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by

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NOTES ON g-FUNCTIONS AND MCP AND QUASI-NAGATA SPACES

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Abstract. We give characterizations of stratifiable spaces and MCP spaces. Moreover, we study some generalized metric spaces which can be defined or characterized by g-functions and present several theorems concerning the metrization of quasi-Nagata spaces.

1. Introduction

The notion of monotonically countably paracompact (MCP) spaces was introduced by Chris Good, Robin Knight, and Ian Stares [6] as a monotone version of countable paracompactness. Ying Ge and Good [5] gave characterizations of stratifiable spaces and MCP spaces from which one can see that stratifiable spaces have similar structures to MCP spaces. In this paper, we shall give other characterizations of stratifiable spaces and MCP spaces.

As is known, in the field of generalized metric spaces, one important task is to find conditions which imply metrizability for certain classes of generalized metric spaces. In [19], Iwao Yoshioka showed that a c-stratifiable quasi-Nagata quasi-γ space is metrizable and, in [16], A. M. Mohamad proved that a quasi-Nagata $w\theta$ space X with a quasi-$G^*_\delta(2)$ diagonal is metrizable. In this paper, we provide

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several other theorems concerning the metrization of quasi-Nagata spaces.

Throughout, a space means a topological space and all spaces in this paper are assumed to be T_1 unless stated otherwise.

Let X be a space. $K(X)$ and $C(X)$ denote the families of all closed subsets and compact subsets of X, respectively. τ is reserved for the topology of a space X. The set of all positive integers is denoted by $\mathbb{N}$. $\langle x_n \rangle$ denotes a sequence.

The following notation is due to Good, Knight, and Stares [6]:

Let $(A_j)_{j\in\mathbb{N}}$ and $(B_j)_{j\in\mathbb{N}}$ be two sequence of subsets of a space X; we write $(A_j) \preceq (B_j)$ if $A_n \subset B_n$ for every $n \in \mathbb{N}$.

Definition 1 ([1])**.** A space X is called a *stratifiable space* if for each $F \in K(X)$ there exists a sequence $\{U(n,F)\}_{n\in\mathbb{N}}$ of open sets satisfying

(1) $F = \bigcap_{n\in\mathbb{N}} U(n,F) = \bigcap_{n\in\mathbb{N}} \overline{U(n,F)}$;

(2) for each $n \in \mathbb{N}$, $U(n,F) \subset U(n,H)$ whenever $F \subset H$.

Definition 2 ([6])**.** A space X is said to be *monotonically countably paracompact* (*MCP*) if there is an operator U assigning to each decreasing sequence $(F_j)_{j\in\mathbb{N}}$ of closed sets with empty intersection, a sequence of open sets $(U(n,(F_j)))_{n\in\mathbb{N}}$ such that

(1) $F_n \subset U(n,(F_j))$ for each $n \in \mathbb{N}$;

(2) if $(F_j) \preceq (H_j)$, then $U(n,(F_j)) \subset U(n,(H_j))$ for all $n \in \mathbb{N}$;

(3) $\bigcap_{n\in\mathbb{N}} \overline{U(n,(F_j))} = \emptyset$.

A *g-function* for a space X is a map $g : \mathbb{N} \times X \to \tau$ such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n,x)$ and $g(n+1,x) \subset g(n,x)$.

Consider the following conditions on g.

($k\beta$) For every $K \in C(X)$, if $K \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(Θ) If $\{x, x_n\} \subset g(n,y_n)$ for all $n \in \mathbb{N}$ and $\langle y_n \rangle$ has a cluster point, then x is a cluster point of $\langle x_n \rangle$.

(θ) If $\{x, x_n\} \subset g(n,y_n)$ and $y_n \in g(n,x)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\langle x_n \rangle$.

($w\theta$) If $\{x, x_n\} \subset g(n,y_n)$ and $y_n \in g(n,x)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(ks) If $y_n \in g(n,x_n)$ for all $n \in \mathbb{N}$ and $y_n \to x$, then $x_n \to x$.

(wcc) If $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and $\langle y_n \rangle$ has a cluster point, then $\langle x_n \rangle$ has a cluster point.

(*quasi-Nagata*) If $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and $\langle y_n \rangle$ converges, then $\langle x_n \rangle$ has a cluster point.

(γ) If $y_n \in g(n, x)$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\langle x_n \rangle$.

($w\gamma$) If $y_n \in g(n, x)$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(*quasi*-γ) If $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$ and $\langle y_n \rangle$ converges, then $\langle x_n \rangle$ has a cluster point.

(q) If $x_n \in g(n, x)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

A space X which has a g-function satisfying condition ($k\beta$) is called a *$k\beta$ space* and the corresponding function is called a *$k\beta$ function*. The others are defined analogously.

First countable spaces can be characterized by g-functions satisfying the following condition:

If $x_n \in g(n, x)$ for all $n \in \mathbb{N}$, then x is a cluster point of $\langle x_n \rangle$.

ks spaces are called strongly quasi-Nagata spaces in [15] and a ks space is equivalent to the condition [that:] there exists a g-function g such that if $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and $y_n \to x$, then x is a cluster point of the sequence $\langle x_n \rangle$.

