U_i) < \min\{\frac{d(p,q)}{5}, \varepsilon\}$ for each $i \in \{1, \ldots, m\}$, and $\mathrm{cl}_Y(U_i) \cap \mathrm{cl}_Y(U_i) \neq \emptyset$ if and only if $|i - j| \leq 1$.

Fix $n \geq 1$ such that $A_n \in \langle U_1, \ldots, U_m\rangle$. Suppose that $p \in U_{i_1}$ and $q \in U_{i_2}$. Notice that $i_1 + 4 \leq i_2$. Fix points $u \in A_n \cap U_1$, $v \in A_n \cap U_{i_1+2}$, and $w \in A_n \cap U_m$. Since $A_n \in C_2(Y)$, A_n has at most two components. Thus, without loss of generality, we may assume that there exists a subcontinuum B of A_n such that $u, v \in B$. By Lemma 2.2, there exists a finite graph D contained in Y such that $A_n \in \mathrm{int}_{C_2(Y)}(C_2(D))$. In particular, $B \subset D$, so B is a finite graph. Let γ be an arc in B which joins u and v.

We show that U_{i_1} is not contained in γ. Suppose to the contrary that $U_{i_1} \subset \gamma$. Let $c, e \in L$ be such that $a < c < p < e < b$ and the subarc $[c, e]$ of L that joins c and e has the property that

$[c,e] \subset U_{i_1}$. Then $[c,e]$ is a subarc of γ, with end points c and e, and the set $K = \gamma - ([c,e] - \{c,e\})$ is a compact subset of γ. So $U_{i_1} - K$ is an open subset of Y containing p. Notice that $U_{i_1} - K \subset [c,e] - \{c,e\} \subset L$. This is a contradiction since $p \notin L^o$.

Therefore, $U_{i_1} - \gamma \neq \emptyset$. Notice that $\gamma \cap U_{i_1} \neq \emptyset$. Since U_{i_1} is arcwise connected (see [30, Theorem 8.26]), there exists an arc β in U_{i_1} such that $\beta \cap \gamma$ is a one-point set. Therefore, there exists a point $x_0 \in R(Y) \cap \gamma \cap U_{i_1}$.

We see that $x_0 \in R_f(Y)$. Since $A_n \in \mathrm{int}_{C_2(Y)}(C_2(D))$ and $x_0 \in A_n$, there exists an open connected subset V of Y such that $x_0 \in V$ and $A_n \cup \mathrm{cl}_Y(V) \subset D$. Since D is a finite graph, we may assume that $\mathrm{cl}_Y(V) \cap R(D) \subset \{x_0\}$. These conditions imply that $V \cap R(Y) = \{x_0\}$. Hence, $x_0 \in R_f(Y)$.

We have shown that for each $\varepsilon > 0$, there exists a point $x \in R_f(Y)$ such that $d(x,p) < \varepsilon$ or $d(x,q) < \varepsilon$. This implies that $\{p,q\} \cap \mathrm{cl}_Y(R_f(Y)) \neq \emptyset$.

So Claim 3 is proved.

Claim 4. Each arc in Y contains a free arc of Y.

Proof of Claim 4: Suppose to the contrary that there exists an arc L in Y such that every subarc of L has empty interior in Y. Let a and b be the end points of L.

First we show that $L \subset \mathrm{cl}_Y(R_f(Y))$. Let $p \in L - \{a,b\}$ and let $\varepsilon > 0$. Let $q \in L - (\{a,b,p\})$ be such that $d(p,q) < \frac{\varepsilon}{2}$. By Claim 3, $\{p,q\} \cap \mathrm{cl}_Y(R_f(Y)) \neq \emptyset$. Thus, there exists $x \in R_f(Y)$ such that $d(x,p) < \varepsilon$. This proves that $L \subset \mathrm{cl}_Y(R_f(Y))$.

