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GENERALIZED METRIC SPACES AND DEVELOPABLE SPACES

YOSHIO TANAKA

ABSTRACT. We survey generalized metric spaces around symmetric spaces, quasi-metric spaces, and developable spaces.

As is well-known, a space X is *metrizable* (or *metric*), if X has a *metric* d, that is, a non-negative real valued function d on X satisfying the following conditions:

(a) $d(x, y) = 0$ iff $x = y$.

(b) $d(x, y) = d(y, x)$ (symmetry).

(c) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

(d) $G \subset X$ is open in X if and only if, for each $x \in G$, some *ball* (or *sphere*) $B_d(x; \epsilon) \subset G$. Here, $B_d(x; \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

A non-negative function d on a space X is an *o-metric* if d satisfies (a) and (d) [24]. For an o-metric d, d is *symmetric* [3] if d satisfies (b), and d is *quasi-metric* (or Δ*-metric* [24]) if d satisfies (c). Clearly, an o-metric d is metric if and only if d is symmetric and quasi-metric. A symmetric d is *semi-metric* if whenever $x \in clA$, $d(x, A) = 0$. An o-metric d is *non-archimedean quasi-metric* (briefly, *n.a.-quasi-metric*) if $d(x, z) \leq Max\,\{d(x, y), d(y, z)\}$ (stronger than (c)).

In [24] and [25], a space X is *o-metrizable* if X has an o-metric d (which is compatible with the topology in X), and such a space is also called *generalized metrizable* in [25]. For general definitions

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(which are not precise) of "generalized metric spaces," see [10]. A space is *symmetrizable*, *semi-metrizable*, *quasi-metrizable*, or *n.a.-quasi-metrizable* if it has a respective o-metric. For these spaces, see [3], [10], [19], [24], [25], and others. In this paper, let us say that a space is "symmetric," "quasi-metric," etc., instead of "symmetrizable," "quasi-metrizable," etc.

For a quasi-metric space (X, d), any $B_d(x; \epsilon)$ is open in X. But, for a symmetric space (X, d), every $B_d(x; \epsilon)$ need not be open. Every semi-metric or quasi-metric space is first countable, but not every symmetric space is first countable.

In this paper, we survey generalized metric spaces around symmetric spaces, quasi-metric spaces, and developable spaces.

We assume that all spaces are regular T_1, and all maps are continuous surjections.

The following basic fact about an o-metric is shown by the proof of [10, Lemma 9.3].

Lemma 1. *Let (X, d) be an o-metric. For a sequence $L = \{x_n : n \in N\}$ in X, L converges to $x \Leftrightarrow d(x, x_n) \to 0 \Leftrightarrow$ any $B_d(x; \epsilon)$ contains L eventually (i.e., $L - B_d(x; \epsilon)$ is at most finite).*

A space X is *sequential* if $A \subset X$ is open in X whenever every sequence converging to a point in A is eventually in A. A space X is *Fréchet* (or *Fréchet-Urysohn*) if whenever $x \in clA$, there exists a sequence in A converging to x. Every Fréchet space is sequential.

Remark 1. (1) A space X satisfies the *weak first axiom of countability* [3] (briefly, X is *g-first countable* [28]) if, to each $x \in X$, one can assign a decreasing sequence $\{Q_n(x) : n \in N\}$ of subsets such that $x \in Q_n(x)$, and $G \subset X$ is open in X if and only if for each $x \in G$, some $Q_n(x) \subset G$. A space is g-first countable if and only if it is an o-metric [25]. Also, a space is first countable if and only if it has an o-metric with any ball open. But, every semi-metric space need not have a symmetric with any ball open [12].

(2) Every o-metric space is sequential. Precisely, let (X, d) be an o-metric (symmetric, respectively); then for $S \subset X$, $(S, d|S)$ is so if and only if S is sequential by means of Lemma 1. Note that not every subset of a symmetric space is necessarily an o-metric (by Example 3(2) later), though any subset of a semi-metric space ((n.a.-) quasi-metric space, respectively) is so.

An o-metric (symmetric, respectively) d is a *strong o-metric* [24] (*strong symmetric*, respectively) if any $intB_d(x;\epsilon) \ni x$. A space is *strongly o-metric* (*strongly symmetric*, respectively) if it has a strong o-metric (strong symmetric, respectively).

The following theorem holds, using Lemma 1; see, for example, [10]. ((a) $\Leftrightarrow$ (c) in (1) is due to [31] (in (2), due to [25], respectively)).

Theorem 1. (1) *For a space X, the following are equivalent.*

(a) *X is semi-metric.*
(b) *X is Fréchet symmetric.*
(c) *X is hereditarily symmetric.*
(d) *X is strongly symmetric.*
(e) *X is symmetric, and for any symmetric d on X, d is a strong symmetric.*

(2) *A space X is first countable if and only if* (b), (c), (d), *or* (e) *in* (1) *holds, replacing "symmetric" with "o-metric."*

A space X is a *developable* (or a *Moore*) *space* if it has a development $\{\mathcal{V}_n : n \in N\}$ (i.e., $\mathcal{V}_n$ are open covers of X such that $\{St(x, \mathcal{V}_n) : n \in N\}$ is a local base at x for each $x \in X$). As is well-known, developable spaces are semi-metric, and also collectionwise normal developable spaces are metric.