It is known that the following implications hold.

$$
\begin{tikzcd}
\text{MCP space} \arrow[rr, "{[20]}"] & & wcc\text{ space} \arrow[d, "\text{Proposition 2.6 of this paper}"] \\
\text{stratifiable space} \arrow[u, "{[5]}"] \arrow[r, "{[20]}"] & ks\text{ space} \arrow[r, "{[17]}"] & k\beta\text{ space} \arrow[d, "{[18]}"] \\
& & \text{quasi-Nagata space}
\end{tikzcd}
$$

2. Characterizations of stratifiable spaces

Theorem 2.1. *For a space X, the following are equivalent.*

(a) *X is stratifiable.*

(b) *For each $F \in K(X)$, there exists a sequence $\{U(n, F)\}_{n \in \mathbb{N}}$ of open sets such that*

(1) $\bigcap_{n\in\mathbb{N}} U(n,F) = F$;
(2) *for each* $n \in \mathbb{N}$, $U(n,F) \subset U(n,H)$ *whenever* $F \subset H$;
(3) *for every* $K \in C(X)$ *and* $F \in K(X)$ *with* $K \cap F = \emptyset$, *there exists* $m \in \mathbb{N}$ *such that* $K \cap \overline{U(m,F)} = \emptyset$.

(c) *For each* $F \in K(X)$, *there exists a sequence* $\{U(n,F)\}_{n\in\mathbb{N}}$ *of open sets such that*
(1) $F \subset U(n,F)$ *for all* $n \in \mathbb{N}$;
(2) *for each* $n \in \mathbb{N}$, $U(n,F) \subset U(n,H)$ *whenever* $F \subset H$;
(3) *let* $\{F_n\}_{n\in\mathbb{N}}$ *be a decreasing sequence of closed sets. Then* $\bigcap_{n\in\mathbb{N}} U(n,F_n) = \bigcap_{n\in\mathbb{N}} F_n$, *and for every* $K \in C(X)$, *if* $K \cap F_n = \emptyset$ *for some* $n \in \mathbb{N}$, *then* $K \cap \overline{U(m,F_m)} = \emptyset$ *for some* $m \in \mathbb{N}$.

(d) *There exists an operator* V *assigning to each decreasing sequence* $(F_j)_{j\in\mathbb{N}}$ *of closed sets, a sequence of open sets* $(V(n,(F_j)))_{n\in\mathbb{N}}$ *such that*
(1) *for each* $n \in \mathbb{N}$, $F_n \subset V(n,(F_j))$;
(2) *if* $(F_j) \preceq (H_j)$, *then* $V(n,(F_j)) \subset V(n,(H_j))$ *for all* $n \in \mathbb{N}$;
(3) $\bigcap_{n\in\mathbb{N}} V(n,(F_j)) = \bigcap_{n\in\mathbb{N}} F_n$, *and for every* $K \in C(X)$, *if* $K \cap F_n = \emptyset$ *for some* $n \in \mathbb{N}$, *then* $K \cap \overline{V(m,(F_j))} = \emptyset$ *for some* $m \in \mathbb{N}$.

Proof: (a) $\Rightarrow$ (b) Suppose that X is a stratifiable space. Then for each $F \in K(X)$, there exists a sequence $\{U(n,F)\}_{n\in\mathbb{N}}$ of open sets satisfying the conditions in Definition 1. Without loss of generality, we may assume that $U(n+1,F) \subset U(n,F)$ for all $n \in \mathbb{N}$ and $F \in K(X)$. Obviously, conditions (1) and (2) are satisfied. Let $K \in C(X)$ and $F \in K(X)$ with $K \cap F = \emptyset$. From $\bigcap_{n\in\mathbb{N}} \overline{U(n,F)} = F$, it follows that $K \cap \bigcap_{n\in\mathbb{N}} \overline{U(n,F)} = \emptyset$. Since $K \in C(X)$, there exist finitely many $n_i, i = 1,2,\cdots,k$ such that $K \cap \bigcap_{i=1}^{k} \overline{U(n_i,F)} = \emptyset$. Let $m = \max\{n_i, i = 1,2,\cdots,k\}$. Then $K \cap \overline{U(m,F)} = \emptyset$.

(b) $\Rightarrow$ (c) Assume (b). We may assume that $U(n+1,F) \subset U(n,F)$ for all $n \in \mathbb{N}$. Then, clearly, conditions (1) and (2) of (c) are satisfied. Let $\{F_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of closed sets. Then $\bigcap_{n\in\mathbb{N}} F_n \subset \bigcap_{n\in\mathbb{N}} U(n,F_n)$. We show that $\bigcap_{n\in\mathbb{N}} U(n,F_n) \subset \bigcap_{n\in\mathbb{N}} F_n$ also holds. Suppose that $x \notin \bigcap_{n\in\mathbb{N}} F_n$. Then there exists $m \in \mathbb{N}$ such that $x \notin F_m = \bigcap_{j\in\mathbb{N}} U(j,F_m)$, and so there is $n \in \mathbb{N}$ such that $x \notin U(n,F_m)$. Setting $N = \max\{m,n\}$, we have

$U(N,F_N) \subset U(N,F_m) \subset U(n,F_m)$. Thus, $x \notin U(N,F_N)$, which implies that $x \notin \bigcap_{n\in\mathbb{N}} U(n,F_n)$. Consequently, $\bigcap_{n\in\mathbb{N}} U(n,F_n) \subset \bigcap_{n\in\mathbb{N}} F_n$.

Suppose now $K \in C(X)$ and $K \cap F_n = \emptyset$ for some $n \in \mathbb{N}$. Then there exists a $k \in \mathbb{N}$ such that $K \cap \overline{U(k,F_n)} = \emptyset$. Let $m = \max\{k,n\}$. Then $K \cap \overline{U(m,F_m)} \subset K \cap \overline{U(k,F_m)} \subset K \cap \overline{U(k,F_n)} = \emptyset$.