Given $y \in L$, by Claim 2, there exists sequences $\{p_n^y\}_{n=1}^{\infty}$ and $\{q_n^y\}_{n=1}^{\infty}$ in $R(X)$ such that $\lim h(\{p_n^y, q_n^y\}) = \{y\}$. We may assume that $\lim p_n^y = p_y$ and $\lim q_n^y = q_y$ for some $p_y, q_y \in X$. Then $h(\{p_y, q_y\}) = \{y\}$. We also may assume that either $\{p_n^y\}_{n=1}^{\infty}$ is constant or the points $p_1^y, p_2^y, \ldots$ are pairwise different, and something similar happens with the sequence $\{q_n^y\}_{n=1}^{\infty}$.

We show that for each $y \in L$, $\{p_y, q_y\} \not\subset R(X)$. Suppose to the contrary that there exists $z \in L$ such that $\{p_z, q_z\} \subset R(X)$. Since $X \in \mathfrak{D}$, by [14, Lemma 6(b)], there exist open connected subsets U and V in X such that $p_z \in U$, $q_z \in V$, $\mathrm{cl}_X(U) \cap R(X) = \{p_z\}$, and $\mathrm{cl}_X(V) \cap R(X) = \{q_z\}$. Let $L_1 = \{y \in L : h^{-1}(\{y\}) \subset U \cup V\}$ and $L_2 = \{y \in L : h^{-1}(\{y\}) \cap (X - (U \cup V)) \neq \emptyset\}$. Clearly, L_2 is a closed subset of L. Given $y \in L_1$, $h^{-1}(\{y\}) \subset U \cup V$. Thus, there exists $N \geq 1$ such that $p_n^y, q_n^y \in (U \cup V) \cap R(X) = \{p_z, q_z\}$, for each $n \geq N$. This implies that $\{p_y, q_y\} \subset \{p_z, q_z\}$. We have seen that $L_1 = \{y \in L : h^{-1}(\{y\}) \subset \{p_z, q_z\}\}$. Hence, L_1 is a closed subset of L. By the connectedness of L, we obtain that $L = L_1$, so L is finite, a contradiction. This completes the proof that for each $y \in L$, $\{p_y, q_y\} \nsubseteq R(X)$.

We may assume that for each $y \in L$, $p_y \notin R(X)$. Thus, the points $p_1^y, p_2^y, \ldots$ are pairwise different elements of $R(X)$. This implies that p_y is an accumulation point of the set $R(X)$. It is easy to show that this implies that p_y is an accumulation point of the set $E(X)$. Since $X \in \mathfrak{D}$, $p_y \in E_a(X)$. We analyze two cases.

Case 1. There exists $z \in L$ such that $q_z \in R(X)$.

Since $X \in \mathfrak{D}$, by [14, Lemma 6(b)], there exists an open connected subset V of X such that $\mathrm{cl}_X(V) \cap R(X) = \{q_z\}$. Let $L_3 = \{y \in L : h^{-1}(\{y\}) \cap V \neq \emptyset\}$ and $L_4 = \{y \in L : h^{-1}(\{y\}) \cap V = \emptyset\}$. Clearly, L_4 is a closed subset of L. Given $y \in L_3$, since p_y is an accumulation point of the set $R(X)$, $p_y \notin V$. Thus, $q_y \in V$. Hence, the sequence $\{q_n^y\}_{n=1}^{\infty}$ is constant. Therefore, $q_y = q_z$. This proves that $L_3 = \{y \in L : h^{-1}(\{y\}) \cap \{q_z\} \neq \emptyset\}$. We have shown that L_3 is also closed in L. By the connectedness of L, $L_3 = L$. Thus, in this case, for each $y \in L$, $h^{-1}(\{y\}) = \{q_z, p_y\}$ and $p_y \in E_a(X)$. In particular, $p_y \neq q_y$, and the function $y \to p_y$ from L into $E_a(X)$ is an embedding. This is a contradiction since $E_a(X)$ does not contain arcs.

Case 2. For each $y \in L$, $q_y \notin R(X)$.