Lemma 2. *A symmetric and quasi-metric space X is developable.*

Proof: Let d be a symmetric (ρ, a quasi-metric, respectively) on X. Since X is first countable, any $intB_d(x, 1/n) \ni x$ by Theorem 1(1). For each $x \in X$ and $n \in N$, let $V_n(x) = intB_d(x, 1/n) \cap B_\rho(x, 1/n)$, and let $\mathcal{V}_n = \{V_n(x); x \in X\}$. Then, for each $x \in X$, any sequence $\{x_n; n \in N\}$ with $x_n \in St(x, \mathcal{V}_n)$ converges to the point x, using Lemma 1. Thus, $\{\mathcal{V}_n : n \in N\}$ is a development for X; hence, X is developable. $\square$

Example 1. (1) The *bow-tie space* has a semi-metric d on R^2 (defined by $d(x, y) = |x - y| + a(x, y)$ if either x or y is on the x-axis; otherwise, $d(x, y) = |x - y|$, where $|x - y|$ is the ordinary distance and $a(x, y)$ is the radian measure of the smallest non-negative angle formed by a line through x and y with a horizontal line). The bow-tie space is not developable, and hence not quasi-metric by Lemma 2.

(2) The *Sorgenfrey line* has a n.a.-quasi-metric d (indeed, for a sequence $\{r_i : i \in N\}$ of all rationals, let $d(x, y) = 1$ if $y < x$; otherwise, $d(x, y) = inf\{1/n : y \in \bigcap_{i=1}^{n}[x, r_i), n \in N\}$ (here, put $[x, r_i) = R$ if $r_i \leq x$)). The line is not developable, hence not symmetric by Lemma 2. The line is separable, but it has no point-countable bases.

(3) The *Michael line* has a n.a.-quasi-metric d (indeed, for a sequence $\{I_i : i \in N\}$ of all open intervals with the end points rational, define $d(x, y) = 1$ if x is irrational; otherwise, $d(x, y) = inf\{1/n : y \in \bigcap_{i=1}^{n}\{I_i : x \in I_i\}$ (here, put $I_i = R$ if $x \notin I_i$)). The line has a σ-disjoint base, but it is not developable, hence not symmetric.

A cover $\mathcal{P}$ of X is a *k-network* [26] if, whenever $K \subset U$ with K compact and U open in X, $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$. When K is a single point, then such a k-network $\mathcal{P}$ is a *network*. Every base is a k-network. For k-networks, see [35], (also [22] or [32]).

Lemma 3. *Let F be a closed discrete set in X and let $D = X - F$. Then the following hold.*

(1) *Suppose F is a G_δ-set in X and D is isolated in X. If X is first countable, then X is developable and n.a.-quasi-metric.*

(2) *Suppose that D is an infinite dense set in X.*

(i) *Let $|F| > |D|$. Then X is not meta-Lindelöf. When X is Fréchet, X has no point-countable k-networks.*

(ii) *If $|F| \geq 2^{|D|}$, then X is neither normal nor meta-Lindelöf.*

Proof: For (1), F is discrete in X; then each point $x \in F$ has a decreasing local base $\{V_n(x) : n \in N\}$ in X with $V_n(x) \cap F = \{x\}$. Let $F = \bigcap\{G_n : n \in N\}$, where the G_n are open in X with $G_{n+1} \subset G_n$. For each $n \in N$, let $\mathcal{A}_n = \{V_n(x) \cap G_n : x \in F\} \cup \{\{d\} : d \in D\}$. Then $\{\mathcal{A}_n : n \in N\}$ is a development for X. For the latter part, let $p, q \in X$. For $p \in F$ and $q \in D$, let $\rho(p, q) = inf\{1/n : q \in V_n(p)\}$ and $\rho(q, p) = 1$. For $p, q \in F$ or $p, q \in D$, let $\rho(p, q) = 1$. Then (X, ρ) is n.a.-quasi-metric.

For (2), suppose X is meta-Lindelöf. Then any open cover of X has a subcover of cardinality $\leq |D|$. Thus, $|F| \leq |D|$, a contradiction. For the latter part, suppose X has a point-countable k-network $\mathcal{P}$. Let $\mathcal{Q} = \{clP : P \in \mathcal{P}$ with $P \cap D \neq \emptyset\}$. X is

Fréchet and D is dense in X. Then $\mathcal{Q}$ is a network for X with $|\mathcal{Q}| \leq |D|$. $|F| \leq |\mathcal{Q}|$; then $|F| \leq |D|$, a contradiction. For (ii), as is well-known, X is not normal by $|F| \geq 2^{|D|}$ (see, for example, [8, 1.7.12(c)]), and X is not meta-Lindelöf by (i). $\square$