(c) $\Rightarrow$ (d) Assume (c). Let $(F_j)_{j\in\mathbb{N}}$ be a decreasing sequence of closed sets. For each $n \in \mathbb{N}$, put $V(n,(F_j)) = U(n,F_n)$. One easily verifies that V is an operator which satisfies all the items of (d).

(d) $\Rightarrow$ (c) Suppose (d). Let V be the operator in (d). For each $x \in X$ and $n \in \mathbb{N}$, put $F_j^n(x) = \{x\}$ whenever $j \le n$ and $F_j^n(x) = \emptyset$, otherwise. Then for fixed $x \in X$ and $n \in \mathbb{N}$, $(F_j^n(x))_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets. For each closed set F, put $U(n,F) = \bigcup_{x\in F} V(n,(F_j^n(x)))$. Since $\{x\} = F_n^n(x) \subset V(n,(F_j^n(x)))$, we have $F \subset U(n,F)$ for all $n \in \mathbb{N}$, and it is clear that $U(n,F) \subset U(n,H)$ whenever $F \subset H$. Suppose now $(F_j)_{j\in\mathbb{N}}$ is a decreasing sequence of closed sets. Since $F \subset U(n,F)$ for all $n \in \mathbb{N}$, we have $\bigcap_{n\in\mathbb{N}} F_n \subset \bigcap_{n\in\mathbb{N}} U(n,F_n)$. For each $n \in \mathbb{N}$ and each $x \in F_n$, since $F_j^n(x) = \emptyset$ for all $j > n$ and $F_j^n(x) = \{x\} \subset F_n \subset F_j$ whenever $j \le n$, we have $F_j^n(x) \subset F_j$ for all $j \in \mathbb{N}$ and so $V(n,(F_j^n(x))) \subset V(n,(F_j))$ for all $n \in \mathbb{N}$. Therefore, $U(n,F_n) = \bigcup_{x\in F_n} V(n,(F_j^n(x))) \subset V(n,(F_j))$ for all $n \in \mathbb{N}$, which shows that $\bigcap_{n\in\mathbb{N}} U(n,F_n) \subset \bigcap_{n\in\mathbb{N}} V(n,(F_j)) = \bigcap_{n\in\mathbb{N}} F_n$.

Suppose now $K \in C(X)$ and $K \cap F_n = \emptyset$ for some $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $K \cap \overline{V(m,(F_j))} = \emptyset$. Thus, $K \cap \overline{U(m,F_m)} \subset K \cap \overline{V(m,(F_j))} = \emptyset$.

(c) $\Rightarrow$ (b) Suppose (c). By letting $F_n = F$ for all $n \in \mathbb{N}$, one readily verifies that all the conditions in (b) are satisfied.

(b) $\Rightarrow$ (a) Suppose (b). Then for each $n \in \mathbb{N}$, $U(n,F) \subset U(n,H)$ whenever $F \subset H$. From (3) of (b), it follows that for each $F \in K(X)$, if $x \notin F$, then there exists an $m \in \mathbb{N}$ such that $x \notin \overline{U(m,F)}$, which implies that $\bigcap_{n\in\mathbb{N}} \overline{U(n,F)} \subset F$. Thus, $\bigcap_{n\in\mathbb{N}} \overline{U(n,F)} = F$ and X is stratifiable. $\square$

From the above theorem, we see that stratifiable spaces have similar structures to k-semi-stratifiable spaces. Since k-semi-stratifiable spaces can be characterized with g functions satisfying (ks) in the introduction, it is natural to ask whether stratifiable spaces can also be characterized with g functions satisfying similar conditions. We have no idea, but we do have the following.

Proposition 2.2. *Let X be a stratifiable space. Then the following statements hold.*

(a) *There exists a g-function g for X such that if $y_n \in \overline{g(n, x_n)}$ for all $n \in \mathbb{N}$ and x is a cluster point of $\langle y_n \rangle$, then x is a cluster point of $\langle x_n \rangle$.*

(b) *There exists a g-function g for X such that if $y_n \in \overline{g(n, x_n)}$ for all $n \in \mathbb{N}$ and $y_n \to x$, then $x_n \to x$.*

(c) *There exists a g-function g for X such that for each $K \in C(X)$, if $K \cap \overline{g(n, x_n)} \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point in K.*

Proof: (a) Suppose that X is a stratifiable space. Then for each $F \in K(X)$, there exists a sequence $\{U(n, F)\}_{n \in \mathbb{N}}$ of open sets satisfying the conditions in Definition 1. We may assume that $U(n+1, F) \subset U(n, F)$ for all $n \in \mathbb{N}$ and $F \in K(X)$. For each $n \in \mathbb{N}$ and $x \in X$, put $g(n, x) = U(n, \{x\})$. Then g is a g-function for X. Suppose now $y_n \in \overline{g(n, x_n)}$ for all $n \in \mathbb{N}$ and x is a cluster point of $\langle y_n \rangle$. If x is not a cluster point of $\langle x_n \rangle$, then there exists $m \in \mathbb{N}$ such that $x \notin \overline{\{x_n : n \geq m\}} = F$, and then there is $k \geq m$ such that $x \notin \overline{U(k, F)}$. As x is a cluster point of $\langle y_n \rangle$, there is $l \geq k$ such that $y_l \notin \overline{U(k, F)}$. But $y_l \in \overline{g(l, x_l)} = \overline{U(l, \{x_l\})} \subset \overline{U(l, F)} \subset \overline{U(k, F)}$, a contradiction.