In this case, for each $y \in L$, the points $q_1^y, q_2^y, \ldots$ are pairwise different elements of $R(X)$. It can be proved that q_y is an accumulation point of elements in $E(X)$. Since $X \in \mathfrak{D}$, $q_y \in E_a(X)$. Thus, $h^{-1}(\{y\}) = \{p_y, q_y\} \subset E_a(X)$. By [9, Lemma 2.2] and [8, Lemma 2.1], the set $W = \bigcup\{h^{-1}(\{y\}) : y \in L\}$ is a locally connected subset of X with at most two components. Since L is an arc, W has a nondegenerate component W_1 which is a locally connected continuum. Therefore, there exists an arc $\lambda \subset W_1 \subset W \subset E_a(X)$, again a contradiction.

This finishes the proof of Claim 4.

Now we prove that Y does not contain simple closed curves. Suppose to the contrary that there exists a simple closed curve

S contained in Y. By Claim 4, there exists a free arc L contained in S. Let p_0 and p_6 be the end points of L. Consider the natural order $<$ in L for which $p_0 < p_6$. Fix points $p_0 < p_1 < p_2 < p_3 < p_4 < p_5 < p_6$. Let L_0 be the subarc of S such that p_0 and p_6 are the end points of L_0, and $L \cap L_0 = \{p_0, p_6\}$. Let L_2 (L_3, L_4, respectively) be the subarc of S with end points p_0 and p_1 (p_2 and p_4, p_5 and p_6, respectively), and satisfying $L_2 \cup L_3 \cup L_4 \subset L$. Let $M = L_2 \cup L_0 \cup L_4$. Choose an open connected subset U of Y such that $M \subset U$ and $\mathrm{cl}_X(U) \cap L_3 = \emptyset$. Let $\mathfrak{U} = \langle U, L_2 - \{p_0, p_1\}, L_4 - \{p_5, p_6\}, L_3 - \{p_2, p_4\}\rangle$, then $\mathfrak{U}$ is an open subset of $C_2(Y)$. Notice that $M \cup \{p_3\} \in \mathfrak{U}$. By [14, 3.16], there exists an element $A \in \mathfrak{U}$ such that $\dim_A[C_2(Y)]$ is finite. Thus, A is contained and intersects each one of the disjoint sets U and $L_3 - \{p_2, p_4\}$. Hence, A has two components: $A_1 \subset U$ and $A_2 \subset L_3 - \{p_2, p_4\}$. Notice that $A_1 \in \langle U, L_2 - \{p_0, p_1\}, L_4 - \{p_5, p_6\}\rangle$ and $A_1 \cap L_3 = \emptyset$. Since $A_1 \cap L_2 - \{p_0, p_1\} \neq \emptyset \neq A_1 \cap L_4 - \{p_5, p_6\}$, there exists a subarc K of L such that $L_3 \subset K$, $K \cap A_1 \cap L_2 - \{p_0, p_1\}$ and $K \cap A_1 \cap L_4 - \{p_5, p_6\}$ are the one-point sets $\{q_1\}$ and $\{q_2\}$, respectively, the end points of K are q_1 and q_2, and $K \cap A_1 = \{q_1, q_2\}$.

By Lemma 2.2, there exists a finite graph D contained in Y such that $A \in \mathrm{int}_{C_2(Y)}(C_2(D))$. In particular, $A_1 \subset D$, so A_1 is a finite graph. Thus, there exists an arc γ in A_1 such that γ joins q_1 and q_2. Notice that the set $T = K \cup \gamma$ is a simple closed curve.

CLAIM 5. $T \cap R(Y) \subset R_f(Y) \cap R(D)$. In particular, $T \cap R(Y)$ is finite.

Proof of Claim 5: Let $p \in T \cap R(Y)$. Since $K \subset L - \{p_0, p_6\}$ and L is a free arc, $p \notin K$. Thus, $p \in \gamma \subset A$. Since $A \in \mathrm{int}_{C_2(Y)}(C_2(D))$, there exists an open connected subset V of Y such that $p \in V$ and $A \cup \mathrm{cl}_Y(V) \subset D$. We may assume that $\mathrm{cl}_Y(V) \cap R(D) \subset \{p\}$. This implies that $p \in R(D)$ and that p has a neighborhood Q in Y such that Q is a simple n-od for some $n \geq 3$ and p is the vertex of Q. Therefore, $p \in R_f(Y) \cap R(D)$.