Example 2. (1) *A metacompact, developable, and n.a.-quasi-metric space X which has a σ-point-finite base (or a point-finite cover) by closed and open metric sets, but X is not normal.*

(2) *A separable developable and n.a.-quasi-metric space X which is locally compact, but X is neither normal nor meta-Lindelöf, and X has no point-countable k-networks.*

(3) *A Lindelöf n.a.-quasi-metric space X which has a σ-disjoint base, but X is neither symmetric nor separable.*

(4) *A separable quasi-metric space X, but X is not symmetric, not n.a.-quasi-metric, not meta-Lindelöf, and not normal.*

(5) *A n.a.-quasi-metric space X which has a σ-disjoint base by closed and open metric sets and a countable open cover by metric sets, but X is not symmetric.*

(6) *A separable developable space X which is locally metric, but X is not quasi-metric, not meta-Lindelöf, and not normal.*

(7) *A developable space X which has a point-countable base by closed and open metric sets, but X is not quasi-metric.*

Proof: For (1), let X be the space Y in [29, Example 3.2] (or [32, Example 5.3]); that is, $Y = \{(x, y) : x \in R, y \geq 0\}$, but for each $r \in R$ and $n \in N$, let $V_n(r) = \{(x, y) : y = |x - r| < 1/n\}$ be a nbd of $(r, 0)$, and let other points be isolated. Then X is developable and n.a.-quasi-metric by Lemma 3(1), and a metacompact space with a σ-point-finite base. But X is not normal by the Baire Category Theorem.

For (2), let X be the space Ψ of Mrowka (see [8, 3.6.I]); that is, let $\mathcal{F}$ be an infinite maximal pairwise almost disjoint collection of infinite subsets of N (thus, $|\mathcal{F}| = 2^{\aleph_0}$), and let $\Psi = \{\omega_F : F \in \mathcal{F}\} \cup N$ with points of N be isolated, and nbds of ω_F are those subsets of Ψ containing ω_F and all but finitely many points of F. Then X is a locally compact, separable space. By Lemma 3, X is developable and n.a.-quasi-metric, but X is neither normal nor meta-Lindelöf, and X has no point-countable k-networks.

For (3), let X be the space obtained from [0, 1] by isolating the points of a Bernstein set (i.e., an uncountable subset of [0, 1] containing no uncountable closed subsets). Then X is Lindelöf, but not separable. X has a n.a.-quasi-metric as in the Michael line, and X has a σ-disjoint base. But X is not symmetric by Lemma 2.

For (4), let X be the space T in [19, Example 1] which is not n.a.-quasi-metric; that is, $T = \{(x, y) : x, y \in R\}$ is the plane such that for each point $p = (a, b) \in R^2$, a nbd of a point p is p together with an open disk tangent to $y = b$ at p, lying above this line. Then T has a quasi-metric d such that $d(p, q) = r$ if q is on the circumference of radius $r \leq 1$ with the southern pole p; otherwise, $d(p, q) = 1$ $(p \neq q)$. Then the separable space X is neither meta-Lindelöf nor normal by Lemma 3(2). Also, X is not symmetric because X contains a closed vertical line which is the Sorgenfrey line.

For (5), (6), and (7), see [34, Example 1.9], [10, Example 10.4], and [34, Example 2.5(2)], respectively. □

A space X is a *$w\Delta$-space* if it has a sequence $\{\mathcal{B}_n : n \in N\}$ of bases for X such that any sequence $\{B_n : n \in N\}$ with $B_n \in \mathcal{B}_n$ satisfies the following condition denoted by ($w\Delta$): if $\{x, x_n\} \subset B_n$ $(n \in N)$, then the sequence $\{x_n : n \in N\}$ has a cluster point in X. When the sequence has the cluster point x, then such a space is precisely developable. Every developable space (or M-space) is a $w\Delta$-space.

As generalizations of $w\Delta$-spaces, let us recall monotonic $w\Delta$-spaces and β-spaces, which are independent. A space X is a *monotonic $w\Delta$-space* (briefly, an *$mw\Delta$-space*) [40] if it has a sequence $\{\mathcal{B}_n : n \in N\}$ of bases for X such that any *decreasing* sequence $\{B_n : n \in N\}$ with $B_n \in \mathcal{B}_n$ satisfies condition ($w\Delta$). $w\Delta$-spaces or p-spaces [3], more generally, quasi-complete spaces [6] are $mw\Delta$-spaces. Conversely, $mw\Delta$-spaces are $w\Delta$-spaces (or p-spaces) if they are submetacompact (= θ-refinable); see [40].

A space X is a *β-space* [15] if there exists a function g from $N \times X$ into the topology such that for any x and n, $x \in g(n, x)$, and if $x \in g(n, x_n)$ $(n \in N)$, then the sequence $\{x_n : n \in N\}$ has a cluster point in X. When the sequence has the cluster point x, then such a space is precisely semi-stratifiable [15] (or pseudostratifiable [17]). σ-spaces or semi-metric spaces are semi-stratifiable. Semi-stratifiable spaces or $w\Delta$-spaces are β-spaces.