(b) Let g be the function in (a). Suppose that $y_n \in \overline{g(n, x_n)}$ for all $n \in \mathbb{N}$ and $y_n \to x$. Let $\langle x_{n_k} \rangle$ be an arbitrary subsequence of $\langle x_n \rangle$. Since $y_{n_k} \in \overline{g(n_k, x_{n_k})} \subset \overline{g(k, x_{n_k})}$ and x is a cluster point of $\langle y_{n_k} \rangle$, x is a cluster point of $\langle x_{n_k} \rangle$. Therefore, $x_n \to x$.

(c) Let g be the function in (a). Take $K \in C(X)$. Suppose that $K \cap \overline{g(n, x_n)} \neq \emptyset$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose $y_n \in K \cap \overline{g(n, x_n)}$. Then $\langle y_n \rangle$ has a cluster point p in K, and p is a cluster point of $\langle x_n \rangle$. □

Lemma 2.3 ([5])**.** *X is an MCP space if and only if for each $F \in K(X)$, there exists a sequence $\{U(n, F)\}_{n \in \mathbb{N}}$ of open sets such that*

(1) *$F \subset U(n,F)$ for all $n \in \mathbb{N}$;*
(2) *for each $n \in \mathbb{N}$, $U(n,F) \subset U(n,H)$ whenever $F \subset H$;*
(3) *let $\{F_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of closed sets with empty intersection. Then $\bigcap_{n\in\mathbb{N}} \overline{U(n,F_n)} = \emptyset$.*

By Lemma 2.3 and with the same method as in the proof of Theorem 2.1, we can prove the following.

Proposition 2.4. *For a space X, the following are equivalent.*

(a) *X is an MCP space.*
(b) *There exists an operator V assigning to each decreasing sequence $(F_j)_{j\in\mathbb{N}}$ of closed sets with empty intersection, a sequence of open sets $(V(n,(F_j)))_{n\in\mathbb{N}}$ such that*
  (1) *for each $n \in \mathbb{N}$, $F_n \subset V(n,(F_j))$;*
  (2) *if $(F_j) \preceq (H_j)$, then $V(n,(F_j)) \subset V(n,(H_j))$ for all $n \in \mathbb{N}$;*
  (3) *for every $K \in C(X)$, there exists $m \in \mathbb{N}$ such that $K \cap \overline{V(m,(F_j))} = \emptyset$.*
(c) *For each $F \in K(X)$, there exists a sequence $\{U(n,F)\}_{n\in\mathbb{N}}$ of open sets such that*
  (1) *$F \subset U(n,F)$ for all $n \in \mathbb{N}$;*
  (2) *for each $n \in \mathbb{N}$, $U(n,F) \subset U(n,H)$ whenever $F \subset H$;*
  (3) *let $\{F_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of closed sets with empty intersection. Then for every $K \in C(X)$, $K \cap \overline{U(m,F_m)} = \emptyset$ for some $m \in \mathbb{N}$.*

It was shown in [20] that an MCP space is a *wcc* space. Actually, we have the following stronger result.

Proposition 2.5. *If X is an MCP space, then there exists a g-function g for X such that if $y_n \in \overline{g(n,x_n)}$ for all $n \in \mathbb{N}$ and $\langle y_n\rangle$ has a cluster point, then $\langle x_n\rangle$ has a cluster point.*

Proof: Suppose that U is the operator in Lemma 2.3 and that $U(n+1,F) \subset U(n,F)$ for all $n \in \mathbb{N}$ and $F \in K(X)$. For each $n \in \mathbb{N}$ and $x \in X$, put $g(n,x) = U(n,\{x\})$. Then g is a g-function for X. Suppose now $y_n \in \overline{g(n,x_n)}$ for all $n \in \mathbb{N}$ and $\langle y_n\rangle$ has a cluster point p. If $\langle x_n\rangle$ has no cluster point, then, by putting $F_n = \{x_m : m \geq n\}$ for all $n \in \mathbb{N}$, we get a decreasing sequence of closed sets $\{F_n\}_{n\in\mathbb{N}}$ and it is obvious that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. From Lemma 2.3, it follows that $\bigcap_{n\in\mathbb{N}} \overline{U(n,F_n)} = \emptyset$. Then there is $k \in \mathbb{N}$

such that $p \notin \overline{U(k, F_k)}$. Since p is a cluster point of $\langle y_n \rangle$, there exists $m \geq k$ such that $y_m \in X - \overline{U(k, F_k)} \subset X - \overline{U(m, F_k)} \subset X - \overline{U(m, F_m)} \subset X - \overline{U(m, \{x_m\})} = X - \overline{g(m, x_m)}$, a contradiction. □

A space X is said to be *weakly subsequential* [9] if each sequence of X with a cluster point has a subsequence with compact closure.

Proposition 2.6. *A weakly subsequential space X is a wcc space if and only if it is a $k\beta$ space.*

Proof: Let g be a *wcc* function and let $K \in C(X)$. Suppose that $K \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose $y_n \in K \cap g(n, x_n)$. Then $\langle y_n \rangle$ has a cluster point and so $\langle x_n \rangle$ has a cluster point. This shows that X is a $k\beta$ space.