This proves Claim 5.

Since Y is not a finite graph, $T \neq Y$. Thus, $T \cap R(Y) \neq \emptyset$. We analyze two cases.

Case 1. $T \cap R(Y) = \{z_0\}$, for some $z_0 \in Y$.

Since $z_0 \in R_f(Y)$, there exists a neighborhood Z_0 of z_0 in Y such that Z_0 is a simple n_0-od, for some $n_0 \geq 3$, and z_0 is the vertex of Z_0. We may assume that $Z_0 \cap R(Y) = \{z_0\}$.

By Claim 2, there exist $w_0, q_0 \in R(X)$ such that $h(\{w_0, q_0\}) = \{z_0\}$. Since $X \in \mathfrak{D}$, by [14, Lemma 6(b)], there exist $r_0, s_0 \geq 3$ and neighborhoods P_0 and Q_0 of w_0 and q_0, respectively, in X such that P_0 is a simple r_0-od, Q_0 is a simple s_0-od, and w_0 and q_0 are the respective vertices of P_0 and Q_0. Here, we consider two subcases.

Subcase 1.1. $w_0 \neq q_0$.

Here, we may assume that $P_0 \cap Q_0 = \emptyset$ and $h(\langle P_0, Q_0 \rangle) \subset \langle Z_0^{\circ} \rangle$. Let η be an arc in X such that $\eta \cap P_0$ and $\eta \cap Q_0$ are one-point sets. Then $G = P_0 \cup \eta \cup Q_0$ is a finite graph. Fix a point $u \in P_0^{\circ} - \{w_0\}$. Then $\{q_0, u\}, \{w_0, q_0\} \in \mathrm{int}_{C_2(X)}(C_2(G)) \cap \langle P_0, Q_0 \rangle$. Thus, by [14, Lemma 1], $\dim_{\{q_0,u\}}[C_2(X)] = \dim_{\{q_0,u\}}[C_2(G)] = s_0 + 2$, $\dim_{\{w_0,q_0\}}[C_2(X)] = \dim_{\{w_0,q_0\}}[C_2(G)] = r_0 + s_0$, and $r_0 + s_0 = \dim_{\{z_0\}}[C_2(Y)] = \dim_{\{z_0\}}[C_2(Z_0)] = n_0 + 2$. Hence, $r_0 + s_0 = n_0 + 2$ and there exists an element $B = h(\{q_0, u\}) \in \langle Z_0^{\circ} \rangle$ such that $\dim_B[C_2(Y)] = s_0 + 2$. Again, applying Lemma 1 of [14], we obtain that $\dim_B[C_2(Y)] = n_0 + 2$ if $z_0 \in B$, and $\dim_B[C_2(Y)] = 4$ if $z_0 \notin B$. Thus, $s_0 + 2 \in \{4, n_0 + 2\}$. This is impossible since $4 < s_0 + 2 < r_0 + s_0 = n_0 + 2$.

Subcase 1.2. $w_0 = q_0$.

Let $Z_1 = Z_0 - \{z \in Z_0 : z \text{ is an end point of } Z_0\}$. Notice that Z_1 is an open subset of Y. Let $\mathfrak{C}$ be a component of $\mathfrak{P}(Y)$ such that $\{z_0\} \in \mathrm{cl}_{C_2(Y)}(\mathfrak{C})$. Since $\{z_0\} \in \langle Z_1 \rangle$, there exists an element $B \in \langle Z_1 \rangle \cap \mathfrak{C}$. By [14, Lemma 2], $z_0 \notin B$. Taking an order arc (see [29, Theorem 1.25]), we can reduce B to either a one-point set or a set of two elements. Thus, we may assume that $B = \{b_1, b_2\}$ for some $b_1, b_2 \in Z_1 - \{z_0\}$. Notice that $\{b_1\}, \{b_2\} \in \mathfrak{P}(Y)$. By Claim 1, there exist $L_1, L_2 \in \mathfrak{A}_S(Y)$ such that $b_1 \in L_1^{\circ}$ and $b_2 \in L_2^{\circ}$. By Lemma 2.1, $\mathfrak{C} = \langle L_1^{\circ}, L_2^{\circ} \rangle$.