In Theorem 2 below, (1) is due to [40, Corollary 2.11]. (2) holds by [10, Theorem 10.7] and Lemma 2, for example. For (3), see [10, Theorem 10.3] (or [19, Theorem 1]). An open collection is *interior-preserving* if the intersection of any subcollection is open.

Theorem 2. (1) *For a space X, X is developable $\Leftrightarrow$ X is a symmetric $mw\Delta$-space $\Leftrightarrow$ X is a semi-stratifiable $mw\Delta$-space.*

(2) *For a quasi-metric space X, X is developable $\Leftrightarrow$ X is symmetric $\Leftrightarrow$ X is a β-space.*

(3) *A space is a n.a.-quasi-metric if and only if it has a σ-interior-preserving base.*

Remark 2. (1) Every n.a.-quasi-metric and symmetric need not be metric, not even normal; see Example 2(1).

(2) Every metric space is a n.a.-quasi-metric by Theorem 2(3), while the (zero-dimensional) Baire space is n.a.-*metric*. A space is n.a.-*metric* if and only if X has a σ-locally finite base by closed and open sets (or balls) [9] (equivalently, Ind $X = 0$; see [8, 7.3.F]). In particular, for a (locally) separable metric space X, X is n.a.-*metric* $\Leftrightarrow$ X has a metric with any ball closed and open $\Leftrightarrow$ X has a base by closed and open sets (equivalently, X is zero-dimensional).

A space is σ-*orthocompact* (σ-*metacompact*, respectively) if every open cover has a σ-interior-preserving (σ-point-finite, respectively) open refinement. Every σ-metacompact space is σ-orthocompact.

Corollary 1. *For a space X, the following are equivalent.*
(a) *X is symmetric and n.a.-quasi-metric.*
(b) *X is a developable space with a σ-interior-preserving base.*
(c) *X is developable and σ-orthocompact.*

In (a), we do not know whether the prefix "n.a." can be deleted, i.e., whether every developable quasi-metric space is n.a.-quasi-metric (see [19, Question 6] or [10, p. 490]).

Locally $mw\Delta$-spaces, in particular locally developable spaces, are $mw\Delta$-spaces [40]. The following holds by Theorem 2.

Corollary 2. *Let X be a symmetric space or a semi-stratifiable space. If X is locally developable or locally quasi-metric, then X is developable.*

Remark 3. Not every locally metric space is symmetric or quasi-metric; see (5) – (7) of Example 2. Below, (ii) is due to [30], but the result for the case "locally symmetric" is due to [31, Theorem 3.6]; (i) is similarly shown (in terms of Remark 1(1)); and (iii) holds by means of [19, Theorem 8].

(i) If X is locally o-metric, then X is an o-metric.

(ii) Let X be submetacompact. If X is locally developable (locally semi-metric, locally symmetric, respectively), then X is developable (semi-metric, symmetric, respectively).

(iii) Let X be σ-metacompact. If X is locally n.a.-quasi-metric (locally quasi-metric, respectively), then X is n.a.-quasi-metric (quasi-metric, respectively).

A cover $\mathcal{P}$ of a space X is a *determining cover* [22] (or, *X is determined by* $\mathcal{P}$ [11]), if $G \subset X$ is open in X if and only if $G \cap P$ is open in P for each $P \in \mathcal{P}$. Every open cover is a determining cover. A sequential space (k-space, respectively) is precisely a space with a determining cover by compact metric (compact, respectively) subsets. A closed cover $\mathcal{F}$ of a space X is a *dominating cover* [22] (or, *X is dominated by* $\mathcal{F}$ [23]), if for any $\mathcal{P} \subset \mathcal{F}$, $S = \bigcup\{P : P \in \mathcal{P}\}$ is closed in X, and $\mathcal{P}$ is a determining cover of S. A closed cover $\mathcal{F}$ is dominating if $\mathcal{F}$ is hereditarily closure-preserving, or increasing countable determining. As is well known, every CW-complex has a dominating cover by compact metric subsets, and every space with a dominating cover by metric subsets is paracompact (actually, stratifiable). For matters related to determining or dominating covers, see [32], [36], or [22].

Example 3. (1) The *sequential fan* S_ω (i.e., the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points) is a Fréchet space with a countable dominating cover by compact metric subsets. S_ω is not first countable, thus not quasi-metric nor symmetric by Theorem 1(2).

(2) The *Arens' space* $S_2 = \bigcup\{L_n : n \in N\} \cup \{0\}$, where $L_n = \{1/n\} \cup \{1/n + 1/k : k \in N\}$, has a symmetric d (defined by $d(0, x) = 1$ if $0 < x \neq 1/n$; otherwise, $d(x, y) = |x - y|$). Then S_2 has a point-finite countable determining (or a countable dominating) cover by compact metric subsets, while a subset $S = S_2 - \{1/n : n \in N\}$ is not sequential. Thus, S_2 is not Fréchet (hence, not quasi-metric), and the set S is not an o-metric by Remark 1(2).