Conversely, let g be a $k\beta$ function. Suppose $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and $\langle y_n \rangle$ has a cluster point. As X is a weakly subsequential space, there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $K = \overline{\{y_{n_k}\}}$ is a compact set. But $y_{n_k} \in g(k, x_{n_k})$ for all $k \in \mathbb{N}$, so $K \cap g(k, x_{n_k}) \neq \emptyset$ for all $k \in \mathbb{N}$. Therefore, $\langle x_{n_k} \rangle$ and hence, $\langle x_n \rangle$ has a cluster point. □

A space X is said to be *subsequential* [9] if each sequence of X with a cluster point has a convergent subsequence.

Proposition 2.7. *A subsequential space X is a wcc space if and only if it is a quasi-Nagata space.*

Proof: Similar to the proof of Proposition 2.6. □

3. Metrization of quasi-Nagata spaces

A space X is called a *strongly α space* [20] (an *α space*, respectively, [7]) if it has a g-function satisfying the following conditions:

(1) $\bigcap_{n \in \mathbb{N}} \overline{g(n, x)} = \{x\}$ ($\bigcap_{n \in \mathbb{N}} g(n, x) = \{x\}$, respectively);

(2) if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$.

Clearly, every strongly α space is Hausdorff.

Lemma 3.1. *A strongly α q space X is first countable.*

Proof: Let h be a strongly α function and let l be a q function. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n, x) = h(n, x) \cap l(n, x)$. We show that g is a first countable function.

Suppose that $x_n \in g(n, x)$ for all $n \in \mathbb{N}$. Then $\langle x_n \rangle$ has a cluster point, say p, because l is a q function. If $x \neq p$, then there exists

$m \in \mathbb{N}$ such that $p \notin \overline{h(m,x)}$. As p is a cluster point of $\langle x_n \rangle$, there is $k \geq m$ such that $x_k \notin \overline{h(m,x)} \supset h(k,x)$, a contradiction. □

Proposition 3.2. *A strongly α quasi-Nagata space X is a ks-space.*

Proof: Let h be a strongly α function and let l be a quasi-Nagata function. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n,x) = h(n,x) \cap l(n,x)$. We show that g is a ks-function.

Suppose that $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and the sequence $\langle y_n \rangle$ converges to p. We show that p is a cluster point of $\langle x_n \rangle$. Since l is a quasi-Nagata function, the sequence $\langle x_n \rangle$ has a cluster point, say q. If $p \neq q$, then there exists $m \in \mathbb{N}$ such that $p \notin \overline{h(m,q)}$. As $y_n \to p$, there is $k \geq m$ such that $y_n \notin \overline{h(m,q)} \supset h(m,q)$ for all $n \geq k$. But q is a cluster point of $\langle x_n \rangle$, so there is $j \geq k$ such that $x_j \in h(m,q)$, and then $y_j \in h(j, x_j) \subset h(m, x_j) \subset h(m,q)$, a contradiction. Therefore, $p = q$ and p is a cluster point of $\langle x_n \rangle$. □

Proposition 3.3. *A strongly α quasi-γ space X is a γ space.*

Proof: Suppose that X is a strongly α quasi-γ space. Yoshioka [19] has shown that a strongly α $w\gamma$ space is a γ space, so it suffices to show that X is a $w\gamma$ space. Since a quasi-γ space is a q space, by Lemma 3.1, X is first countable, and thus, X is a $w\gamma$ space. □

Yoshioka [20] proved that a strongly α $w\theta$ space is a θ space. Actually, we have the following stronger result.

Proposition 3.4. (a) *An α θ space X is a Θ space.* (b) *A strongly α $w\theta$ space X is a Θ space.*

Proof: (a) Let h be an α function and let l be a θ function. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n,x) = h(n,x) \cap l(n,x)$. Suppose that $\{p, x_n\} \subset g(n, y_n)$ for all $n \in \mathbb{N}$ and $\langle y_n \rangle$ has a cluster point q. We show that p is a cluster point of $\langle x_n \rangle$. Since q is a cluster point of $\langle y_n \rangle$, there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in h(k,q)$ and thus, $h(k, y_{n_k}) \subset h(k,q)$ for all $k \in \mathbb{N}$. Now, from $p \in g(n_k, y_{n_k}) \subset h(k, y_{n_k})$, it follows that $p \in \bigcap_{k \in \mathbb{N}} h(k,q) = \{q\}$. Hence, $p = q$ and then p is a cluster point of $\langle y_n \rangle$. Again, there exists a subsequence $\langle y_{n_i} \rangle$ of $\langle y_n \rangle$ such that $y_{n_i} \in l(i,q)$ for all $i \in \mathbb{N}$. Since $\{p, x_{n_i}\} \subset l(i, y_{n_i})$ for all $i \in \mathbb{N}$ and l is a θ function, p is a cluster point of $\langle x_{n_i} \rangle$ and hence of $\langle x_n \rangle$.

(b) Since a strongly α $w\theta$ space is a θ space [20] and a strongly α space is an α space, by (a), one readily sees that (b) holds. □

A space X is said to have a *quasi-$G^*_\delta(2)$ diagonal* (*quasi-G^*_δ diagonal*, respectively) [15] if there exists a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of open families of X such that for any pairs of distinct points $x, y \in X$, there exists $m \in \mathbb{N}$ such that $x \in \overline{st^2(x, \mathcal{G}_m)} \subset X - \{y\}$ ($x \in \overline{st(x, \mathcal{G}_m)} \subset X - \{y\}$, respectively).

It is clear that a space X which has a quasi-$G^*_\delta(2)$ diagonal is Hausdorff and if X has a quasi-$G^*_\delta(2)$ diagonal, then it has a quasi-G^*_δ diagonal.