On the other hand, it is easy to check that if $I_1, I_2 \in \mathfrak{A}_S(Y)$ and $z_0 \in I_1 \cap I_2$, then $\langle I_1^{\circ}, I_2^{\circ} \rangle$ is a component of $\mathfrak{P}(Y)$ such that $z_0 \in \mathrm{cl}_{C_2(Y)}(\langle I_1^{\circ}, I_2^{\circ} \rangle)$.

Let $I_1, \ldots, I_{k_0}$ be the maximal free arcs in Y containing z_0 and let $L_1, \ldots, L_{m_0}$ be the free circles in Y containing z_0. Notice that $m_0 \geq 1$ and $n_0 = k_0 + 2m_0$. From the two paragraphs above, the number of components of $\mathfrak{P}(Y)$ containing $\{z_0\}$ in its closure is $k_0 + m_0 + \frac{(k_0+m_0)(k_0+m_0-1)}{2}$. Similarly, since X is a dendrite, the number of components of $\mathfrak{P}(X)$ containing $\{w_0\}$ in its closure is $r_0 + \frac{(r_0)(r_0-1)}{2}$. Thus, $r_0 + \frac{(r_0)(r_0-1)}{2} = k_0 + m_0 + \frac{(k_0+m_0)(k_0+m_0-1)}{2}$ and $r_0 = k_0 + m_0$.

By [14, Lemma 1], $\dim_{\{z_0\}}[C_2(Y)] = \dim_{\{z_0\}}[C_2(Z_0)] = n_0 + 2$ and $\dim_{\{w_0\}}[C_2(X)] = \dim_{\{w_0\}}[C_2(P_0)] = r_0 + 2$. This implies that $n_0 = r_0$. This contradicts the facts that $r_0 = k_0 + m_0$ and $n_0 = k_0 + 2m_0$.

We have shown that Case 1 is impossible.

Case 2. $T \cap R(Y) = \{z_1, \ldots, z_m\}$, where $m \geq 2$.

We may assume that there exist subarcs $T_1, \ldots, T_m$ of T such that for each $i \in \{1, \ldots, m\}$, T_i joins z_i and z_{i+1} (where $z_{m+1} = z_1$) and $T_i \cap \{z_1, \ldots, z_m\} = \{z_i, z_{i+1}\}$. Notice that for each $i \in \{1, \ldots, m\}$, T_i is a free arc joining two points in $R_f(Y)$ (by Claim 5). This implies that T_i is a maximal free arc and then $T_i \in \mathfrak{A}_{R_f}(Y)$. By [15, Theorem 10], there exist subarcs $J_1, \ldots, J_m \in \mathfrak{A}_{R_f}(X)$ such that $h(\langle J_i^{\mathrm{o}} \rangle) = \langle T_i^{\mathrm{o}} \rangle$ for each $i \in \{1, \ldots, m\}$. Let $T_{m+1} = T_1$ and $J_{m+1} = J_1$. We consider two subcases.

Subcase 2.1. $m \geq 3$.