The following theorem is due to [34], but for the last equivalence in (ii) of (1), the X_α have a point-countable base; thus, X has a point-countable k-network, using [11, Proposition 2.1]. Then, since X is a k-space, the equivalence holds by [21, Corollary 2.13].

Theorem 3. (1) *Let X have a point-finite determining cover $\{X_\alpha : \alpha \in I\}$.*

(i) *If the X_α are symmetric (o-metric, respectively), then so is X.*

(ii) *If the X_α are metacompact developable, then X is (metacompact) developable $\Leftrightarrow$ X is (n.a.-) quasi-metric $\Leftrightarrow$ X is Fréchet $\Leftrightarrow$ X contains no closed copy of S_2.*

(2) *Let X have a dominating cover $\{X_\alpha : \alpha \in I\}$.*

(i) *If the X_α are semi-metric, then X is symmetric $\Leftrightarrow$ X is an o-metric $\Leftrightarrow$ X contains no closed copy of S_ω.*

(ii) *If the X_α are metric (semi-metric, quasi-metric, n.a.-quasi-metric, developable, respectively), then so is X $\Leftrightarrow$ X is first countable $\Leftrightarrow$ X contains no closed copy of S_ω and no S_2.*

Remark 4. (1) In Theorem 3(1), we can't replace "point-finite determining cover" with "countable dominating cover" (by Example 3(1)), or with "point-countable open cover" (by Example 2(5), (7)).

(2) Every Fréchet space with a point-countable determining cover by (locally) separable metric subsets is paracompact in view of [11, Corollary 8.9]. But every separable symmetric space with a point-finite determining cover by compact metric subsets need not be meta-Lindelöf, and not normal (indeed, the space Y in [11, Example 9.3] has the desired properties, using Lemma 3(2)).

(3) In [34], the author asked if every first countable space with a point-finite determining cover by (n.a.-) quasi-metric subsets (developable subsets, respectively) is a (n.a.-) quasi-metric (developable, respectively). Also, he asked if every o-metric space with a dominating cover by symmetric subsets is symmetric.

Let (X, d) be an o-metric space. For $A \subset X$, let the diameter $D(A) = sup\{d(p, q) : p, q \in A\}$. For a sequence $L = \{x_n : n \in N\}$ converging to $x \in X$, we say that L is *d-Cauchy* if for each $\epsilon > 0$, some $D(\{x_n; n \geq k\} \cup \{x\}) < \epsilon$. We say that (X, d) satisfies the *condition of Cauchy* (briefly, (X, d) is *Cauchy*) if every convergent sequence is d-Cauchy. Also, (X, d) satisfies the *weak condition of Cauchy* (briefly, (X, d) is *weak Cauchy*) when every convergent sequence has a d-Cauchy subsequence. For Cauchy or weak Cauchy symmetric spaces, see, for example, [5] and [33].

Let $f : X \to Y$ be a map such that (X, d) is metric. Then f is a *π-map* [2], [3] (or *P-map* [13]) (with respect to d), if for any $y \in Y$ and any nbd V of y, $d(f^{-1}(y), X - f^{-1}(V)) > 0$; equivalently, if $d(a_n, x_n) \to 0$ with $f(a_n) = y$, then $f(x_n) \to y$. Every compact map f (i.e., all $f^{-1}(y)$ are compact) from a metric space X is a π-map (with respect to any metric on X). Developable spaces are precisely open π-images of metric spaces [13].

Remark 5. (1) Let $f : X \to Y$ be a quotient π-map such that (X, d) is metric. Then Y has a weak Cauchy symmetric ρ defined by $\rho(y, y') = d(f^{-1}(y), f^{-1}(y'))$ (see, for example, [3]).

(2) Every symmetric space with any ball open is weak Cauchy. Every semi-metric has a weak Cauchy semi-metric (see [5] or [18]).

An o-metric d is a *strong* o-metric* if any $intB^*_d(x; \epsilon) \ni x$. Here, $B^*_d(x; \epsilon) = \{y \in X : d(y, x) < \epsilon\}$. Every semi-metric d is a strong* o-metric. A space is a *strongly* o-metric* space [20] if it has a strongly* o-metric.

Theorem 4. *For a space X,* (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) *below holds (*(a) $\Leftrightarrow$ (b) *is due to* [20], *and* (b) $\Rightarrow$ (c) *is due to* [18]*).*

(a) *X is strongly* o-metric.*
(b) *X is semi-stratifiable o-metric.*
(c) *X is weak Cauchy symmetric.*

Remark 6. In the previous theorem, (c) $\Rightarrow$ (a) or (b) need not hold. Indeed, in view of [33, Theorem 2.3 and Theorem 2.8], there exists a Cauchy symmetric space X which is a quotient finite-to-one image of a metric space, but X is not perfect; hence, X is not semi-stratifiable. Here, a space is *perfect* if every closed set is a G_δ-set.