Proposition 3.5. (a) *A $w\gamma$ space X with a quasi-$G^*_\delta(2)$ diagonal is a γ space.* (b) *A quasi-γ space X with a quasi-$G^*_\delta(2)$ diagonal is a γ space.*

Proof: (a) Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a quasi-$G^*_\delta(2)$ diagonal sequence and let k be a $w\gamma$ function. For each $x \in X$ and $n \in \mathbb{N}$, put

$$f(n,x) = \begin{cases} st(x, \mathcal{G}_n), & x \in \cup\mathcal{G}_n; \\ X, & x \notin \cup\mathcal{G}_n. \end{cases}$$

Let $h(n,x) = \bigcap_{i=1}^n f(i,x)$ and let $g(n,x) = h(n,x) \cap k(n,x)$. We show that g is a γ function. Suppose that $x_n \in g(n, y_n)$ and $y_n \in g(n,p)$ for all $n \in \mathbb{N}$. Since k is a $w\gamma$ function, the sequence $\langle x_n\rangle$ has a cluster point, say q. For each $n \in c(p) = \{i \in \mathbb{N} : p \in \cup\mathcal{G}_i\}$, we have $y_m \in g(m,p) \subset g(n,p) \subset f(n,p) = st(p, \mathcal{G}_n) \subset \cup\mathcal{G}_n$ whenever $m \geq n$, and then $x_m \in g(m, y_m) \subset g(n, y_m) \subset f(n, y_m) = st(y_m, \mathcal{G}_n)$. Consequently, $x_m \in st^2(p, \mathcal{G}_n)$ whenever $m \geq n$. Since q is a cluster point of $\langle x_n\rangle$, $q \in \overline{\{x_m : m \geq n\}} \subset \overline{st^2(p, \mathcal{G}_n)}$ for all $n \in c(p)$. Thus, $q \in \bigcap_{n\in c(p)} \overline{st^2(p, \mathcal{G}_n)} = \{p\}$, which implies that $p = q$ and p is a cluster point of $\langle x_n\rangle$.

(b) Suppose that X is a quasi-γ space with a quasi-$G^*_\delta(2)$ diagonal. Mohamad [15] proved that a q space with a quasi-G^*_δ diagonal is first countable and it is easy to show that a first countable quasi-γ space is a $w\gamma$ space. Since a quasi-γ space is a q space and X has a quasi-G^*_δ diagonal, we conclude that X is a $w\gamma$ space. Now, by (a), X is a γ space. $\square$

A space X is called a *c-stratifiable* [10] (*c-semi-stratifiable* [13], respectively,) space if it has a g-function g such that for any $K \in C(X)$, $\bigcap_{n\in\mathbb{N}} \overline{g(n,K)} = K$ ($\bigcap_{n\in\mathbb{N}} g(n,K) = K$, respectively), where $g(n,K) = \cup\{g(n,x) : x \in K\}$.

Obviously, a c-stratifiable space is Hausdorff.

Proposition 3.6. *A c-stratifiable quasi-Nagata space X is a ks-space.*

Proof: Let h be a c-stratifiable function and let l be a quasi-Nagata function. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n,x) = h(n,x) \cap l(n,x)$. We show that g is a ks-function.

Suppose that $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and the sequence $\langle y_n \rangle$ converges to p. We show that p is a cluster point of $\langle x_n \rangle$. Since l is a quasi-Nagata function, every subsequence of $\langle x_n \rangle$ has at least a cluster point. Let q be a cluster point of $\langle x_n \rangle$ and suppose that $p \neq q$. Then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \in h(k,q)$ and $x_{n_k} \neq p$ for all $k \in \mathbb{N}$. We now show that q is a unique cluster point of $\langle x_{n_k} \rangle$. Let r be a cluster point of $\langle x_{n_k} \rangle$ with $r \neq q$. Since h is a c-stratifiable function, there is $m \in \mathbb{N}$ such that $r \notin \overline{h(m,q)}$, but r is a cluster point of $\langle x_{n_k} \rangle$, so there is $j \geq m$ such that $x_{n_j} \notin \overline{h(m,q)} \supset \overline{h(j,q)}$. The contradiction shows that $r = q$, and from the above process of the proof, we know that q is a unique cluster point of $\langle x_{n_k} \rangle$. Since every subsequence of $\langle x_{n_k} \rangle$ has a cluster point, q is a cluster point of every subsequence of $\langle x_{n_k} \rangle$, which implies that $\langle x_{n_k} \rangle$ converges to q.

Now, let $K = \{x_{n_k} : k \in \mathbb{N}\} \cup \{q\}$. Then $K \in C(X)$ and $p \notin K$. Since h is a c-stratifiable function, we have $p \notin \bigcap_{n \in \mathbb{N}} \overline{h(n,K)}$ and hence, there exists $m \in \mathbf{N}$ such that $p \notin \overline{h(m,K)}$. As $\langle y_n \rangle$ converges to p, which shows that p is a cluster point of $\langle y_{n_k} \rangle$, there exists $j \geq m$ such that $y_{n_j} \notin \overline{h(m,K)} \supset \overline{h(j,K)} \supset h(j,K) \supset \bigcup_{k \in \mathbb{N}} h(j, x_{n_k})$. This implies that $y_{n_j} \notin h(j, x_{n_k})$ for all $k \in \mathbb{N}$. As a special case, we have $y_{n_j} \notin h(j, x_{n_j})$, a contradiction. Thus, $p = q$ and then p is a cluster point of $\langle x_n \rangle$. □

Proposition 3.7. (a) *A Hausdorff c-semi-stratifiable θ space X is a Θ space* [4]. (b) *A c-stratifiable $w\theta$ space X is a Θ space.*

Proof: (a) This has been proved in [4].