Given $i \in \{1, \ldots, m\}$, $T_i \cap T_{i+1} = \{z_i\}$. This implies that $\mathrm{cl}_{C_2(Y)}(\langle T_i^{\mathrm{o}} \rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle T_{i+1}^{\mathrm{o}} \rangle) = \{\{z_i\}\}$. Thus, $\mathrm{cl}_{C_2(X)}(\langle J_i^{\mathrm{o}} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_{i+1}^{\mathrm{o}} \rangle)$ is a one-point set. This implies that $J_i \cap J_{i+1} \neq \emptyset$ and $J_i \neq J_{i+1}$. Thus, $J_i \cap J_{i+1} = \{x_i\}$ for some $x_i \in X$. If there exists $j \in \{1, \ldots, m\}$ such that $J_j \notin \{J_i, J_{i+1}\}$ and $x_i \in J_j$, then

$$\{x_i\} \in \mathrm{cl}_{C_2(X)}(\langle J_i^{\mathrm{o}} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_{i+1}^{\mathrm{o}} \rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_j^{\mathrm{o}} \rangle).$$

This implies that

$$\mathrm{cl}_{C_2(Y)}(\langle T_i^{\mathrm{o}} \rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle T_{i+1}^{\mathrm{o}} \rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle T_j^{\mathrm{o}} \rangle) \neq \emptyset.$$

Thus, $T_i \cap T_{i+1} \cap T_j \neq \emptyset$, a contradiction. Hence, for each $j \in \{1, \ldots, m\}$, such that $J_j \notin \{J_i, J_{i+1}\}$, $x_i \notin J_j$. This implies that $J_1 \cup \ldots \cup J_m$ is a simple closed curve in X. This contradicts the fact that X is a dendrite and proves that this case is impossible.

Subcase 2.2. $m = 2$.

Here, $\mathrm{cl}_{C_2(Y)}(\langle T_1^{\circ}\rangle) \cap \mathrm{cl}_{C_2(Y)}(\langle T_2^{\circ}\rangle) = \{\{z_1\}, \{z_2\}, \{z_1, z_2\}\}$. This implies that $\mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}\rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_2^{\circ}\rangle)$ is nonempty and finite. Thus, $J_1 \cap J_2 \neq \emptyset$ and $J_1 \neq J_2$. Since X is a dendrite, $J_1 \cap J_2 = \{x\}$, for some $x \in X$. Thus, $\mathrm{cl}_{C_2(X)}(\langle J_1^{\circ}\rangle) \cap \mathrm{cl}_{C_2(X)}(\langle J_2^{\circ}\rangle) = \{x\}$. Hence, $h(\{\{x\}\}) = \{\{z_1\}, \{z_2\}, \{z_1, z_2\}\}$, a contradiction.

This concludes the analysis of Case 2.

And the proof of Theorem 3.1 is complete. □

Problem 3.2. *Let* $\mathfrak{UH} = \{X : X$ *is a locally connected continuum for which the hyperspace* $C_n(X)$ *is unique for each* $n \geq 1\}$. *Find a characterization of the elements of the class* $\mathfrak{UH}$.

By the main result of [20] and Theorem 3.1, $\mathfrak{UH}$ includes the finite graphs (different from an arc and a simple closed curve) and the elements of $\mathfrak{D}$. By the main result of [15], dendrites which are not in $\mathfrak{D}$ do not belong to $\mathfrak{UH}$.

References

[1] Gerardo Acosta, *On compactifications of the real line and unique hyperspace*, Topology Proc. **25** (2000), Spring, 1–25.

[2] ______, *Continua with unique hyperspace*, in Continuum Theory. Lecture Notes in Pure and Applied Mathematics, 230. New York: Dekker, 2002. 33–49

[3] ______, *Continua with almost unique hyperspace*, Topology Appl. **117** (2002), no. 2, 175–189.

[4] ______, *On smooth fans and unique hyperspaces*, Houston J. Math. **30** (2004), no. 1, 99–115.

[5] Gerardo Acosta, Janusz J. Charatonik, and Alejandro Illanes, *Irreducible continua of type* λ *with almost unique hyperspace*, Rocky Mountain J. Math. **31** (2001), no. 3, 745–772.

[6] Gerardo Acosta, Rodrigo Hernández-Gutiérrez, and Verónica Martínez-de-la-Vega, *Dendrites and symmetric products*. To appear in Glasnik Matematicki.

[7] Enrique Castañeda and Alejandro Illanes, *Finite graphs have unique symmetric products*, Topology Appl. **153** (2006), no. 9, 1434–1450.