Corollary 3. (1) *For a space X, the following are equivalent. (*(a) $\Leftrightarrow$ (b) *is due to* [6]*.)*

(a) *X is semi-metric.*
(b) *X is first countable and semi-stratifiable.*
(c) *X is Fréchet and strongly* o-metric.*

(d) *X has a strong o-metric and a strong* o-metric.*
(e) *X has an o-metric d such that for each $x \in X$, any $\{y \in X : Max\{d(x,y), d(y,x)\} < \epsilon\}$ contains a nbd of x.*

(2) *For a quasi-metric space X, the following are equivalent.*
(a) *X is developable.*
(b) *X is strongly* o-metric.*
(c) *X is semi-stratifiable.*

A sequence $\{\mathcal{A}_n : n \in N\}$ $(\mathcal{A}_{n+1} < \mathcal{A}_n)$ of covers of X is *semi-refined* if $U \subset X$ is open if and only if for each $x \in U$, some $st(x, \mathcal{A}_n) \subset U$; see, for example, [33]. A space is symmetric if and only if it has a semi-refined sequence (see, for example, [42]). We say that a space X is *g-developable* if it has a semi-refined sequence $\{\mathcal{A}_n : n \in N\}$ satisfying condition (*) for each $n \in N$, and for each convergent sequence L in X, some $A \in \mathcal{A}_n$ contains L eventually, and its limit point. Also, X is *weakly g-developable* if we replace "eventually" with "frequently." A space is g-developable if and only if it is g-developable in the sense of [20]; see [33].

A map $f : X \to Y$ is *sequence-covering* [28] if each convergent sequence in Y is an image of some convergent sequence in X under f. Every open map from a first countable space is sequence-covering.

Theorem 5. *For a space X, the following are equivalent.*
(a) *X is g-developable (weakly g-developable, respectively).*
(b) *X is Cauchy symmetric (weak Cauchy symmetric, respectively).*
(c) *X is Cauchy o-metric (weak Cauchy o-metric, respectively).*
(d) *X is a quotient sequence-covering (quotient, respectively) π-image of a (locally compact) metric space.*

Proof: The equivalence among (a), (b), and (d) is due to [33].

For (c) $\Rightarrow$ (b), let (X, d) be Cauchy o-metric (weak Cauchy o-metric, respectively). For each $n \in N$, let $\mathcal{A}_n = \{A : D(A) < 1/n\}$. Since X is sequential, $\{\mathcal{A}_n : n \in N\}$ is a semi-refined sequence in X satisfying condition (*). For $x, y \in X$, define $\rho(x, y) = inf\{1/n : y \in St(x, \mathcal{A}_n)\}$. Then (X, ρ) is Cauchy symmetric. The implication for the parenthetic part is similarly shown. $\square$

Remark 7. A space is weak Cauchy symmetric if and only if it is a quotient π-image of a metric space; see, for example, [17] and

[33]. But every symmetric space need not be a quotient π-image of a metric space because every symmetric space need not be weak Cauchy [18]. A characterization for a symmetric space by a certain π-image of a metric space is given in [25]. However, we do not know any characterization for a (n.a.-) quasi-metric space by an image of a metric space under a (nice) map.

A map $f : X \to Y$ is *pseudo-open* if, for $f^{-1}(y) \subset G$ with G open in X, $y \in int f(G)$; equivalently, f is *hereditarily quotient* (i.e., $f|f^{-1}(A)$ is quotient for any $A \subset Y$) [1]. Open or closed maps are pseudo-open.

Lemma 4 ([1])**.** *Let $f : X \to Y$ be a quotient map with X metric. Then f is pseudo-open if and only if Y is Fréchet.*

Corollary 4. *For a Fréchet space X, the following are equivalent.*

(a) *X is developable (semi-metric, respectively).*
(b) *X is g-developable (weakly g-developable, respectively).*
(c) *X is Cauchy o-metric (weak Cauchy o-metric, respectively).*
(d) *X is a pseudo-open sequence-covering (pseudo-open, respectively) π-image of a (locally compact) metric space.*

Proof: For (c) $\Rightarrow$ (a), let (X, d) be a Cauchy o-metric. For any $x \in X$, $D(B_d(x; 1/n)) \to 0$ since d is Cauchy, and $intB_d(x; 1/n) \ni x$ by Theorem 1(2). For each $n \in N$, let $\mathcal{V}_n = \{intB_d(x; 1/n) : x \in X\}$. Then $\{\mathcal{V}_n : n \in N\}$ is a development for X; thus, X is developable.

Other implications hold by Theorem 1(1), Theorem 5, Lemma 4, and Remark 5(2). $\square$

Corollary 5. *A space is developable if it is Cauchy semi-metric or weak Cauchy quasi-metric.*

Lemma 5 below holds by [14, Theorem 1] (with Theorem 1(1)), but for the latter case, recall that every first countable space with a point-countable k-network has a point-countable base [11].