(b) Since c-stratifiable spaces are c-semi-stratifiable spaces and Hausdorff, by (a), it suffices to show that a c-stratifiable $w\theta$ space is a θ space.

Let h be a c-stratifiable function and let l be a $w\theta$ function. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n,x) = h(n,x) \cap l(n,x)$. We show that g is a θ function.

Suppose that $\{p, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, p)$ for all $n \in \mathbb{N}$. Then $\langle x_n \rangle$ has a cluster point q. Similar to Lemma 3.1, one can verify that a c-stratifiable q space is first countable. Since a $w\theta$ space is a q space, g is a first countable function. Then there is a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \to q$. Also, since $y_{n_k} \in g(k, p)$ for all $k \in \mathbb{N}$, we have $y_{n_k} \to p$. If $p \neq q$, then there is a subsequence $\langle y_{n_{k_i}} \rangle$ of $\langle y_{n_k} \rangle$ such that $y_{n_{k_i}} \to p$ and $y_{n_{k_i}} \neq q$ for all $i \in \mathbb{N}$.

Now, set $K = \{y_{n_{k_i}} : i \in \mathbb{N}\} \cup \{p\}$; then K is compact and $q \notin K$ and hence, there exists $m \in \mathbf{N}$ such that $q \notin \overline{h(m, K)}$. Since q is a cluster point of $\langle x_{n_{k_i}} \rangle$, there is $j \geq m$ such that $x_{n_{k_j}} \notin \overline{h(m, K)} \supset h(j, K)$, which implies that $x_{n_{k_j}} \notin h(j, y_{n_{k_i}})$ for all $i \in \mathbb{N}$. As a special case, $x_{n_{k_j}} \notin h(j, y_{n_{k_j}})$, which contradicts the fact that $x_{n_{k_j}} \in g(n_{k_j}, y_{n_{k_j}}) \subset h(j, y_{n_{k_j}})$. The contradiction shows that $p = q$ and then p is a cluster point of $\langle x_n \rangle$. □

A space X is said to have *property* A' (*property* A, respectively) [11] if there is a sequence $\langle V_n \rangle$ of relations on X satisfying the following.

(1) For each $x \in X$ and $n \in \mathbb{N}$, $x \in V_{n+1}(x) \subset V_n(x) \in \tau$.

(2) For each $x \in X$, $\bigcap_{n \in \mathbb{N}} \overline{V_n^2(x)} = \{x\}$ ($\bigcap_{n \in \mathbb{N}} V_n^2(x) = \{x\}$, respectively).

It is easy to see that a space X having property A' is Hausdorff, and similar to the proof of Lemma 3.1, one can prove that a q space with property A' is first countable. W. F. Lindgren and P. Fletcher [11] proved that a $w\gamma$ space which has property A' is a γ-space. Since a quasi-γ space is a q space and a first countable quasi-γ space is a $w\gamma$ space, we have the following.

Proposition 3.8. *A quasi-γ space X that has property A' is a γ space.*

Proposition 3.9. *A quasi-Nagata space X that has property A' is a ks-space.*

Proof: Let h be a quasi-Nagata function and let $\langle V_n \rangle$ be a sequence of relations satisfying the conditions of property A'. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n, x) = h(n, x) \cap V_n(x)$. We show that g is a ks-function.

Suppose that $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and the sequence $\langle y_n \rangle$ converges to p. Then $\langle x_n \rangle$ has a cluster point q, and hence, there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \in V_k(q)$ for all $k \in \mathbb{N}$. From $y_{n_k} \in V_{n_k}(x_{n_k}) \subset V_k(x_{n_k})$, it follows that $y_{n_k} \in V_k^2(q)$ for all $k \in \mathbb{N}$.

Suppose $p \neq q$. Then there exists $m \in \mathbb{N}$ such that $p \notin \overline{V_m^2(q)}$. As $y_n \to p$, p is a cluster point of $\langle y_{n_k} \rangle$ and thus, there is $j \geq m$ such that $y_{n_j} \notin \overline{V_m^2(q)} \supset \overline{V_j^2(q)} \supset V_j^2(q)$. The contradiction shows that $p = q$ and thus, p is a cluster point of $\langle x_n \rangle$. □

Proposition 3.10. (a) *A θ space X that has property A is a Θ-space.* (b) *A $w\theta$ space X that has property A' is a Θ-space.*

Proof: (a) Let h be a θ function and let $\langle V_n \rangle$ be a sequence of relations satisfying the conditions of property A. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n, x) = h(n, x) \cap V_n(x)$.

Suppose that $\{p, x_n\} \subset g(n, y_n)$ for all $n \in \mathbb{N}$ and q is a cluster point of $\langle y_n \rangle$. Then there exists a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in V_k(q)$ for all $k \in \mathbb{N}$. From $p \in V_k(y_{n_k})$, it follows that $p \in \bigcap_{k \in \mathbb{N}} V_k^2(q) = \{q\}$. Hence, $p = q$ and then p is a cluster point of $\langle y_n \rangle$. Now, since h is a θ function, p is a cluster point of $\langle x_n \rangle$. Thus, g is a Θ function.