[8] Janusz J. Charatonik and Alejandro Illanes, *Local connectedness in hyperspaces*, Rocky Mountain J. Math. **36** (2006), no. 3, 811–856.

[9] Doug Curtis and Nguyen To Nhu, *Hyperspaces of finite subsets which are homeomorphic to* $\aleph_0$*-dimensional linear metric spaces*, Topology Appl. **19** (1985), no. 3, 251–260.

[10] R. Duda, *On the hyperspace of subcontinua of a finite graph, I*, Fund. Math. **62** (1968), 265–286.

[11] ______, *Correction to the paper: "On the hyperspace of subcontinua of a finite graph, I,"* Fund. Math. **69** (1970), 207–211.

[12] Carl Eberhart and Sam B. Nadler, Jr., *Hyperspaces of cones and fans*, Proc. Amer. Math. Soc. **77** (1979), no. 2, 279–288.

[13] David Herrera-Carrasco, *Dendrites with unique hyperspace*, Houston J. Math. **33** (2007), no. 3, 795–805.

[14] David Herrera-Carrasco, Alejandro Illanes, María de J. López, and Fernando Macías-Romero, *Dendrites with unique hyperspace $C_2(X)$*, Topology Appl. **156** (2009), no. 3, 549–557.

[15] David Herrera-Carrasco and Fernando Macías-Romero, *Dendrites with unique n-fold hyperspace*, Topology Proc. **32** (2008), 301–320.

[16] Alejandro Illanes, *Chainable continua are not C-determined*, Topology Appl. **98** (1999), no. 1-3, 211–216.

[17] ______, *Fans are not C-determined*, Colloq. Math. **81** (1999), no. 2, 299–308.

[18] ______, *Dendrites with unique hyperspace $F_2(X)$*, JP J. Geom. Topol. **2** (2002), no. 1, 75–96.

[19] ______, *The hyperspace $C_2(X)$ for a finite graph X is unique*, Glas. Mat. Ser. III **37(57)** (2002), no. 2, 347–363.

[20] ______, *Finite graphs X have unique hyperspaces $C_n(X)$*, Topology Proc. **27** (2003), no. 1, 179–188.

[21] ______, *Hyperspaces of continua*, in Open Problems in Topology II. Ed. Elliott Pearl. Amsterdam–Boston: Elsevier, 2007. 279-288

[22] Alejandro Illanes and Jorge M. Martínez-Montejano, *Compactifications of $[0,\infty)$ with unique hyperspace $F_n(X)$*. To appear in Glasnik Matematicki.

[23] Alejandro Illanes and Sam B. Nadler, Jr. *Hyperspaces: Fundamentals and Recent Advances*. Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[24] Ivan Lončar, *Non-metric rim-metrizable continua and unique hyperspace*, Publ. Inst. Math. (Beograd) (N.S.) **73(87)** (2003), 97–113.

[25] Sergio Macías, *On C-determined continua*, Glas. Mat. Ser. III **32(52)** (1997), no. 2, 259–262.

[26] ______, *Hereditarily indecomposable continua have unique hyperspace 2^X*, Bol. Soc. Mat. Mexicana (3) **5** (1999), no. 2, 415–418.

[27] ______, *On the hyperspaces $C_n(X)$ of a continuum X*, Topology Appl. **109** (2001), no. 2, 237–256.

[28] Verónica Martínez-de-la-Vega, *Dimension of n-fold hyperspaces of graphs*, Houston J. Math. **32** (2006), no. 3, 783–799.

[29] Sam B. Nadler, Jr., *Hyperspaces of Sets: A Text with Research Questions.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49. New York-Basel: Marcel Dekker, Inc., 1978.

[30] ———, *Continuum Theory: An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics, 158. New York: Marcel Dekker, Inc., 1992.

Instituto de Matemáticas; Universidad Nacional Autónoma de México; Área de la Investigación Científica; Circuito Exterior, Ciudad Universitaria; 04510, México, D.F. México

E-mail address: `illanes@matem.unam.mx`