Lemma 5. *A space X is developable if X is a symmetric space with a point-countable base, or X is a semi-metric space with a point-countable k-network.*

Lemma 6 ([13])**.** *Let X have a development $\{\mathcal{B}_n : n \in N\}$, and let $\mathcal{B}_n = \{B_\alpha : \alpha \in I_n\}$. Let $M = \{\sigma = (\sigma(n)) \in \Pi I_n : \bigcap\{B_{\sigma(n)} :$*

$n \in N\} = \{x_\sigma\}\}$. Let M be a subspace of the Baire space $(\Pi I_n, \rho)$, where the I_n are discrete spaces. Define $f : M \to X$ by $f(\sigma) = x_\sigma$. Then f is an open π-map with respect to ρ, and $f^{-1}(x) = \Pi\{\alpha \in I_n : x \in B_\alpha\}$.

Theorem 6. *For a space X, the following are equivalent.*

(a) *X is developable and metacompact (meta-Lindelöf, respectively).*
(b) *X is a developable space with a σ-point-finite (point-countable, respectively) base.*
(c) *Same as (b), but replace "developable" with "symmetric."*
(d) *X is an open compact image (open s-and-π-image, respectively) of a metric space.*
(e) *Same as (d), but replace "open" with "pseudo-open" twice.*

Proof: (a) $\Rightarrow$ (b) is obvious.

(c) $\Leftrightarrow$ (b) holds by Lemma 5.

For (b) $\Rightarrow$ (d), we show that X has a development by point-finite (point-countable, respectively) open covers. The implication for the parenthetic part holds by Lemma 6, so let X have a σ-point-finite base $\{\mathcal{B}_n : n \in N\}$. For each $m \in N$, let $\mathcal{B}_{nm}$ be the collection of all intersections of m distinct elements of $\mathcal{B}_n$, and let $\{\mathcal{B}_{nm} : n, m \in N\} = \{\mathcal{G}_i : i \in N\}$. Since X is semi-metric, it is perfect, so each $\bigcup\{G : G \in \mathcal{G}_i\} = \bigcup\{F_{ij} : j \in N\}$ for some closed sets F_{ij} in X. Let $\mathcal{D}_{ij} = \mathcal{G}_i \cup \{(X - F_{ij})\}$. Then $\{\mathcal{D}_{ij} : i, j \in N\}$ is a development by point-finite open covers. Thus, (b) $\Rightarrow$ (d) holds by Lemma 6.

For (e) $\Rightarrow$ (a), let X be a pseudo-open compact image of a metric space. Then X is metacompact by [4, Proposition 1]. To see that X is developable, let X be, more generally, a pseudo-open s-and-π-image of a metric space. Then X is symmetric by Remark 5(1). Since X is Fréchet by Lemma 4, X is semi-metric by Theorem 1(1). Also, X is a quotient s-image of a metric space, so X has a point-countable k-network in view of [11]. Thus, X is developable by Lemma 5. □

Corollary 6. *For a space X, the following are equivalent.*

(a) *X is developable and (σ-) metacompact.*
(b) *X is a perfect space with a σ-point-finite base.*

(c) *X is symmetric, (n.a.-) quasi-metric, and (σ-) metacompact.*

Remark 8. (1) Every paracompact, perfect space with a point-countable base need not be developable; see [41].

(2) Every semi-metric space is perfect (but need not be a σ-space; see [10, Example 9.10]). But every symmetric space need not be perfect by [7, Example 3.1] or Remark 6, and neither must every quasi-metric space by Example 1(3) or by Example 2(3), (4) with Corollary 6. However, we do not know whether points are G_δ-sets among symmetric spaces, in particular, among spaces with a point-finite determining cover by compact metric subsets; equivalently, quotient compact images of locally compact metric spaces; see [22].

Corollary 7. *Let X be a Fréchet space which is a quotient compact image of a metric space. Then X is a developable, n.a.-quasi-metric space with a σ-point-finite base.*

For a symmetric space X, X is ω_1-compact (i.e., any uncountable subset has an accumulation point in X) $\Leftrightarrow$ X is Lindelöf $\Leftrightarrow$ X is hereditarily Lindelöf ($\Leftarrow$ X is hereditarily separable); see [25]. For a semi-metric (generally, semi-stratifiable) space X, X is Lindelöf if and only if X is hereditarily separable [6], [17]. The existence of a Lindelöf, non-separable, symmetric space is consistent with ZFC [27].

Theorem 7. (1) *Let X be a weak Cauchy symmetric space. If X is Lindelöf, then X is (hereditarily) separable.*

(2) *Let X be a symmetric space with a point-countable k-network. If X is Lindelöf, then X is separable.*

Proof: (1) is due to [25, Corollary 13].