(b) Since property A' implies property A, by (a), it suffices to show that a $w\theta$ space that has property A' is a θ-space.

Let h be a $w\theta$ function and let $\langle V_n \rangle$ be a sequence of relations satisfying the conditions of property A'. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n, x) = h(n, x) \cap V_n(x)$. We show that g is a θ function.

Suppose that $\{p, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, p)$ for all $n \in \mathbb{N}$. Then $x_n \in V_n^2(p)$ for all $n \in \mathbb{N}$. Since h is a $w\theta$ function, $\langle x_n \rangle$ has a cluster point q. If $p \neq q$, then there exists $m \in \mathbb{N}$ such that $q \notin \overline{V_m^2(p)}$, and thus, there is $k \geq m$ such that $x_k \notin \overline{V_m^2(p)} \supset V_k^2(p)$, a contradiction. Therefore, p is a cluster point of $\langle x_n \rangle$ and g is a θ function. □

Lemma 3.11 ([19])**.** *For a T_0 space X, the following are equivalent.*

(a) *X is metrizable;*
(b) *X is a ks γ space;*
(c) *X is a ks θ space.*

Statement (c) of Theorem 3.12 and (b) of Theorem 3.13 have been proved in [19] and [16], respectively. We restate them here for completeness.

Theorem 3.12. (a) *A strongly α quasi-Nagata quasi-γ space X is metrizable.*

(b) *A quasi-Nagata quasi-γ space X with a quasi-$G^*_\delta(2)$ diagonal is metrizable.*

(c) *A c-stratifiable quasi-Nagata quasi-γ space X is metrizable* [18].

(d) *A quasi-Nagata quasi-γ space X that has property A' is metrizable.*

Proof: (a) follows from Proposition 3.2, Proposition 3.3, and Lemma 3.11.

(b) follows from Theorem 4.10 in [15], Proposition 3.5, and Lemma 3.11.

(c) has been proved in [19].

(d) follows from Proposition 3.8, Proposition 3.9, and Lemma 3.11. □

Theorem 3.13. (a) *A strongly α quasi-Nagata $w\theta$ space X is metrizable.*

(b) *A quasi-Nagata $w\theta$ space X with a quasi-$G^*_\delta(2)$ diagonal is metrizable* [16].

(c) *A c-stratifiable quasi-Nagata $w\theta$ space X is metrizable.*

(d) *A quasi-Nagata $w\theta$ space X that has property A' is metrizable.*

Proof: (a) follows from Proposition 3.2, Proposition 4.7 in [20] (a strongly α $w\theta$ space X is a θ space), and Lemma 3.11.

(b) has been proved in [16].

(c) follows from Proposition 3.6, Proposition 3.7, and Lemma 3.11.

(d) follows from Proposition 3.9, Proposition 3.10, and Lemma 3.11. □

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References

[1] Carlos J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

[2] ______, *On metrizability of topological spaces*, Canad. J. Math. **20** (1968), 795–804.

[3] Geoffrey D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. **32** (1970), 47–54.

[4] Peter Fletcher and William F. Lindgren, *On $w\Delta$-spaces, $w\sigma$-spaces and $\Sigma^\sharp$-spaces*, Pacific J. Math. **71** (1977), no. 2, 419–428.

[5] Ying Ge [Ge Ying] and Chris Good, *A note on monotone countable paracompactness*, Comment. Math. Univ. Carolin. **42** (2001), no. 4, 771–778.

[6] Chris Good, Robin Knight, and Ian Stares, *Monotone countable paracompactness*, Topology Appl. **101** (2000), no. 3, 281–298.

[7] R. E. Hodel, *Moore spaces and $w\ \Delta$-spaces*, Pacific J. Math. **38** (1971), 641–652.

[8] ______, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J. **39** (1972), 253–263.

[9] V. Dwight House, *Countable products of generalized countably compact spaces*, Pacific J. Math. **57** (1975), no. 1, 183–197.

[10] Kyung Bai Lee, *Spaces in which compacta are uniformly regular G_δ*, Pacific J. Math. **81** (1979), no. 2, 435–446.

[11] W. F. Lindgren and P. Fletcher, *Locally quasi-uniform spaces with countable bases*, Duke Math. J. **41** (1974), 231–240.

[12] D. J. Lutzer, *Semimetrizable and stratifiable spaces*, General Topology and Appl. **1** (1971), no. 1, 43–48.

[13] Harold W. Martin, *Metrizability of M-spaces*, Canad. J. Math. **25** (1973), 840–841.

[14] ______, *A note on the metrization of γ-spaces*, Proc. Amer. Math. Soc. **57** (1976), no. 2, 332–336.

[15] A. M. Mohamad, *Conditions which imply metrizability in some generalized metric spaces*, Topology Proc. **24** (1999), Spring, 215–232.

[16] ______, *Some results on quasi-σ and θ spaces*, Houston J. Math. **27** (2001), no. 1, 59–65.

[17] Lisheng Wu, *On k-semi-stratifiable spaces* (Chinese), J. Suzhou Univ. **1** (1983), 1–4.

[18] Shengxiang Xia, *On monotone countable mesocompactness*, Questions Answers Gen. Topology **23** (2005), no. 2, 113–120.

[19] Iwao Yoshioka, *On the metrizations of γ-spaces and ks-spaces*, Questions Answers Gen. Topology **19** (2001), no. 1, 55–72.

[20] ______, *Closed images of spaces having g-functions*, Topology Appl. **154** (2007), no. 9, 1980–1992.

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