For (2), let $\mathcal{P}$ be a point-countable k-network for X; here assume $\mathcal{P}$ is closed under finite intersections. For each $n \in N$, let $\mathcal{P}_n = \{P \in \mathcal{P} : P$ is contained in some $B_d(x; 1/n)\}$, and let $\mathcal{Q}_n = \{Q : Q$ is a finite union of elements of $\mathcal{P}_n\}$. Let $x \in X$ and $n \in N$. Then, since $\mathcal{P}$ is a point-countable k-network for X, each sequence converging to x is eventually contained in some $Q \in \mathcal{Q}_n$ with $x \in Q \subset B_d(x; 1/n)$, in view of the proof of Lemma 1.5 in [39] (using Lemma 1 there). Thus, for each $n \in N$, $\mathcal{P}_n$ is a point-countable cover of X, and $\mathcal{Q}_n$ is a determining cover of X since X is sequential by Remark 1(2). Then any $\mathcal{P}_n$ has a countable subcover of X. Indeed, suppose some $\mathcal{P}_n$ has no countable subcovers. Let $\mathcal{P}_n = \{P_\alpha : \alpha < \gamma\}$, and let $A(x) = \{\alpha : x \in P_\alpha\}$ for each $x \in X$. Since each $A(x)$ is countable, by induction, take a subset $D = \{x_\beta : \beta < \omega_1\}$ of X such that $x_\beta \in X - \bigcup\{P_\alpha : \alpha \in \bigcup\{A(x_\rho) : \rho < \beta\}\}$. Then $D \cap Q$ is at most finite for each $Q \in \mathcal{Q}_n$. Thus, D is closed discrete in X with $|D| = \omega_1$. But since X is Lindelöf, D has an accumulation point in X. This is a contradiction. Hence, any $\mathcal{P}_n$ has a countable subcover of X. Thus, X has a countable cover $\{B_d(x_{nm}; 1/n) : n, m \in N\}$. Then $\{x_{nm} : n, m \in N\}$ is dense in X, so X is separable. □

Remark 9. (1) Every separable Cauchy symmetric space need not be Lindelöf even if it has a n.a.-quasi-metric (by Example 2(2)), or has a point-countable k-network (by the space Y in Remark 4(2)).

(2) Among n.a.-quasi-metric spaces, the Lindelöf property and separability are independent in view of Example 2(2), (3), but we do not know whether every hereditarily separable (n.a.-) quasi-metric space is Lindelöf (or ω_1-compact).

Corollary 8. *Let X be a symmetric space which is a quotient s-image of a metric space. Then X is Lindelöf if and only if every closed subset is separable.*

Any countable product of semi-metric spaces ((n.a.-) quasi-metric spaces, respectively) is so. But not every product of two symmetric spaces is an *o*-metric; see, for example, [31]. The following holds in view of [39].

Theorem 8. (1) *Let X be a symmetric space such that* (a) *X has a point-countable k-network,* (b) *each point of X is a G_δ-set, or* (c) *X is meta-Lindelöf; otherwise,* CH *holds. Let Y be semi-metric.*

(i) *$X \times Y$ is symmetric if and only if X is semi-metric (developable for* (a)*), or Y is locally compact.*

(ii) *X^ω is symmetric if and only if X is semi-metric (developable for* (a)*).*

(2) *Let X be an o-metric space satisfying* (a) *or* (b) *in* (1)*. Let Y be first countable.*

(i) *$X \times Y$ is o-metric if and only if X is first countable, or Y is locally countably compact.*

(ii) *X^ω is o-metric if and only if X is first countable.*

Theorem 9. (1) *Let P be the product of countably many symmetric spaces. Then P is symmetric if and only if P is a k-space* [25].

(2) *Same as* (1)*, but replace "symmetric" with "o-metric" twice.*

Proof: The "only if" parts for (1) and (2) hold because every o-metric space is sequential by Remark 1(2), hence a k-space.

For the "if" part for (1), let $P = \Pi\{X_n : n \in N\}$ be a k-space, and let each X_n have a symmetric d_n such that each $D(X_n) \leq 1/n$. Since P is the product of sequential spaces X_n, P is sequential in view of [38, Theorem 2.15]. Now, for $x = (x_n), y = (y_n) \in P$, let $d(x,y) = sup\{d_n(x_n, y_n) : n \in N\}$. Then d is a symmetric on P. Indeed, let $G \subset P$. If G is open, then for each $x \in G$, $B_d(x;\epsilon) \subset G$ for some $\epsilon > 0$. The converse also holds. To see that G is open, since P is sequential, it suffices to show that each sequence L in P converging to a point $x = (x_n) \in G$ is eventually in G. Let $B_d(x;\epsilon) \subset G$ for some $\epsilon > 0$. For $m \in N$ with $1/m < \epsilon$, $B(x) = B_{d_1}(x_1;\epsilon) \times \cdots \times B_{d_m}(x_m;\epsilon) \times \Pi\{X_n : n > m\} \subset B_d(x;\epsilon)$. Thus, the sequence L is eventually in $B(x)$ by Lemma 1, and hence is also in G.

The "if" part for (2) holds by replacing "symmetric" with "o-metric" in the above. $\square$

Remark 10. If X^ω is an o-metric, then X need not be first countable under CH. Indeed, under CH, there exists a compact o-metric space which is not first countable [16]. Then X^ω is o-metric by Theorem 9(2), but X is not first countable. However, we do not know if X needs to be first countable for X^ω to be symmetric [22].

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